Etale homotopy theory (after Artin-Mazur, Friedlander et al.)

Heidelberg
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Lecture 1: Motivation

March 18, 2014

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For example, the idea of a Riemann surface grew out of the attempt to understand integrals of rational functions over the complex numbers.

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Algebraic geometry underwent an algebraization: e.g. varieties over fields of positive characteristic and schemes over more general bases.

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For a variety $X \subset P^n$ over the complex numbers: take the complex points $X(C)$ and topologize it as a subspace in complex projective space.
Write $X_{cl} := X(C)$ for this topological space.

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This "cycle map" is the subject of the famous Hodge conjecture.

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Now take the complex manifold $X_C(C)$ as before and get a homotopy type and topological invariants.

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Wait: there is not just one embedding $K \subset C$ and we have to make a choice!

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Serre's answer:

Yes, it does!

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Theorem (Serre):
There is a smooth projective variety V defined over a number field K and there are embeddings ϕ and ψ of K into $\mathbb{C}$ such that

$$\pi_1(V_{cl}^{\phi}) \not\approx \pi_1(V_{cl}^{\psi}).$$

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There is a smooth projective variety V defined over a number field K and there are embeddings ϕ and ψ of K into C such that

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Thus, even though the two complex varieties V^{ϕ} and V^{ψ} are conjugate, they have different homotopy types.

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Key observation:

Given an embedding $\phi: K \subset C$, $\pi_1(E_{cl}^{\phi})$ is a projective R-module of rank one. Thus $\pi_1(E_{cl}^{\phi})$ corresponds to an element e_{ϕ} in Cl.

Conversely, every element of Cl is of the form e_{ϕ} for some embedding $\phi: K \subset C$ and we have $e_{\phi} = e_{\phi'}$ if and only if ϕ' is either equal to ϕ or to its complex conjugate.

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$\pi_1(E^{\psi}_{cl})$ is not a free R-module (exists since $h>1$).

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Now we can define: V := (Y x A)/G.

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Finally: $\pi_1(V_{cl}^{\phi}) \not\approx \pi_1(V_{cl}^{\psi})$.

(Because: An isomorphism would imply that $\pi_1(A_{cl}^{\phi})$ and $\pi_1(A_{cl}^{\psi})$ were isomorphic as S-modules. ⚡)

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the etale homotopy type.

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- an intrinsic invariant, i.e., only depend on the isomorphism type of the scheme;
- a space whose cohomology and fundamental group should be equal to Grothendieck’s etale cohomology and etale fundamental group;
- functorial; in particular, for varieties over fields there should be a Galois action.

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- Quillen's idea for a proof of the Adams conjecture, a purely topological statement.

Proofs of the Adams conjecture:

We will discuss two methods to prove the Adams conjecture (and there are more). Both involve etale homotopy theory in an essential way.

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- In Lecture 3: Sullivan's approach.

Galois symmetries on profinite completions of spaces are induced by etale homotopy types.

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By endowing E with a Hermitean metric and looking at vectors of length 1 in $E-0$ we get a fiber bundle

$$S(E) \to X$$

with fiber a $2n-1$-sphere S^{2n-1}.

Fiber homotopy equivalence:

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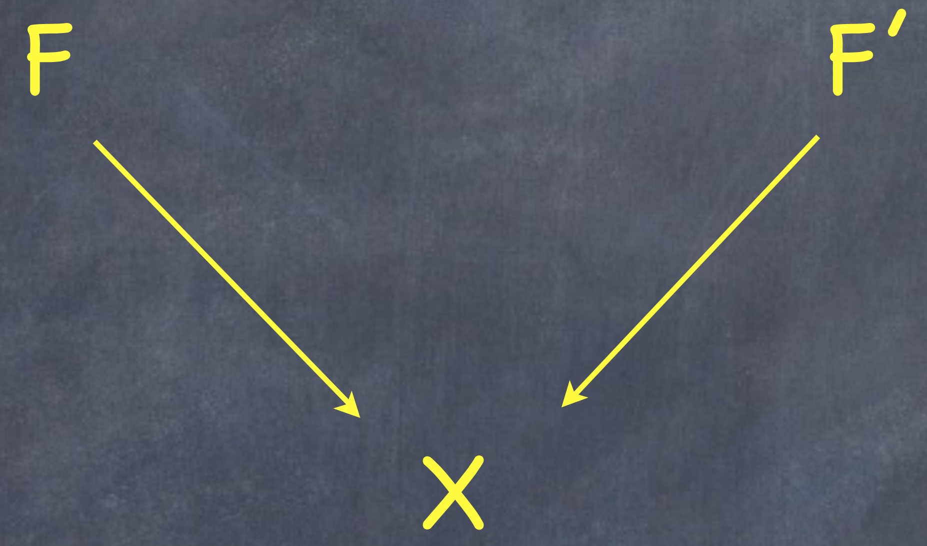
We say that two fiber bundles F and F' over X



