Homology, Homotopy and Applications, vol. ??(??), ????, pp.1Doc-Start–30

ON THE COKERNEL OF THE THOM MORPHISM FOR COMPACT LIE GROUPS

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Abstract

We give a complete description of the potential failure of the surjectivity of the Thom morphism from complex cobordism to integral cohomology for compact Lie groups via a detailed study of the Atiyah–Hirzebruch spectral sequence and the action of the Steenrod algebra. We show how the failure of the surjectivity of the topological Thom morphism can be used to find examples of non-trivial elements in the kernel of the induced differential Thom morphism from differential cobordism of Hopkins and Singer to differential cohomology. These arguments are based on the particular algebraic structure and interplay of the torsion and non-torsion parts of the cohomology and cobordism rings of a given compact Lie group. We then use the geometry of special orthogonal groups to construct concrete cobordism classes in the non-trivial part of the kernel of the differential Thom morphism.

1. Introduction

The Thom morphism $\tau\colon MU \longrightarrow H\mathbb{Z}$ from complex cobordism to integral singular cohomology is of fundamental importance for the study of the stable homotopy category. A special feature of τ is that it encodes both deep algebraic and geometric structures. This is a common theme of the present paper and is reflected in the following two ways τ may be described. On the one hand, τ interpolates between two extreme ends of the spectrum of oriented cohomology theories which may be classified by their formal group laws, as τ corresponds to the unique morphism from the universal formal group law to the additive one (see [**1**, II Example (4.7)]). On the other hand, τ may be described geometrically in the following way. Let X be a smooth manifold. By Quillen's work in [**23**], classes in $MU^*(X)$ can be represented by proper complex-oriented maps $g\colon M \to X$. The Thom morphism sends the cobordism class $[g]$ to the pushforward $g_*[M]$ of the Poincaré dual $[M]$ of the fundamental class of M. Thus, roughly speaking, a cohomology class is in the image of τ if it can be expressed by a fundamental class of an almost-complex manifold. Hence the question whether τ is surjective or not is directly connected to concrete geometric phenomena (see also

The second-named author was partially supported by the RCN Project No. 313472 *Equations in Motivic Homotopy.*

2020 Mathematics Subject Classification: 57R77, 57T10, 55N22, 55S05, 55S10

Article available at `http://dx.doi.org/10.4310/HHA.????.v??.n??.a?`

[**25**]). In cohomological degrees $i = 0, 1, 2$, the Thom morphism is surjective for all spaces, since the Eilenberg–MacLane spaces $K(\mathbb{Z}, i)$ are torsion-free for $i = 0, 1, 2$. In cohomological degrees $i \geqslant 3$, however, τ may fail to be surjective, even though the coefficient ring of MU is much larger than the one of $H\mathbb{Z}$. It is well-known that the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(X; MU^q) \implies MU^{p+q}(X)$$

both provides a way to show that τ may be surjective and that its differentials may yield obstructions to the surjectivity of τ (see [**2**]). However, the image of the Thom morphism has not been studied for many types of spaces.

The purpose of the present paper is to give a complete description of the potential failure of the surjectivity of the Thom morphism for compact connected Lie groups which provide an important class of examples of smooth manifolds. Our first main result is the following:

Theorem 1.1. *Let G be a compact connected Lie group with simple Lie algebra. Then Table 1 below shows the minimal cohomological degree q for which the Thom morphism $\tau\colon MU^q(G) \longrightarrow H^q(G;\mathbb{Z})$ fails to be surjective.*

In fact, for each minimal cohomological degree where τ fails to be surjective, we provide concrete non-torsion classes in $H^k(G;\mathbb{Z})$ which are not in the image of τ. The methods to prove Theorem 1.1 are described in sections 2.1 and 2.2, and the study of the individual types of Lie groups occupies section 3. We note that generalised cohomology groups for some types of compact Lie groups are well-known, for example for complex K-theory from [**10**], for exceptional Lie groups and Morava K-theory from [**13, 20**], and in Brown–Peterson cohomology from [**28, 29, 30**] (see for example also [**14, 15**]). Some of our computations could have been deduced from these results. However, in order to give a unified and self-contained picture we provide direct proofs for all groups we consider. In [**16**] we study the case of classifying spaces for exceptional Lie groups and of certain gauge groups.

Remark 1.2. In section 2.1 we recall why τ is surjective whenever $H^*(G;\mathbb{Z})$ is torsion-free. However, we point out that this argument is not sufficient to explain the cases in Table 1 where τ is surjective. The pattern we observe in Table 1 indicates that Lie groups of type $\mathfrak{a}_\mathfrak{n}$ and $\mathfrak{c}_\mathfrak{n}$ tend to have a surjective Thom morphism, while groups of type $\mathfrak{b}_\mathfrak{n}$ and $\mathfrak{d}_\mathfrak{n}$ do not have a surjective Thom morphism in sufficiently high dimensions. The exceptional Lie groups on the other hand show a clear pattern. We note, however, that the behaviors of E_7 and E_8 are slightly different from the one of the other groups (see section 3.4). We do not know of a general geometric explanation for why τ is surjective or not surjective for a given Lie group. In section 4, however, we use the geometry and cell structure of special orthogonal groups to construct concrete geometric elements in $MU^*(SO(n))$.

Remark 1.3. We note that in the cases where τ fails to be surjective in cohomological degree 3, the generator $e_3 \in H^3(G;\mathbb{Z})$ which is not hit by τ is not in the image of the homomorphism

$$ku^3(G) \to H^3(G;\mathbb{Z})$$

from connective complex K-theory ku either. This is due to the fact that the Milnor operation Q_1 and the Steenrod operation Sq^3 provide obstructions which are

Table 1: Summary of the results of Theorem 1.1

Lie Algebra	Lie Group	Surjective	Min. degree where surjectivity fails
$\mathfrak{a}_n$	$SU(n)$	yes	–
	$SU(n)/\Gamma_l$	not for $4 \mid n$ and $l \equiv 2 \pmod 4$, yes otherwise	$2^r - 1$ where $r \in \mathbb{Z}$ is max. st. $2^r \mid n$
$\mathfrak{c}_n$	$Sp(n)$	yes	–
	$PSp(n)$	not for n even, yes for n odd	$2^{r+1} - 1$ where $r \in \mathbb{Z}$ is max. st. $2^r \mid n$
$\mathfrak{b}_n$, $\mathfrak{d}_n$	$Spin(n)$	not for $n \geqslant 7$	3
	$SO(n)$	not for $n \geqslant 5$	3
	$Ss(n)$	not for $n \geqslant 4$	3 if $8 \mid n$; 7 else
	$PSO(n)$	not for $n \geqslant 8$	3 if $8 \mid n$; 7 else
$\mathfrak{g}_2$	G_2	no	3
$\mathfrak{f}_4$	F_4	no	3
$\mathfrak{e}_6$	E_6, simply-connected	no	3
	E_6/Γ_3, centerless	no	3
$\mathfrak{e}_7$	E_7, simply-connected	no	3
	E_7/Γ_2, centerless	no	3
$\mathfrak{e}_8$	E_8	no	3

differentials in the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(G; ku^q) \implies ku^{p+q}(G).$$

This applies to several of the groups of type $\mathfrak{b}_n$ and $\mathfrak{d}_n$ and to all exceptional Lie groups (see Table 1 for the specific groups). In the other cases, however, surjectivity may not fail for ku but only on a higher stage in the tower of cohomology theories $MU \to \cdots \to MU\langle 2\rangle \to MU\langle 1\rangle = ku \to MU\langle 0\rangle = H\mathbb{Z}$.

