Etale homotopy theory (after Artin-Mazur, Friedlander et al.)

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Lecture 3: Applications

March 20, 2014

Our selection of applications:

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- Comparisons

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For more applications see Friedlander's great book.

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There is a canonical map $\varepsilon : X_{cl} \to X_{et}$ in pro-H.

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For X geometrically unibranch:

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If X is geometrically unibranch and X_{cl} is simply connected:

$\pi_n(X_{cl})\hat{} \approx \pi_n(X_{et})$ for all n.

Serre's example revisited:

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Then after profinite completion there is an isomorphism in pro-H:

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Thus the possible difference of the homotopy types of $X_{1,cl}$ and $X_{2,cl}$ vanishes after completion. To prove this we use etale homotopy theory.

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where ^ denotes completion away from char k.

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The "complex K-theory" of T with Z/m-coefficients can be defined as

$$K^0(T;Z/m) = \mathrm{Hom}_H\,(C(m) \wedge T, BU) \text{ and}$$
$$K^1(T;Z/m) = \mathrm{Hom}_H\,(S^1 \wedge C(m) \wedge T, BU).$$

where BU is the infinite complex Grassmannian.

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If Y is a pro-space, its (complex) K-theory is defined by

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If X is a scheme of finite type over a complete discrete valuation ring with separably closed residue field, Friedlander defines the "etale K-theory of X" to be the K-theory of X_{et}.

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If X is a complex variety, then

$$K^*_{et}(X;Z/m) \approx K^*(X_{cl};Z/m).$$

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Dwyer and Friedlander interpreted important arithmetic questions in terms of this Galois action on etale K-theory.

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After the first construction by Friedlander there were more sophisticated definitions of etale K-theory by Friedlander, Dwyer-Friedlander and Thomason.

They all come equipped with natural maps

$$K^*_{alg}(X;Z/l^n) \to K^*_{et}(X;Z/l^n)$$

where l is a prime invertible on X.

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Thomason: If X is a smooth quasi-projective variety over a field of characteristic ≠ l of finite mod-l etale cohomological dimension, then

$$K_*^{alg}(X;Z/l^n)[\beta^{-1}] \to K_*^{et}(X;Z/l^n)$$

is an isomorphism.

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"Quillen-Lichtenbaum conjecture":

If X is a smooth variety over a field and n is invertible in k, then the natural map

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Note: The "Quillen-Lichtenbaum conjecture" follows from the "Bloch-Kato conjecture".

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Let $BU(n)$ be the Grassmannian of complex n-planes, BU be the infinite complex Grassmannian.

Let BG be the classifying space of (stable) spherical fibrations.

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Adams: For all k, the map

$$J \circ (\psi^k - 1) : BU(n) \to BU \to BG[1/k]$$

is null-homotopic, i.e., homotopic to a constant map.

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First step: As in Lecture 1, it suffices to consider the p-completed maps (for each p with $(k,p)=1$)

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Sullivan's amazing idea:

Interpret the Adams operations as "Galois symmetries" on profinitely completed homotopy types of classifying spaces.

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Concretely: $\sigma \in Gal_Q$ acts on $\pi_2(P^n(C)\hat{})=Z_p$ by multiplication with $\chi(\sigma)$ where χ denotes the cyclotomoic character.

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Key fact: The etale homotopy type tells us how to read off the action on finite covers.

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In the same way: There is a nice action of Gal_Q on $P^\infty(C)$^ ($\approx K(Z_p,2)$) and on $BU(n)$^:

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Concretely: $\sigma \in Gal_Q$ acts on $BU(n)$^ such that

$$\sigma(c_i) = \chi(\sigma)^{-i} \cdot c_i$$

on cohomology, where c_i is the ith Chern class.

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Key observation: This σ is an "unstable version" of the Adams operation ψ^k. (Use splitting principle and compute the effect on line bundles.)

This is very remarkable: Without completions, ψ^k is an endomorphism of BU and not BU(n).

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is homotopy commutative and cartesian.

Thus, twisting by ψ^k does not change the corresponding spherical fibration. This completes the sketch of Sullivan's proof of the Adams conjecture.

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Let k be a field. A "sums-of-squares formula" of type $[r,s,n]$ is an identity of the form

$$(x_1^2 + \ldots + x_r^2)\cdot(y_1^2 + \ldots + y_s^2) = z_1^2 + \ldots + z_n^2$$

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where each z_i is a bilinear expression in the x's and y's with coefficients in k.

For $k=\mathbb{R}$ such an identity corresponds to an "axial map"

$$\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}.$$

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Davis found improved results using BP-theory.

Sums of squares in positive characteristic:

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Etale realizations and BP-theory for pro-spaces: The topological obstructions do not depend on the field k ($\mathrm{char}\ k \neq 2$).

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- Schmidt's extension of Artin-Mazur's etale type

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The etale homotopy type of M is the pro-object

$$\pi Triv_{/M} \to H$$

$$U_\bullet \mapsto \pi_0(U_\bullet).$$

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This defines a functor

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But: in general, the map $A^1_S \to S$ does not induce an isomorphism of etale fundamental groups.

The functor ht only factors through A^1-localization if we complete away from the residue characteristics.

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Isaksen's "rigidified" etale realization:

Isaksen extends Friedlander's etale topological type to motivic spaces.

The etale type of simplicial presheaves on Sm_S is the formal extension of a colimit preserving functor of the etale type of schemes.