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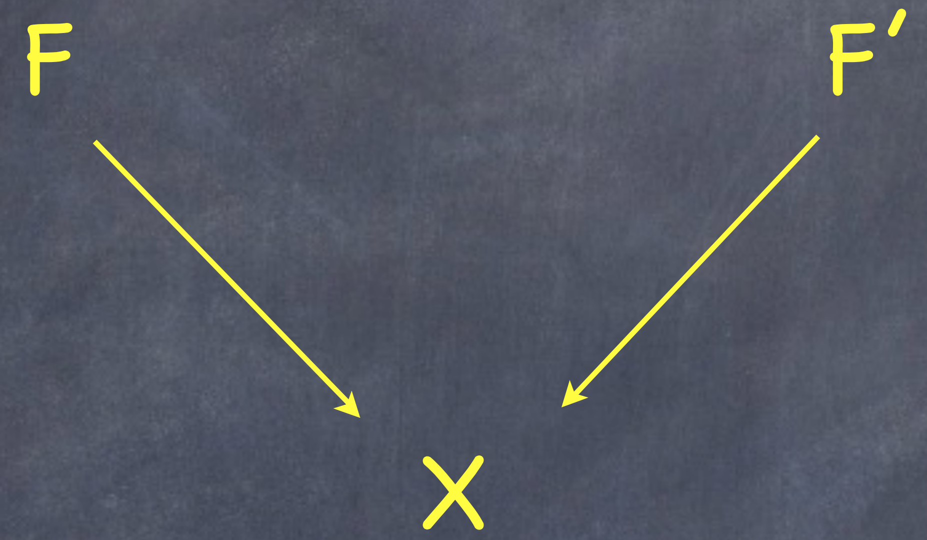

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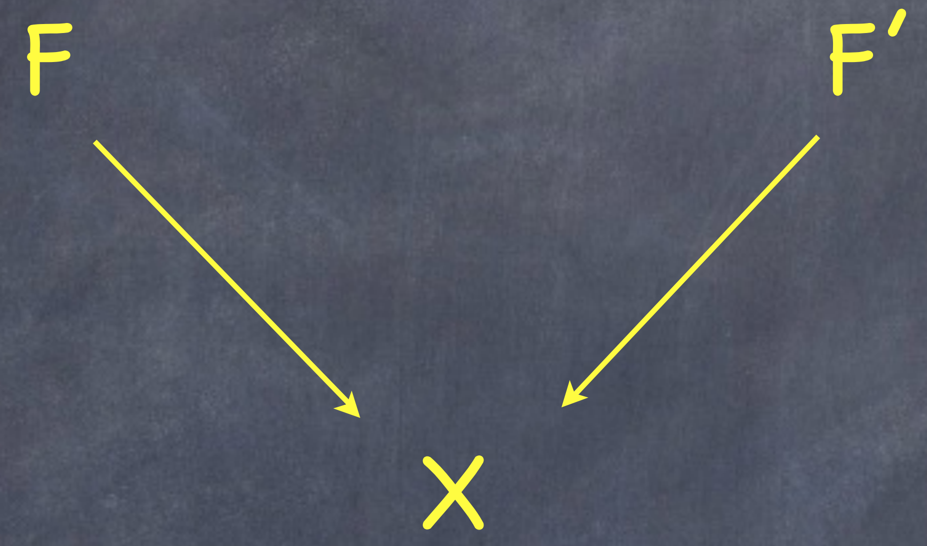

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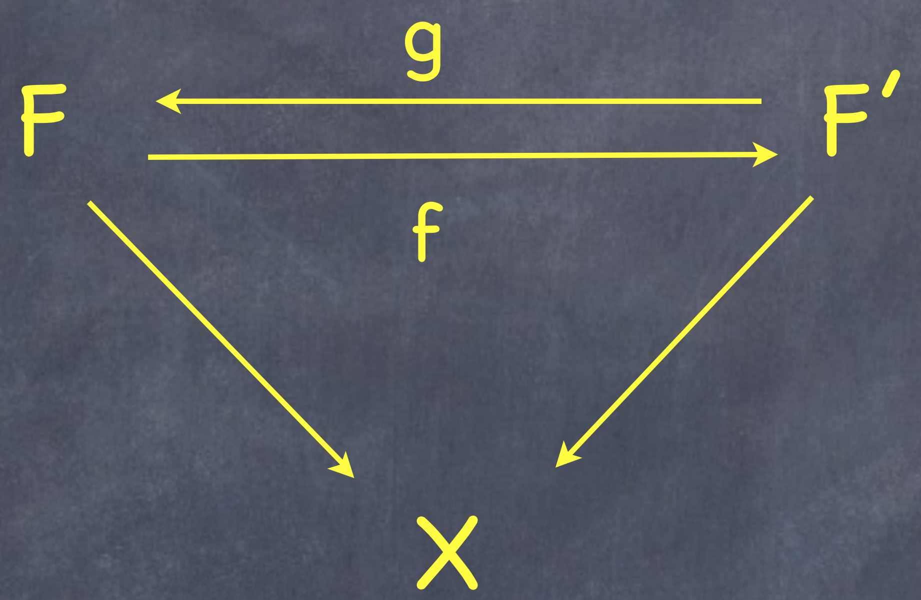