A concrete motivation for our study of the Thom morphism arises from the theory of generalised differential cohomology theories for smooth manifolds developed by Hopkins and Singer in [**12**]. For a rationally even spectrum E and a smooth manifold X, the differential E-cohomology groups are denoted by $\check{E}(q)^n(X)$. The most interesting choice of degrees is $n = q$. The group $\check{E}(q)^q(X)$ then sits in several short exact sequences as described in [**12**, diagram (4.57)]. In particular, the natural homomorphism $\check{E}(q)^q(X) \to E^q(X)$ is surjective. Hence the Thom morphism $\tau\colon MU \to H\mathbb{Z}$

induces a commutative diagram

$$\begin{array}{ccc} \check{MU}(q)^q(X) & \longrightarrow & MU^q(X) \\ \downarrow{\scriptstyle \check{\tau}} & & \downarrow{\scriptstyle \tau} \\ \check{H}(q)^q(X) & \longrightarrow & H^q(X;\mathbb{Z}) \end{array}$$

in which the horizontal maps are surjective. Thus, if τ is not surjective, then $\check{\tau}$ fails to be surjective as well. We note that Grady and Sati study in [**7**] the surjectivity of the differential analog of the map from complex K-theory to cohomology using a differential version of the Atiyah–Hirzebruch spectral sequence.

However, the failure of the surjectivity of τ also allows us to find non-trivial elements in the kernel of $\check{\tau}$. For every rationally even spectrum E, $\check{E}(q)^q(X)$ sits in a short exact sequence of the form

$$0 \to E^{q-1}(X) \otimes \mathbb{R}/\mathbb{Z} \to \check{E}(q)^q(X) \to A^q_E(X) \to 0$$

where the group $A^q_E(X)$ is defined by the following pullback square in which the group $\Omega^*(X;\pi_*E\otimes\mathbb{R})^q_{\mathrm{cl}}$ denotes closed forms on X of total degree q:

$$\begin{array}{ccc} A^q_E(X) & \longrightarrow & \Omega^*(X;\pi_*E\otimes\mathbb{R})^q_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ E^q(X) & \longrightarrow & H^q(X;\pi_*E\otimes\mathbb{R}). \end{array}$$

The Thom morphism $\tau\colon MU \to H\mathbb{Z}$ induces a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & MU^{q-1}(X)\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z} & \longrightarrow & \check{MU}(q)^q(X) & \longrightarrow & A^q_{MU}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_{\mathbb{R}/\mathbb{Z}}} & & \downarrow{\scriptstyle \check{\tau}} & & \downarrow{\scriptstyle \tau_A} & & \\ 0 & \longrightarrow & H^{q-1}(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z} & \longrightarrow & \check{H}(q)^q(X) & \longrightarrow & A^q_H(X) & \longrightarrow & 0. \end{array}$$

Recall that the kernel of the Thom morphism always contains the ideal $MU^{*<0}\cdot MU^*(X)$ of $MU^*(X)$, since τ is a natural transformation of oriented cohomology theories. We therefore use the following terminology:

Definition 1.4. We say that an element in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ or $\check{\tau}$ is *non-trivial* if it is not contained in the respective ideal generated by $MU^{*<0}$.

We will explain in section 2.3 how the failure of τ to be surjective enables us to find non-trivial elements in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$. This leads to the following result, for which we emphasise that the assumption applies to a large class of compact Lie groups by Theorem 1.1:

Theorem 1.5. *Let G be a compact Lie group G and q an integer such that the Thom morphism $\tau\colon MU^{q-1}(G)\longrightarrow H^{q-1}(G;\mathbb{Z})$ fails to be surjective on a non-torsion class. Then the kernel of the differential Thom morphism*

$$\check{\tau}\colon \check{MU}(q)^q(G) \to \check{H}(q)^q(G)$$

is non-trivial in the sense of Definition 1.4.

The significance of Theorem 1.5 is that, together with Theorem 1.1, it provides examples of classes on smooth manifolds which can be studied using differential cobordism but not using differential cohomology. We thus demonstrate by concrete examples that the generalized differential invariants of [**12**] are stronger than invariants that can be obtained by just using differential cohomology. In section 2.4 we explain how we can use the Atiyah–Hirzebruch spectral sequence to find non-trivial elements in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ and $\check{\tau}$ whenever τ is not surjective.

In section 4 we switch perspectives and give a concrete and geometric construction of a non-trivial element in the kernel of $\check{\tau}$ for special orthogonal groups. From Proposition 3.1 we know that the generator $e_3 \in H^3(SO(5);\mathbb{Z})$ is not hit by τ. In section 4.1 we show that the class $2e_3$, however, is in the image of τ by constructing a proper complex-oriented smooth map

$$g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \longrightarrow SO(5)$$

such that $\tau([g]) = 2e_3$ where $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ denotes the Grassmannian of *oriented* 2-planes in $\mathbb{R}^5$. In section 4.1 we prove the following result which we generalise in section 4.2 to higher dimensional $SO(n)$:

Theorem 1.6. *The class $\frac{1}{2}[g]$ is a non-trivial element in the kernel of*

$$\check{\tau}\colon \check{MU}(4)^4(SO(5)) \longrightarrow \check{H}(4)^4(SO(5)).$$

As noted in Remark 1.3, we could have formulated Theorem 1.5 for ku instead of MU as well, and the corresponding assumption would apply to the groups where surjectivity fails for ku already. The geometric construction of Theorem 1.6, however, and its generalisation to higher $SO(n)$ are particular to MU. Moreover, since the Thom morphism allows for a unified picture, we formulate our findings for τ.

Finally, we note that the phenomenon the example of Theorem 1.6 detects bears a certain similarity with the example used in [**12**, §2.7] to explain the behavior of a certain partition function in mathematical physics. We refer for example to [**7**, Example 48] for other interesting phenomena in mathematical physics related to the study of the morphisms between generalised differential cohomology theories. We do not know of a potential similar application of Theorem 1.6 yet. We also hope that the techniques to prove Theorem 1.6 will be useful to shed new light on the Abel–Jacobi invariant for complex cobordism of [**9**] and [**11**].

Acknowledgements: The first-named author would like to thank Tobias Barthel for helpful discussions and the Max Planck Institute for Mathematics in Bonn for its hospitality where parts of the work on this paper have been carried out. Both authors thank the anonymous referee for helpful comments and Knut Bjarte Haus for helpful discussions on the ideas used in section 4.

2. Obstructions and detecting elements in the kernel

In this section we explain the techniques that we use in section 3 to study the cokernel of τ and the kernel of $\check{\tau}$. We assume that X is a finite CW-complex for simplicity. We are going to say that a homomorphism between abelian groups is *torsion* if its image is contained in the subgroup of torsion elements.

2.1. The Thom morphism is an edge map

A key tool in our study of the Thom homomorphism is the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(X; MU^q) \implies MU^{p+q}(X).$$

Since $MU^* \cong \mathbb{Z}[x_{-2}, x_{-4}, \ldots]$, this spectral sequence is concentrated in the fourth quadrant. Since the top row of the E_2-page is the integral cohomology of X, there is a well-defined edge map such that the composition

$$MU^p(X) \longrightarrow E_\infty^{p,0} \longrightarrow E_2^{p,0} \cong H^p(X; \mathbb{Z})$$

can be identified with the Thom morphism. It then follows from the general theory of spectral sequences that the Thom morphism is surjective if and only if all the differentials starting in the top row of the spectral sequence are trivial. Since all the differentials are torsion, the Thom morphism is surjective whenever $H^*(X; \mathbb{Z})$ has no torsion.

If $H^*(X; \mathbb{Z})$ has torsion, the Thom morphism may still be surjective. Since the construction of the spectral sequence is functorial, the first non-trivial differentials starting in the top row of the E_2-page are cohomology operations given by maps of the form $d\colon H^*(X; \mathbb{Z}) \to H^*(X; A)$ where A is a finitely generated free abelian group. If a differential d is p-torsion, then so is the composition $\rho \circ d$, where ρ is the map induced by the reduction modulo p homomorphism of A. Thus we can describe all first non-trivial differentials using cohomology operations of type $(\mathbb{Z}, m; \mathbb{Z}/p, n)$. These operations correspond to the elements in the cohomology group $H^n(K(\mathbb{Z}, m); \mathbb{Z}/p)$.

