Etale homotopy theory (after Artin-Mazur, Friedlander et al.)

Heidelberg
March 18-20, 2014

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Lecture 2: Construction

March 19, 2014

Towards the idea:

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These etale open sets are great for defining sheaf cohomology.

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For each $n \geq 0$, form

$$U_{i0,\ldots,in} = U_{i0} \times_X \ldots \times_X U_{in}$$

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(Would need: all $U_{i0,\dots,in}$ are contractible.)

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where the colimit ranges over all etale covers.

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Observation: The global sections $H^0(U_{i0,\dots,in};F)$ only depend on the set of connected components $\pi_0(U_{i0,\dots,in})$.

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In order to make this idea work in full generality we need some preparations.

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category.

We get a category pro-C by defining the morphisms to be

$$\mathrm{Hom}\,(X, Y) = \lim_j \operatorname{colim}_i \mathrm{Hom}\,(X_i, Y_j).$$

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For A an abelian group, the homology groups of X are

$$H_n(X;A) = \{H_n(X_i;A)\}_{\{i\in I\}}.$$

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If A is has an action by $\pi_1(X)$, then there are also cohomology groups of X with local coefficients in A.

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such that Hom (X,W) ≈ Hom (X^,W) for W in LH.

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- of cohomology groups

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if A is a finite abelian L-group.

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To see this, let $cosk_n$: $H \rightarrow H$ be the coskeleton functor which kills homotopy in dimension $\geq n$.

Let X be a space and let $X^\#$ be the inverse system

$$X^\# = \{cosk_n X\}.$$

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Hence the inverse exists if and only if

$X = cosk_n X$ for some integer n.

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Theorem (Artin-Mazur): A map $f{:}X \to Y$ in pro-H is a #-isomorphism if and only if f induces an

$$\pi_n(f){:}\ \pi_n(X) \xrightarrow{\approx} \pi_n(Y) \text{ for all } n \geq 0.$$

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#-isomorphisms and a Whitehead theorem:

Let $X \to Y$ be map in pro-H and L a set of primes.

Then $f^{\wedge}: X^{\wedge} \to Y^{\wedge}$ is a #-isomorphism if and only if

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- $\pi_1(X)^{\wedge} \approx \pi_1(Y)^{\wedge}$ and
- $H^n(Y;A) \approx H^n(X;A)$ for every $n \geq 0$ and every $\pi_1(Y)$-twisted coefficient group A which is a finite abelian L-group such that the action of $\pi_1(Y)$ factors through $\pi_1(Y)^{\wedge}$.

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But: Suppose that X is simply-connected and all $\pi_n(X)$'s are "L-good" groups. Then

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(There are improvements by Sullivan.)

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Serre calls G "L-good" if μ^* is an isomorphism for all $n \geq 0$ and every $G^\wedge$-module A which is a finite abelian L-group.

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For example: finitely gen. abelian groups are good; $\pi_1(X)$ of a smooth connected curve X is good.

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This is an improvement due to Sullivan of the results by Artin-Mazur.

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Let $G=\{G_i\}$ be a pro-group and $K(G,1)=\{K(G_i,1)\}$ its classifying pro- space such that

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Then G is L-good if and only if the canonical map of pro-groups $G \to G\hat{}$ induces a #-isomorphism

$$K(G,1) \approx K(G\hat{},1).$$

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Sullivan: If each X_i has finite homotopy groups, then the functor

$$\lim_i\, [-,X_i]: H^{op} \to \text{Sets}$$

is representable in H by a CW-complex, which he denotes by $\lim_i X_i$.

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For example: X is locally noetherian.

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$$U \;\begin{array}{c}\leftarrow\\\leftarrow\\\rightarrow\end{array}\; U \times_X U \;\begin{array}{c}\rightarrow\\\leftarrow\\\leftarrow\\\leftarrow\\\rightarrow\end{array}\; U \times_X U \times_X U \;\begin{array}{c}\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow\end{array}\; U \times_X U \times_X U \times_X U \quad \cdots$$

i.e. U_n is the $n+1$-fold fiber product of U over X.

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Idea: Choose coverings in each dimensions for forming $U_\bullet$.

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- Continuing this process leads to a hypercovering of X.

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But: This category is not cofiltering !

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Verdier's theorem: Let F be an etale sheaf on X. Then for every $n \geq 0$ there is an isomorphism

$$H^n(X;F) \approx \mathrm{colim}_{U \in HR(X)} H^n(F(U_\bullet)).$$

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Note: Since we had to take homotopy classes of maps of hypercoverings, X_{et} is only a pro-object in the homotopy category H.

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The cohomology of X_{et} is the etale cohomology of X:

$H^n_{et}(X;F) \approx H^n(X_{et};F)$ for all $n \geq 0$ and every locally constant etale sheaf F on X.

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In general: $\pi_1(X_{et})$ is different from the profinite etale fundamental group of Grothendieck in SGA 1 (but it is the one of SGA 3).

Etale homotopy groups:

The etale homotopy groups are defined as

$$\pi_n(X) := \pi_n(X_{et}) \text{ for all } n \geq 0.$$

In general: $\pi_1(X_{et})$ is different from the profinite etale fundamental group of Grothendieck in SGA 1 (but it is the one of SGA 3).

For: $\pi_1(X_{et})$ takes all etale covers into account, not just finite ones.

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For example: every normal scheme (local rings are integrally closed) is a geometrically unibranch.

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- It contains the information of known etale topological invariants.