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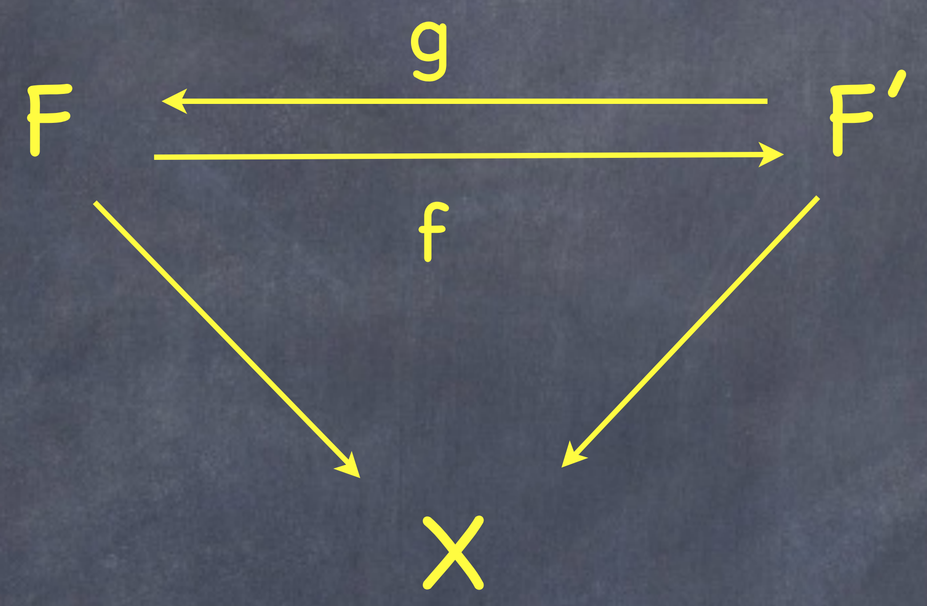

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and homotopy equivalences $gf \simeq id_F$ and $fg \simeq id_{F'}$ which at each time t are maps of fiber bundles.

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The functor S(-) induces the J-homomorphism

$$J: K(X) \to SF(X).$$

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Adams' conjecture: Let E be a complex vector bundle over a finite CW-complex X and k an integer. Then there is an integer n such that $k^n(\psi^k E-E)$ maps to zero under J.

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Adams' conjecture: Let E be a complex vector bundle over a finite CW-complex X and k an integer. Then there is an integer n such that $k^n(\psi^k E-E)$ maps to zero under J.

(In fact, Adams conjectures also the case of real vector bundles.)

The Quillen-Friedlander approach:

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The idea of the proof is based on three obeservations:

Quillen's observation 1:

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Let R be a strict henselization of $\mathbb{Z}$ at p, $R \subset \mathbb{C}$ an embedding and $k=\bar{\mathbb{F}}_p$ the closed point of R, V_R a proper smooth scheme over R.

Then there are canonical equivalences of spaces

$$\hat{V_{\mathbb{C},cl}} \xrightarrow{\sim} \hat{V_{\mathbb{C},et}} \xrightarrow{\sim} \hat{V_{R,et}} \xleftarrow{\sim} \hat{V_{k,et}}$$

where ^ denotes profinite completion away from p.

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Let $F\colon V \to V$ be the Frobenius map and write

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Then we have an equality in $K(V)$

$$\psi^p(E) = E^{(p)}.$$

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and induces an equivalence

$$(E-0)^{\wedge}_{et} \approx (E^{(p)}-0)^{\wedge}_{et}.$$

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Crucial point: The Grassmannian and the canonical bundle can be defined as schemes over the integers.

Then we should be able to apply the observations in the following way:

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$$
\begin{array}{ccccccccc}
K(V_{C,cl}) & \longleftarrow & & & K(V_C) & \longleftarrow & K(V) & \longrightarrow & K(V_k) \\
\downarrow J & & & & \downarrow J & & \downarrow J & & \downarrow J \\
SF(V_{C,cl}) & \xrightarrow{\Theta_L} & SF(V^{\wedge}_{C,cl}) & \longleftarrow & SF(V^{\wedge}_{C,et}) & \longleftarrow & SF(V^{\wedge}_{et}) & \longrightarrow & SF(V^{\wedge}_{k,et})
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\downarrow J & & & & \downarrow J & & \downarrow J & & \downarrow J \\
SF(V_{C,cl}) & \xrightarrow{\Theta_L} & SF(V^{\wedge}_{C,cl}) & \longleftarrow & SF(V^{\wedge}_{C,et}) & \longleftarrow & SF(V^{\wedge}_{et}) & \longrightarrow & SF(V^{\wedge}_{k,et})
\end{array}
$$

We need to show: $J(\psi^p(E_C) - E_C) = 0$ in $SF(V^{\wedge}_{C,cl})$.

By the comparison of classical and etale homotopy types,

The Quillen-Friedlander proof:

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By “Frobenius = Adams operation” it suffices to show:

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This holds by Observation 3 and we are done!

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- If V is a scheme over R and E an algebraic vector bundle of dimension n, then

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In his thesis, Friedlander proved that geometric and homotopy fibers behave well under etale homotopy types, thereby proved the Adams conjecture.