For $p = 2$, the cohomology ring $H^*(K(\mathbb{Z}, m); \mathbb{Z}/2)$ is a polynomial ring over generators of the form $\mathrm{Sq}^I(\iota_m)$, where I is an admissible sequence where the last term is different from 1, and ι_m is the fundamental class of $K(\mathbb{Z}, m)$ as explained in [**19**, Chapter 9, Theorem 3]. Thus, in order to prove that there are no non-trivial differentials that are 2-torsion, it suffices to check that all Steenrod operations of odd degree are trivial (except Sq^1, since a non-trivial differential increases the cohomological degree by at least 3). For odd primes p, the cohomology operations we have to study can all be described using reduced power operations P^k combined with Bocksteins β (see [**5**] for a complete description). In order to prove surjectivity it therefore suffices to show that all sequences of reduced power operations and Bocksteins that increase the cohomological degree by an odd number greater than 1 must be trivial. The fact that this also works in cases where there is torsion of the form $\mathbb{Z}/p^k$ with $k > 1$ can be deduced by considering short exact sequences of the form

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1} \to 0.$$

2.2. Obstructions and Bockstein cohomology

Now we explain how we can find cohomology classes which are not in the image of the Thom homomorphism. From the description of τ as an edge map we know that an element $x \in H^n(X; \mathbb{Z})$ is not in the image of τ if there is at least one differential d on the E_2-page of the Atiyah–Hirzebruch spectral sequence with $(\rho \circ d)(x) \neq 0$ where

$$\rho\colon H^*(X; \mathbb{Z}) \longrightarrow H^*(X; \mathbb{Z}/p)$$

is the homomorphism induced by reduction mod p. Suppose now we know how the Steenrod algebra acts on $H^*(X; \mathbb{Z}/p)$. In fact, all Steenrod operations of odd degree vanish on the image of $MU^*(X)$ in $H^*(X;\mathbb{Z}/p)$ for all prime numbers p (see for example [**26**, page 468], [**4**, Proposition 3.6], [**6**]). Then it remains to understand how ρ acts. The tool we use to find the concrete element in $H^*(X;\mathbb{Z}/p)$ a given $x \in H^*(X;\mathbb{Z})$ maps to is *Bockstein cohomology*, the definition of which we now recall from [**8**, Chapter 3E]: The Bockstein homomorphism $\beta\colon H^n(X;\mathbb{Z}/p) \longrightarrow H^{n+1}(X;\mathbb{Z}/p)$ is the connecting homomorphism in the long exact sequence induced in cohomology by the short exact sequence $0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$. It satisfies $\beta^2 = 0$, and thus defines a cochain complex

$$\cdots \xrightarrow{\beta_{n-1}} H^n(X;\mathbb{Z}/p) \xrightarrow{\beta_n} H^{n+1}(X;\mathbb{Z}/p) \xrightarrow{\beta_{n+1}} \cdots .$$

The *nth Bockstein cohomology* of X denoted by $BH^n(X;\mathbb{Z}/p)$ is defined as the nth cohomology of this complex. We compute the groups $BH^n(X;\mathbb{Z}/p)$ by providing concrete descriptions of the Bockstein complex. Since X is assumed to be a finite CW-complex, all cohomology groups of X are finitely generated. By [**8**, Proposition 3E.3] the relationship between $H^*(X;\mathbb{Z})$ and $BH^*(X;\mathbb{Z}/p)$ is then given as follows: Each $\mathbb{Z}$-summand of $H^n(X;\mathbb{Z})$ contributes one $\mathbb{Z}/p$-summand to $BH^n(X;\mathbb{Z}/p)$, and each $\mathbb{Z}/p^k$-summand (with $k \geqslant 2$) of $H^n(X;\mathbb{Z})$ contributes one $\mathbb{Z}/p$-summand to $BH^{n-1}(X;\mathbb{Z}/p)$ and one $\mathbb{Z}/p$-summand to $BH^n(X;\mathbb{Z}/p)$. The $\mathbb{Z}/p$-summands of $H^n(X;\mathbb{Z})$, however, do not contribute to $BH^n(X;\mathbb{Z}/p)$. Finally, for an odd prime p, we will also use the following obstruction (see also [**15**] and [**22**]):

Lemma 2.1. *Let $Q_1\colon H^*(X;\mathbb{Z}/p) \to H^{*+2p-1}(X;\mathbb{Z}/p)$ be the first Milnor operation and let $x \in H^i(X;\mathbb{Z})$ be a non-torsion class. If $Q_1(\rho(x)) \neq 0$, then x is not in the image of $ku^i(X) \to H^i(X;\mathbb{Z})$ and hence not in the image of the Thom morphism.*

Proof. By [**27**, Proposition 1.7] (see also [**24**, Proposition 4-4]), there is a commutative diagram

$$\begin{array}{ccccc} ku^i_{(p)}(X) & \xrightarrow{\tau_{ku_{(p)}}} & H^i(X;\mathbb{Z}_{(p)}) & \xrightarrow{\cdot v_1} & ku^{i+2p-1}_{(p)}(X) \\ & & \downarrow & & \downarrow \\ & & H^i(X;\mathbb{Z}/p) & \xrightarrow[\pm Q_1]{} & H^{i+2p-1}(X;\mathbb{Z}/p) \end{array}$$

in which the top row is exact, where $ku^i_{(p)}(X)$ denotes p-local connective complex K-theory and the map $\tau_{ku_{(p)}}$ is the map which factors the canonical morphism $\tau_{BP}\colon BP \to H\mathbb{Z}_{(p)}$ for Brown–Peterson theory. Thus, if $Q_1(\rho(x)) \neq 0$, then the image of x in $H^i(X;\mathbb{Z}_{(p)})$ cannot be lifted to $ku^i_{(p)}(X)$. This implies that x cannot be lifted to $ku^i(X)$ either, and hence the assertion. ∎

2.3. The kernel of the differential Thom morphism

We will now explain how the failure of τ to be surjective enables us to find nontrivial elements in the kernel of $\check{\tau}$. We write $MU^{*<0} \cdot MU^k(X)$ for the subgroup of $MU^k(X)$ consisting of elements of the form $\gamma \cdot \mu$ where $\gamma \in MU^{k-s}$ with $s > k$ and $\mu \in MU^s(X)$. The sum over all k defines an ideal in $MU^*(X)$ which we denote by $MU^{*<0} \cdot MU^*(X)$. Since $\tau(MU^{*<0}) = 0$, we get that τ induces a well-defined homomorphism

$$\tau\colon MU^*(X)/(MU^{*<0}\cdot MU^*(X)) \to H^*(X;\mathbb{Z}),$$

which we also denote by τ. Consider the homomorphism $MU^* \to \mathbb{Z}$ which sends $n\cdot 1 \in MU^0$ to $n\in\mathbb{Z}$ and $\gamma \in MU^{*<0}$ to 0. Then there is an isomorphism of rings

$$MU^*(X)/(MU^{*<0}\cdot MU^*(X)) \cong MU^*(X)\otimes_{MU^*}\mathbb{Z}.$$

By slight abuse of notation, we then also write $MU^k(X)\otimes_{MU^*}\mathbb{Z}$ for the group $MU^k(X)/(\oplus_s MU^s\cdot MU^{k-s}(X))$. We let the graded ring MU^* act on $\mathbb{R}/\mathbb{Z}$ by the map $MU^*\otimes\mathbb{R}/\mathbb{Z}\longrightarrow\mathbb{R}/\mathbb{Z}$ defined by

$$\begin{aligned} n\otimes a &\longmapsto na, \text{ for } n\in MU^0\cong\mathbb{Z}\\ \gamma\otimes a &\longmapsto 0, \text{ for } \gamma\in MU^{*<0}. \end{aligned}$$

Then we get a canonical isomorphism

$$(MU^*(X)\otimes_{MU^*}\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z} \xrightarrow{\cong} MU^*(X)\otimes_{MU^*}\mathbb{R}/\mathbb{Z}.$$

We will now explain how the information on the cokernel of τ helps to understand the kernel of the induced Thom homomorphism

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}}\colon MU^*(X)\otimes_{MU^*}\mathbb{R}/\mathbb{Z}\longrightarrow H^*(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}$$

in differential cobordism.