Using a Z/l-model structure, the etale type becomes a left Quillen functor from motivic spaces to the pro-category of simplicial sets.

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Its elements are called "cycles".

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Denote $CH^p(X) := Z^p(X)/_{\sim rat}$ for cycles modulo "rational equivalence".

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In the 1990's Totaro showed that cl_H factors via a quotient of complex cobordism $MU^*(X_{cl})$:

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$$CH^p(X) \xrightarrow{\ cl_{MU}\ } MU^{2p}(X)\otimes_{MU^*}\mathbb{Z} \xrightarrow{\ \vartheta\ } H^{2p}(X;\mathbb{Z})$$

$$Z\subset X \mapsto [Z_{sm}]_{MU\text{-fund. class}} \mapsto [Z_{sm}]_{H\text{-fund. class}}$$

Totaro's factorization:

$$\begin{array}{ccccc} CH^p(X) & & \xrightarrow{\ cl_H\ } & & H^{2p}(X;Z) \\ & \searrow^{cl_{MU}} & & \nearrow^{\vartheta} & \\ & & MU^{2p}(X)\otimes_{MU^*}Z & & \end{array}$$

$$\begin{array}{ccccc} Z\subset X & & & & [Z_{sm}]_{H\text{-fund. class}} \\ & \searrow & & \nearrow & \\ & & [Z_{sm}]_{MU\text{-fund. class}} & & \end{array}$$

This is diagram commutes.

Consequences:

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- A topological obstruction on the image of cl_H: image of cl_H is contained in image of ϑ.

In particular, all odd degree cohomology operations must vanish on the image of cl_H.

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- A topological obstruction on the image of cl_H: image of cl_H is contained in image of ϑ.
In particular, all odd degree cohomology operations must vanish on the image of cl_H.
- More importantly: We can study the kernel of cl_H by finding elements in the kernel of ϑ that are in the image of cl_{MU}; good candidates are polynomials in Chern classes.
Totaro used this method to find important new examples of elements in the Griffiths group.

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Algebraic cycles and etale cobordism:

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Integral Tate “conjecture”: Is

$$CH^i(X)\otimes\mathbb{Z}_l \xrightarrow{cl_{Het}} H^{2i}(X_{\bar{k}};\mathbb{Z}_l(i))^{G_k}$$

surjective? The answer is “no” as we will explain now.

Etale cobordism (Q.):

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For a variety X over an alg. closed field we define the l-adic etale cobordism of X to be

$$\hat{MU}^n_{et}(X) := \mathrm{Hom}_{\hat{SH}}(\Sigma^\infty(\hat{X}_{et}), \Sigma^n \hat{MU})$$

where $\hat{SH}$ is the stable l-adic homotopy category of profinite spectra.

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Note: The construction of cl_{MUet} uses that there are "tubular neighborhoods" in etale homotopy.

Consequences:

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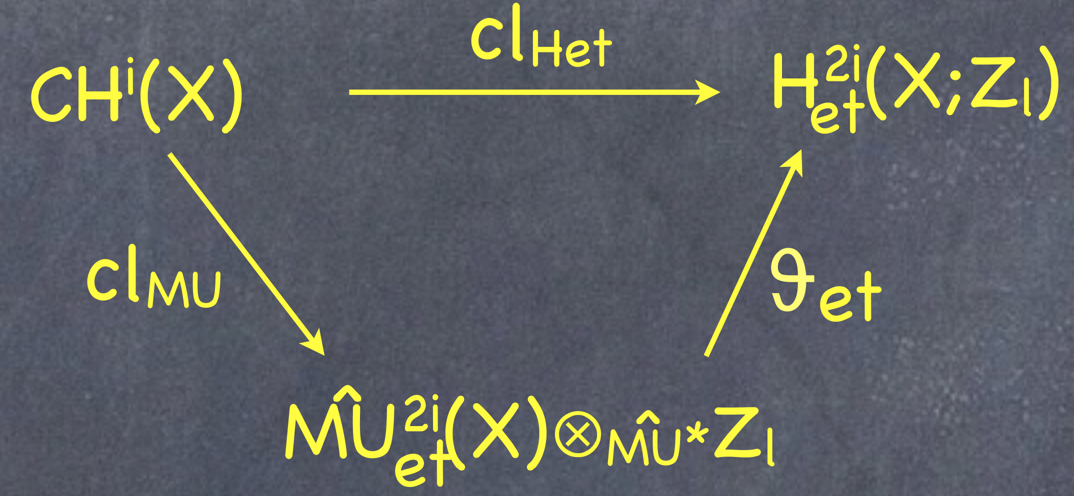

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- Cycles of Atiyah and Hirzebruch provide counter-examples to the integral version of the Tate conjecture for varieties over finite fields.

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where $\hat{H}_{ket}$ is a suitable homotopy category of "profinite spaces" over k_{et}.

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We interpret this set as the set $\pi_0((X_{et})^{hG_k})$ of connected components of the "continuous homotopy fixed points of X_{et}".

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Pal and Harpaz-Schlank use the map

$$X(k) \to \pi_0((X_{et})^{hG_k})$$

to reinterpret obstructions to the existence of rational points in terms of etale homotopy theory.

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The hope: Homotopy methods give us a chance to understand the set $\pi_0((X_{et})^{hG_k})$ and the above map. But so far, we don't know if this works.

Thank you!