Lemma 2.2. *Let $\alpha\in H^k(X;\mathbb{Z})$ be a non-torsion class. Assume that the image of the Thom morphism*

$$\tau\colon MU^k(X)\longrightarrow H^k(X;\mathbb{Z})$$

contains $n\alpha$ for some integer $n>1$, but not α itself or an element of the form $\alpha+y$, where $n\cdot y=0$. Let $\mu\in MU^k(X)$ be an element such that $\tau(\mu)=n\alpha$. Then

$$\mu\otimes\frac{1}{n}\in MU^k(X)\otimes_{MU^*}\mathbb{R}/\mathbb{Z}$$

is a non-trivial element in the kernel of the induced map

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}}\colon MU^k(X)\otimes_{MU^*}\mathbb{R}/\mathbb{Z}\longrightarrow H^*(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}.$$

Proof. The element $\mu\otimes\frac{1}{n}$ maps to 0 under $\bar{\tau}_{\mathbb{R}/\mathbb{Z}}$ since $n\alpha\otimes\frac{1}{n}=\alpha\otimes 1=0$. However, $\mu\otimes\frac{1}{n}$ cannot be 0 in $MU^k(X)\otimes_{MU^*}\mathbb{R}/\mathbb{Z}$, since if μ had been of the form $n\gamma$, then γ would map to α or $\alpha+y$. □

2.4. Detecting elements in the kernel of the differential Thom morphism

We now describe a procedure to find an element μ as in Lemma 2.2 using the Atiyah–Hirzebruch spectral sequence. In section 4 we will give a geometric construction for special orthogonal groups. For the other cases, we can proceed as follows. Assume that we have a non-torsion cohomology class $\alpha\in H^k(X;\mathbb{Z})$ which is not in the image of the Thom morphism, while an integer multiple $n\alpha$ is in the image. We will now explain how we can then find a cobordism class which maps to $n\alpha$. Since α is not in the image of the Thom morphism, there must be at least one non-trivial differential starting at $H^k(X;\mathbb{Z})$. If this differential is, say, m-torsion, then $m\alpha$ is in the kernel of the differential and survives to the next page of the spectral sequence. Since X is assumed to be finite dimensional, the spectral sequence is bounded on the right, and there can only be finitely many non-trivial differentials starting at any one position. By counting how much torsion there is in cohomological degrees greater than k, we can then determine an integer n for which $n\alpha$ must be in the image of the Thom morphism. Once we have reached the E_∞-page of the spectral sequence, the position $(k, 0)$ contains the desired cobordism class.

3. The cokernel for compact Lie groups

The goal of this section is to determine whether or not the Thom morphism is surjective for a given compact, connected, simple Lie group. Such a Lie group has a simple Lie algebra. Given a simple Lie algebra $\mathfrak{g}$, we find the associated Lie groups using the following method based on [**21**, 10.7.2, Theorem 4]. We first determine the unique (up to isomorphism) compact, simply-connected Lie group G with Lie algebra $\mathfrak{g}$. The center $Z(G)$ is always finite. The other compact, connected, simple Lie groups with the same Lie algebra are of the form G/K, where K is a subgroup of $Z(G)$. Organising our analysis by the associated Lie algebra is justified by the following observation. Given a Lie group G, we denote by $H^*_{\text{free}}(G;\mathbb{Z})$ the non-torsion part of the cohomology $H^*(G;\mathbb{Z})$. Then there is an isomorphism $H^*_{\text{free}}(G;\mathbb{Z}) \cong H^*_{\text{free}}(H;\mathbb{Z})$ if G and H are Lie groups with the same Lie algebra. We will therefore recall the non-torsion cohomology part only once in the section for a given Lie algebra. Unless otherwise stated, the computation of the cohomology rings can be found in one of the following two sources: The cohomology of the groups $SU(n)$, $Sp(n)$, $\text{Spin}(n)$, $SO(n)$ as well as all the exceptional Lie groups and classifying spaces can be found in [**18**], while the cohomology of $Ss(n)$, $PSO(n)$, $PSp(n)$ and the quotients of $SU(n)$ can be found in [**3**]. Finally, given a ring R we write $\Lambda_R(x_{i_1}, \dots, x_{i_n}) := R[x_{i_1}, \dots, x_{i_n}]/(x_{i_1}^2, \dots, x_{i_n}^2)$, and unless otherwise stated x_{i_j} is an element of degree i_j. When the choice of ring is clear from the context, we omit R from the notation.

3.1. Groups with Lie algebra $\mathfrak{b}_\mathfrak{n}$ and $\mathfrak{d}_\mathfrak{n}$

The simply-connected Lie groups that correspond to the Lie algebras of type $\mathfrak{b}_\mathfrak{n}$ and $\mathfrak{d}_\mathfrak{n}$ are the *spin groups* $\text{Spin}(2n+1)$ and $\text{Spin}(2n)$, respectively. We will consider both types of spin groups together, since their cohomology rings are similar. However, the possible quotients are different in the odd and even cases. The center of the group $\text{Spin}(2n+1)$ is isomorphic to $\mathbb{Z}/2$, which gives us only one possible quotient, the *odd special orthogonal group*, denoted by $SO(2n+1)$.

For the even case, we know by [**18**, Chapter II, Theorem 4.14] that the centers are given by $Z(\text{Spin}(4n+2)) \cong \mathbb{Z}/4$ and $Z(\text{Spin}(4n)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. For $\text{Spin}(4n+2)$, taking the quotient by the subgroup of order 2 yields the *even special orthogonal group* $SO(4n+2)$, while taking the quotient by the whole center gives the *projective special orthogonal group* $PSO(4n+2)$. For $\text{Spin}(4n)$, the center $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ has three subgroups of order 2. Taking the quotient by the whole center once again produces the projective special orthogonal group $PSO(4n)$. One of the subgroups of order 2 will again give us a special orthogonal group $SO(4n)$. The remaining two subgroups produce isomorphic quotient groups, known as the *semi-spin group* $Ss(4n)$ (see [**18**, Chapter II, Theorem 4.15]). In total we have to consider four different types of groups.

3.1.1. Special orthogonal groups

The non-torsion cohomology of the special orthogonal groups is given by

$$H^*_{\text{free}}(SO(n);\mathbb{Z}) \cong \begin{cases} \Lambda(e_3, e_7, \ldots, e_{2n-3}), & n \text{ odd} \\ \Lambda(e_3, e_7, \ldots, e_{2n-5}, y_{n-1}), & n \text{ even.} \end{cases}$$

Proposition 3.1. *For $n \geqslant 5$, the generator $e_3 \in H^3(SO(n);\mathbb{Z})$ is not in the image of the Thom homomorphism.*

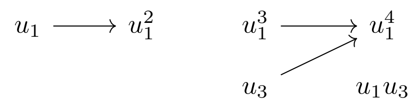
Proof. The $\mathbb{Z}/2$-cohomology of $SO(n)$ is given by

$$H^*(SO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[u_1, u_3, \ldots, u_{2m-1}]/(u_1^{k_1}, u_3^{k_3}, \ldots, u_{2m-1}^{k_{2m-1}}), \tag{1}$$

where $m = \lfloor \frac{n}{2} \rfloor$ and k_i is the least power of 2 such that $|u_i^{k_i}| \geqslant n$. In order to find the image of e_3 in $H^3(SO(n);\mathbb{Z}/2)$ under ρ, we need to analyse the Bockstein homomorphism β. From [**8**], we have

$$\beta(u_{2i-1}) = u_{2i} \text{ and } \beta(u_{2i}) = 0,$$

where we interpret u_{2i} as u_i^2 and iterate if necessary. Assuming $n \geqslant 5$, we have the following table for the generators for $H^*(SO(n);\mathbb{Z}/2)$ in low degrees, where the arrows denote the non-trivial Bockstein homomorphisms.



We see that $BH^3(SO(n);\mathbb{Z}/2)$ is generated by $u_1^3 + u_3$. It follows that the reduction homomorphism to $\mathbb{Z}/2$-cohomology maps e_3 to $u_1^3 + u_3$. We can then deduce that e_3 is not in the image of the Thom homomorphism, since

$$\mathrm{Sq}^3(u_1^3 + u_3) = u_1^6 + u_3^2 \neq 0.$$

Note that $u_3^2 = 0$ if $n = 5$, but u_1^6 is nonzero. □

Remark 3.2. Using the same methods, we can show that any given generator $e_{4k+3} \in H^{4k+3}(SO(n);\mathbb{Z})$ is not in the image of the Thom morphism for sufficiently large n. However, we do not know of an efficient way to determine a minimal n for each generator e_{4k+3}, apart from analysing the Bockstein diagrams on a case by case basis. We return to this question in section 4.2.

Proposition 3.3. *For $SO(n)$ with $n \leqslant 4$, the Thom morphism is surjective in all degrees.*

Proof. We have the homeomorphisms

$$SO(1) \cong \text{pt},\ SO(2) \cong S^1,\ SO(3) \cong \mathbb{RP}^3,\ \text{and } SO(4) \cong \mathbb{RP}^3 \times S^3.$$

The Thom morphism is always surjective in degrees $\leqslant 2$ (see for example [**26**, Theorem 2.2]). This proves the assertion for $SO(1)$ and $SO(2)$. For $SO(3)$, there is no nontrivial differential in the Atiyah–Hirzebruch spectral sequence starting in cohomological degree 3, since $\mathbb{RP}^3$ is 3-dimensional. This shows that the Thom morphism is surjective in all degrees for $SO(3)$. Finally, $H^k(SO(4);\mathbb{Z})$ has torsion only if $k = 2$

or $k = 5$. Thus, there are no differentials which start in degree $\geqslant 3$, increase the cohomological degree by at least 3, and which end in torsion. It follows that the Thom morphism is surjective for $SO(4)$. □

3.1.2. Spin groups

For the rest of this section, we use the following notation. Given $n \in \mathbb{N}$, we let q be the greatest power of 2 such that $q|n$, and let t be the least power of 2 such that $n \leqslant t$. The cohomology of the spin groups is given by

$$H^*(\mathrm{Spin}(n); \mathbb{Z}/2) \cong \Lambda(z) \otimes \mathbb{Z}/2[u_3, u_5, \ldots, u_{2m-1}]/(u_3^{k_3}, \ldots, u_{2m-1}^{k_{2m-1}}),$$

where $|z| = t - 1$ and where m and the k_i's are as in (1).

Proposition 3.4. *For $n \geqslant 7$, the generator $e_3 \in H^3(\mathrm{Spin}(n); \mathbb{Z})$ is not in the image of the Thom morphism. For $n \leqslant 6$, the Thom morphism is surjective in all degrees.*

Proof. For $n \leqslant 7$, there is no torsion in the integral cohomology of $\mathrm{Spin}(n)$, and it follows that the Thom morphism is surjective. However, for $n \geqslant 7$, the generator $e_3 \in H^3(\mathrm{Spin}(n); \mathbb{Z})$ maps to $u_3 \in H^3(\mathrm{Spin}(n); \mathbb{Z}/2)$, for which

$$\mathrm{Sq}^3 u_3 = u_3^2 \neq 0.$$

Thus, e_3 is not in the image of the Thom morphism for $n \geqslant 7$. □

3.1.3. Semi-spin groups

The cohomology ring $H^*(Ss(n); \mathbb{Z}/2)$ of the semi-spin groups with coefficients in $\mathbb{Z}/2$ is isomorphic to

$$\mathbb{Z}/2[v]/(v^q) \otimes \Lambda(z) \otimes \mathbb{Z}/2[u_3, u_5, \ldots, \widehat{u}_{q-1}, \ldots, u_{n-1}, u_{2q-2}]/(u_3^{k_3}, \ldots, u_{n-1}^{k_{n-1}}, u_{2q-2}^{k_{2q-2}}),$$

where $|v| = 1$ and $|z| = t - 1$. The Steenrod operations are given by

$$\mathrm{Sq}^j(u_k) = \binom{k}{j} u_{k+j}$$

wherever it makes sense, with the exception that $\mathrm{Sq}^1(u_k) = v^{k+1}$, if $q \geqslant 8$ and $k = \frac{q}{2} - 1$. Recall that, for semi-spin groups, n must be a multiple of 4. Hence we do not need to separate the cases for even and odd n. Note also that the class u_{2q-2} will only be included if $2q - 2 < n$.

Proposition 3.5. *For $Ss(4)$, the Thom morphism is surjective in all degrees. For $k \geqslant 2$, we have: If $8 \mid n$, then the generator $e_3 \in H^3(Ss(4k); \mathbb{Z})$ is not in the image of τ. If $8 \nmid n$, then the generator $e_7 \in H^7(Ss(4k); \mathbb{Z})$ is not in the image of τ.*

Proof. For $n = 4$, the cohomology ring together with its Steenrod operations of $Ss(4)$ is isomorphic to the cohomology ring with Steenrod operations of $SO(4)$. It then follows from Proposition 3.3 that the Thom morphism is surjective for $Ss(4)$.

Now we assume $n = 4k$ and $k \geqslant 2$. There are three cases to consider:

Case 1: $\mathbf{n \equiv 8 \pmod{16}}$ In this case we have $q = 8$, and the $\mathbb{Z}/2$-cohomology ring is given by

$$H^*(Ss(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^8) \otimes \Lambda(z) \otimes \mathbb{Z}/2[u_3, \ldots, \widehat{u}_7, \ldots, u_{n-1}, u_{14}]/(u_3^{k_3}, \ldots),$$

where $|z| \geqslant 7$ since t is at least 8. Since $q = 8$ and $\mathrm{Sq}^1(u_k) = v^{k+1}$, we conclude that $\mathrm{Sq}^1(u_3) = v^4$. The other Sq^1s are easy to work out. The non-torsion generator

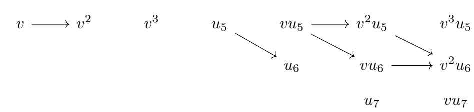
$e_3 \in H^3(Ss(n);\mathbb{Z})$ maps to $v^3 + u_3 \in H^3(Ss(n);\mathbb{Z}/2)$, and we can check that Sq^3 does not act trivially on this class as $\mathrm{Sq}^3(v^3 + u_3) = v^6 + u_3^2 \neq 0$.

Case 2: $\mathbf{n \equiv 0}$ (mod 16) Since q is greater than 8, the Bockstein homomorphism acts trivially on u_3. Moreover, the Bockstein cohomology in degree 3 is generated by u_3 alone. This implies that e_3 is sent to u_3, and we see that u_3 is not in the image of the Thom morphism.

Case 3: $\mathbf{n \equiv 4}$ (mod 8) We have $q = 4$ and $n \geqslant 12$, which means that the $\mathbb{Z}/2$-cohomology is given by

$$H^*(Ss(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^4) \otimes \Lambda(z) \otimes \mathbb{Z}/2[u_5, u_6, u_7, u_9, \ldots, u_{n-1}]/(u_5^{k_5}, \ldots),$$

where $|z| \geqslant 15$. We get the Bockstein diagram



where the two arrows starting in vu_5 indicate that the Bockstein is given by a sum, i.e., $\beta(vu_5) = v^2u_5 + vu_6$. Hence, in this case, e_3 is in the image of the Thom morphism since its reduction v^3 does not survive any Steenrod operation. However, we can find a suitable class in degree 7. From the Bockstein diagram we see that e_7 is sent to either u_7 or $u_7 + v^2u_5 + vu_6$, and we can check that both classes survive Sq^3:

$$\mathrm{Sq}^3(u_7) = \binom{7}{3} u_{10} = u_5^2 \neq 0, \ \mathrm{Sq}^3(u_7 + v^2u_5 + vu_6) = u_5^2 \neq 0. \qquad \square$$

3.1.4. Projective special orthogonal groups

The $\mathbb{Z}/2$-cohomology ring $H^*(PSO(n);\mathbb{Z}/2)$ of the projective special orthogonal groups is given by

$$\mathbb{Z}/2[v]/(v^q) \otimes \mathbb{Z}/2[u_1, u_3, \ldots, \widehat{u}_{q-1}, \ldots, u_{n-1}, u_{2q-2}]/(u_1^{k_1}, \ldots, u_{n-1}^{k_{n-1}}, u_{2q-2}^{k_{2q-2}}),$$

where $|v| = 1$, with the Steenrod operations acting by

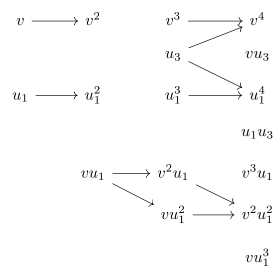
$$\mathrm{Sq}^j u_k = \binom{k}{j} u_{k+j},$$

whenever it makes sense, except when $j = 1,\ k = \frac{q}{2} - 1$ and $q \geqslant 8$. In the latter case we have $\mathrm{Sq}^1(u_k) = u_{k+1} + v^{k+1}$. Note that u_{2q-2} is only be included if $2q - 2 < n$, as for the semi-spin groups. If n is odd, then $PSO(n) = SO(n)$. Hence we will focus on the case that n is even.

Proposition 3.6. *Let $n \geqslant 8$ be even. If $8 \mid n$, then the generator $e_3 \in H^3(PSO(n);\mathbb{Z})$ is not in the image of τ. If $8 \nmid n$, then the generator $e_7 \in H^7(PSO(n);\mathbb{Z})$ is not in the image of τ.*

Proof. We have to consider the following cases:

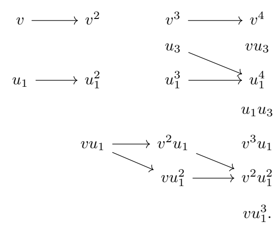
Case 1: $\mathbf{n \equiv 8}$ (mod 16) We have $q = 8$ and note that $\mathrm{Sq}^1(u_3) = u_1^4 + v^4$. We get the following diagram of Bockstein homomorphisms:



The non-torsion class $e_3 \in H^3(PSO(n);\mathbb{Z})$ maps to either the element $v^3+u_1^3+u_3$ or $v^3+u_1^3+u_3+v^2u_1+vu_1^2$ in $H^3(PSO(n);\mathbb{Z}/2)$, and we have

$$\mathrm{Sq}^3(v^3+u_1^3+u_3) = v^6+u_1^6+u_3^2 \neq 0$$
$$\mathrm{Sq}^3(v^3+u_1^3+u_3v^2u_1+vu_1^2) = v^6+u_1^6+u_3^2+v^4u_1^2+v^2u_1^4 \neq 0.$$

Case 2: n $\equiv$ 0 (mod 16) In low degrees, this is almost the same as the previous case, with the exception that $\mathrm{Sq}^1(u_3)=u_1^4$ since $q \geqslant 16$. This gives us the diagram

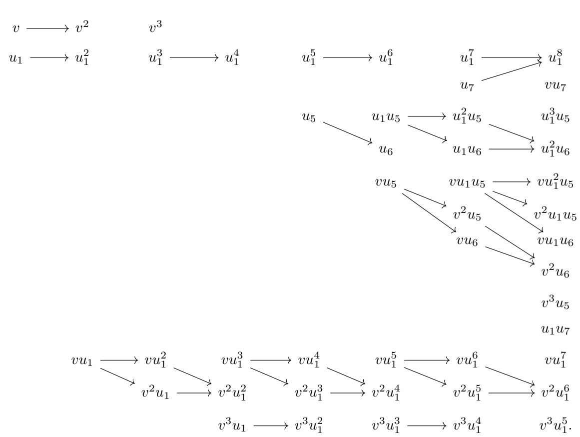


Then $e_3 \in H^3(PSO(n);\mathbb{Z})$ maps to $u_1^3+u_3$ or $u_1^3+u_3+v^2u_1+vu_1^2$, for which

$$\mathrm{Sq}^3(u_1^3+u_3) = u_1^6+u_3^2,\ \mathrm{Sq}^3(u_1^3+u_3+v^2u_1+vu_1^2) = u_1^6+u_3^2+v^4u_1^2+v^2u_1^4$$

which are both nonzero.

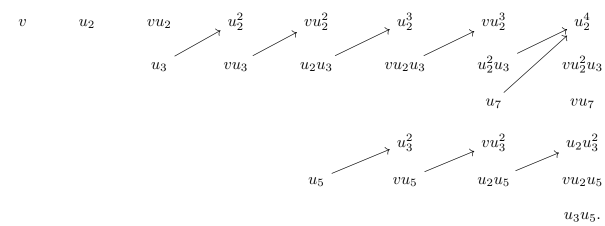
Case 3: n $\equiv$ 4 (mod 8) Assume $n \geqslant 12$. In this case we have $q=4$, and consequently the class u_3 does not exist. The Bockstein diagram has the form



We see that the class e_7 can reduce to several different classes in $H^7(PSO(n);\mathbb{Z}/2)$ depending on our choice of isomorphism $H^7(PSO(n);\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2^4$. We know that e_7 maps to $u_7 + u_1^7$ or $u_7 + u_1^7$ plus any of the classes $u_1^2u_5 + u_1u_6$, $v^2u_5 + vu_6$, $vu_1^6 + v^2u_1^5$, $v^3u_1^5$. We have $\mathrm{Sq}^3(u_7 + u_1^7) = u_{10} + u_1^{10} \neq 0$. We then note that applying Sq^3 to any of the torsion classes $u_1^2u_5 + u_1u_6$, $v^2u_5 + vu_6$, $vu_1^6 + v^2u_1^5$, $v^3u_1^5$ cannot yield u_{10}. Hence they cannot cancel out the nonzero contribution we got from $\mathrm{Sq}^3(u_7 + u_1^7)$. This proves that e_7 and e_7 plus torsion are not in the image of the Thom morphism.

Case 4: n ≡ 2 (mod 4) Assume $n \geqslant 10$. Since $q = 2$, there is no class u_1, but instead a class u_2. The cohomology and Bockstein diagrams are therefore

$$H^*(PSO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^2) \otimes \mathbb{Z}/2[u_2, u_3, u_5, \ldots, u_{n-1}]/(u_2^{k_2}, \ldots)$$



The class e_7 is sent to either $u_7 + u_2^2u_3$, or $u_7 + u_2^2u_3$ plus one or both of the classes vu_2^3, vu_3^2. As above, we may use Sq^3 as an obstruction, but the computation is easier for Sq^7. We get

$$\mathrm{Sq}^7(u_7 + u_2^2u_3) = u_7^2 + u_2^4u_3^2 \neq 0,\ \mathrm{Sq}^7(vu_2^3) = v^2u_2^6 = 0,\ \mathrm{Sq}^7(vu_3^2) = v^2u_3^4 = 0.$$

Hence e_7 is not in the image of the Thom morphism. Note that $u_7^2 = 0$ if $n \leqslant 14$, but $u_2^4u_3^2$ is nonzero. □

Proposition 3.7. *For $PSO(2)$, $PSO(4)$ and $PSO(6)$, the Thom morphism is surjective in all degrees.*

Proof. The isomorphism $PSO(2) \cong SO(2)$ implies that the Thom morphism is surjective for $PSO(2)$, by Proposition 3.3. We recall that the integral cohomology of the group $PSO(4) \cong SO(3) \times SO(3)$ is given by

$$H^k(PSO(4);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 6 \\ \mathbb{Z}/2, & k = 4 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & k = 2, 5 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, & k = 3 \\ 0, & \text{else.} \end{cases}$$

Since the differentials in the Atiyah–Hirzebruch spectral sequence increase the cohomological degree by at least 3, the only differentials which start at a non-trivial cohomology group to a group with torsion are

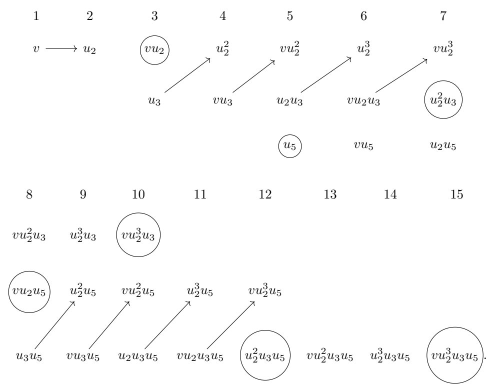
$$d_3\colon H^0(PSO(4);\mathbb{Z}) \longrightarrow H^3(PSO(4);\mathbb{Z})$$
$$d_3\colon H^2(PSO(4);\mathbb{Z}) \longrightarrow H^5(PSO(4);\mathbb{Z}).$$

However, since the Thom morphism is surjective in degrees $\leqslant 2$, these differentials must also be trivial. Thus we can conclude that the Thom morphism is surjective for

$PSO(4)$. For the group $PSO(6) \cong PU(4)$ we have

$$H^*_{\text{free}}(PSO(6);\mathbb{Z}) \cong \Lambda(e_3, e_7, y_5)$$
$$H^*(PSO(6);\mathbb{Z}/2) \cong \mathbb{Z}[v]/(v^2) \otimes \mathbb{Z}/2[u_2, u_3, u_5]/(u_2^4, u_3^2, u_5^2),$$

and the following Bockstein diagram in which the top numbers denote the degree of the respective elements



The elements that correspond to non-torsion classes in $H^*(PSO(6);\mathbb{Z})$ have been circled. This yields the integral cohomology groups

$$H^k(PSO(6);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 3, 8, 15 \\ \mathbb{Z}/4, & k = 2, 14 \\ \mathbb{Z}/2, & k = 4, 6, 11 \\ \mathbb{Z} \oplus \mathbb{Z}/2, & k = 5, 10, 12 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2, & k = 9 \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2, & k = 7 \\ 0, & \text{else.} \end{cases}$$

It is now straight-forward to check that all Steenrod operations of odd degree greater than 1 act trivially on the image of the reduction homomorphism $H^*(PSO(6);\mathbb{Z}) \to H^*(PSO(6);\mathbb{Z}/2)$. This shows that all differentials in the Atiyah–Hirzebruch spectral sequence are trivial. Thus the Thom morphism is surjective. □

3.2. Groups with Lie algebra $\mathfrak{c}_n$

The integral cohomology of the simply-connected Lie group $Sp(n)$ is given by

$$H^*(Sp(n);\mathbb{Z}) \cong \Lambda(e_3, e_7, \ldots, e_{4n-1}).$$

Since the cohomology is torsion-free, the arguments explained in section 2.1 imply:

Proposition 3.8. *The Thom morphism is surjective in all cohomological degrees for $Sp(n)$.* □

The center of $\mathrm{Sp}(n)$ is isomorphic to $\mathbb{Z}/2$, consisting of the positive and negative of the identity matrix. It follows that there is only one other compact Lie group with the same Lie algebra which we obtain by dividing out by $Z(\mathrm{Sp}(n))$ (see [**3**]). This group is known as the *projective symplectic group*, denoted by $PSp(n)$. The $\mathbb{Z}/2$-cohomology of $PSp(n)$ is given by

$$H^*(PSp(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^{4q}) \otimes \Lambda(b_3, b_7, \dots, \widehat{b}_{4q-1}, \dots, b_{4n-1}),$$

where q is the largest power of 2 dividing n. The Steenrod squares are given by

$$\mathrm{Sq}^{4j}(b_{4k+3}) = \binom{k}{j} b_{4k+4j+3}$$

with all other Steenrod squares trivial, except $\mathrm{Sq}^1(b_{2q-1}) = v^{2q}$ when n is even. The latter implies that we have to distinguish between whether n is even or odd. We start with the even case.

Proposition 3.9. *For all even $n \geqslant 2$, the generator $e_{2q-1} \in H^{2q-1}(PSp(n);\mathbb{Z})$ is not in the image of the Thom morphism.*

Proof. It follows from the Bockstein diagram that the generator e_{2q-1} is mapped to $v^{2q-1} + b_{2q-1}$ plus torsion. We deduce that $\mathrm{Sq}^{2q-1}(v^{2q-1} + b_{2q-1}) = v^{4q-2} \neq 0$, and hence the assertion. □

Proposition 3.10. *For all odd n, the Thom morphism is surjective for $PSp(n)$ in all cohomological degrees.*

Proof. Assume that n is odd. We have the cohomology ring

$$H^*(PSp(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^4) \otimes \Lambda(b_7, b_{11}, \dots, b_{4n-1}).$$

Thus, the only non-trivial Bocksteins are the ones that go from a term containing a factor v to a term containing a factor v^2. The non-torsion elements of $H^*_{\mathrm{free}}(PSp(n);\mathbb{Z})$ map to elements of the form $b_{i_1} \cdots b_{i_k}$ or of the form $v^3 b_{i_1} \cdots b_{i_k}$ in $H^*(PSp(n);\mathbb{Z}/2)$, while the torsion elements are sent to elements of the form $v^2 b_{i_1} \cdots b_{i_k}$. We claim that none of these elements can survive an odd-dimensional Steenrod square. Assume that $\alpha \in H^*(PSp(n);\mathbb{Z}/2)$ is such that $\mathrm{Sq}^{2n+1}(\alpha) \neq 0$. Then $\mathrm{Sq}^1\mathrm{Sq}^{2n}\alpha \neq 0$, which implies that $\mathrm{Sq}^{2n}\alpha$ is of the form $v b_{i_1} \cdots b_{i_k}$. Since the number of v's cannot be changed by a Steenrod square of even degree, the class α is a product of b_i's and precisely one v. However, as seen in the Bockstein diagram, such elements do not correspond to elements of the integral cohomology of $PSp(n)$. This implies the assertion. □

3.3. Groups with Lie algebra $\mathfrak{a}_n$

The integral cohomology of the simply-connected Lie group $SU(n)$ is given by

$$H^*(SU(n);\mathbb{Z}) \cong \Lambda(e_3, e_5, \dots, e_{2n-1}),$$

and is torsion-free. Hence we can conclude:

Proposition 3.11. *The Thom morphism is surjective in all cohomological degrees for $SU(n)$.* □

The center of $SU(n)$ is isomorphic to $\mathbb{Z}/n$. Hence, depending on n, we can take several different quotients of this group. The case where we divide out be the entire center is known as the *projective special unitary group* $PSU(n)$. The cohomology groups of the various quotients are as follows. Let l be a natural number dividing n.

Let Γ_l be the subgroup of $Z(SU(n))$ of order l, and set $G(n,l) := SU(n)/\Gamma_l$. Suppose p is an odd prime dividing l and p^r is the largest power of p dividing n. Then

$$H^*(G(n,l);\mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^{p^r}) \otimes \Lambda(z_1, z_3, \ldots, \widehat{z}_{2p^r-1}, \ldots, z_{2n-1}),$$

where $|y| = 2$. The power operations are given by

$$P^k(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k(p-1)}, \text{ and } \beta(z_{2p^{r-1}-1}) = y^{p^{r-1}}.$$

Similarly, if $p = 2$ and $4 \mid l$, then

$$H^*(G(n,l);\mathbb{Z}/2) \cong \mathbb{Z}/2[y]/(y^{2^r}) \otimes \Lambda(z_1, z_3, \ldots, \widehat{z}_{2^{r+1}-1}, \ldots, z_{2n-1}),$$

with

$$\mathrm{Sq}^{2k}(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k}, \text{ and } \mathrm{Sq}^1(z_{2^r-1}) = y^{2^{r-1}},$$

and all other odd-degree Steenrod operations are trivial. Finally, if $l \equiv 2 \pmod 4$, then

$$H^*(G(n,l);\mathbb{Z}/2) \cong \mathbb{Z}/2[z_1]/({z_1}^{2^{r+1}}) \otimes \Lambda(z_3, z_5, \ldots, \widehat{z}_{2^{r+1}-1}, \ldots, z_{2n-1}), \qquad (2)$$

with the Steenrod operations

$$\mathrm{Sq}^{2k}(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k} \text{ and } \mathrm{Sq}^1(z_{2^r-1}) = z_1^{2^r}.$$

Although there is significant torsion in the cohomology of $G(n,l)$, the Thom morphism is only non-surjective in specific cases.

Proposition 3.12. *Let $n, l \geqslant 1$ be integers with $4 \mid n$ and $l \equiv 2 \pmod 4$. Then the generator $e_{2^r-1} \in H^{2^r-1}(G(n,l);\mathbb{Z})$ is not in the image of the Thom morphism.*

Proof. The cohomology of $G(n,l)$ is as in (2), with $r \geqslant 2$. The Bockstein diagram yields that $e_{2^r-1} \in H^{2^r-1}(G(n,l);\mathbb{Z})$ maps to ${z_1}^{2^r-1} + z_{2^r-1}$ (possibly plus torsion), for which $\mathrm{Sq}^{2^r-1}({z_1}^{2^r-1} + z_{2^r-1}) = {z_1}^{2^{r+1}-2} \neq 0$. Thus, e_{2^r-1} is not in the image of the Thom morphism. □

Proposition 3.13. *Let n, l be positive integers such that $4 \nmid n$ or $l \not\equiv 2 \pmod 4$. Then the Thom morphism is surjective in all cohomological degrees for $G(n,l)$.*

Proof. We will show that all the differentials starting in the top row in the Atiyah–Hirzebruch spectral sequence for $G(n,l)$ must be trivial. Suppose a sequence of power operations and Bocksteins of odd degree $\geqslant 3$ is non-trivial when evaluated on an element of $H^*(G(n,l);\mathbb{Z}/p)$, where p is an odd prime. Since the only non-trivial Bockstein is

$$\beta(z_{2p^{r-1}-1}) = y^{p^{r-1}},$$

this can only occur if there is some z_{2i-1} such that

$$P^k(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k(p-1)} = z_{2p^{r-1}-1} \qquad (3)$$

for some k. We will now show that this is impossible by showing that for any i, k, r satisfying (3), the binomial coefficient is divisible by p.

To simplify the notation, let $j = i - 1$. The binomial coefficient $\binom{j}{k}$ can be computed using Lucas' theorem, which says that if

$$j = j_0 + j_1 p + \dots j_m p^m, \text{ and } k = k_0 + k_1 p + \dots k_m p^m$$

are the base p expansions of j and k, then

$$\binom{j}{k} \equiv \prod_{t=0}^{m} \binom{j_t}{k_t} \pmod{p}.$$

Here we use the convention that $\binom{j}{k} = 0$ if $j < k$. From equation (3) we see that

$$j = p^{r-1} + k - kp - 1. \tag{4}$$

We can assume that $r > 1$, since otherwise the only non-trivial Bockstein is $\beta(z_1) = y$, which leads to a surjective Thom morphism. Now, let s be the smallest natural number such that $p^{s-1} \mid k$, but $p^s \nmid k$. From equation (4) we see that $s < r$, since otherwise j would be negative. We then get

$$j \equiv k - 1 \pmod{p^s}.$$

Since $k \not\equiv 0 \pmod{p^s}$, we get $j_v < k_v$ for some $v < s$, and it follows from Lucas' theorem that $\binom{j}{k} \equiv 0 \pmod{p}$. This proves that there are no non-trivial differentials of odd torsion. It remains to show that there are no nontrivial differentials of 2-torsion in the remaining cases. There are two such cases: when n and l are both multiples of 4 and when $n \equiv 2 \pmod 4$. In the former case, the only non-trivial Bockstein is $\mathrm{Sq}^1(z_{2^r-1}) = y^{2^{r-1}}$, and the surjectivity of the Thom morphism follows from the same argument as with the odd torsion, by setting $P^k = \mathrm{Sq}^{2k}$. In the latter case, if $n \equiv 2 \pmod 4$ and $l \equiv 2 \pmod 4$, the cohomology ring is

$$H^*(G(n,l);\mathbb{Z}/2) \cong \mathbb{Z}/2[z_1]/(z_1^4) \otimes \Lambda(z_5, z_7, \dots, z_{2n-1}),$$

and the only Bockstein is $\mathrm{Sq}^1(z_1) = z_1^2$. The surjectivity of the Thom morphism then follows from the same argument as in the case $PSp(n)$ with n odd. □

3.4. Groups with exceptional Lie algebras

We will now consider Lie groups with exceptional Lie algebras. It turns out that the cases $\mathfrak{g}_2$, $\mathfrak{f}_4$ and $\mathfrak{e}_6$ follow the same pattern, while we can say a bit more on $\mathfrak{e}_7$ and $\mathfrak{e}_8$. We will therefore split our analysis into three subsections.

3.4.1. Groups with Lie algebra $\mathfrak{g}_2$, $\mathfrak{f}_4$ and $\mathfrak{e}_6$.

The free cohomologies of the exceptional Lie groups G_2, F_4 and E_6 are given by

$$\begin{aligned} H^*_{\mathrm{free}}(G_2;\mathbb{Z}) &\cong \Lambda(e_3, e_{11}), \\ H^*_{\mathrm{free}}(F_4;\mathbb{Z}) &\cong \Lambda(e_3, e_{11}, e_{15}, e_{23}), \\ H^*_{\mathrm{free}}(E_6;\mathbb{Z}) &\cong \Lambda(e_3, e_9, e_{11}, e_{15}, e_{17}, e_{23}). \end{aligned}$$

The center of E_6 is isomorphic to $\mathbb{Z}/3$ (see [**18**]). Hence there is another group with Lie algebra $\mathfrak{e}_6$, which we refer to as the *centerless* E_6 and denote by E_6/Γ_3.

Proposition 3.14. *For G_2, F_4, E_6 and E_6/Γ_3, the generator e_3 in integral cohomology is not in the image of the Thom morphism.*

Proof. The $\mathbb{Z}/2$-cohomologies of G_2, F_4 and E_6 are given by

$$\begin{aligned} H^*(G_2;\mathbb{Z}/2) &\cong \mathbb{Z}/2[x_3]/(x_3^4)\otimes\Lambda(x_5),\\ H^*(F_4;\mathbb{Z}/2) &\cong \mathbb{Z}/2[x_3]/(x_3^4)\otimes\Lambda(x_5,x_{15},x_{23}),\\ H^*(E_6;\mathbb{Z}/2) &\cong \mathbb{Z}/2[x_3]/(x_3^4)\otimes\Lambda(x_5,x_9,x_{15},x_{17},x_{23}). \end{aligned}$$

In each case, the generator e_3 in integral cohomology reduces to x_3 in $\mathbb{Z}/2$-cohomology and $\mathrm{Sq}^3(x_3)=x_3^2\neq 0$. Hence e_3 is not in the image of the Thom morphism. Since neither the free cohomology nor the cohomology with coefficients in $\mathbb{Z}/2$ are altered by dividing out by a subgroup of order 3, the same argument as for E_6 applies to the quotient E_6/Γ_3. □

Remark 3.15. We note that the other generators in the integral cohomology groups of G_2, F_4, E_6 and E_6/Γ_3 are in the image the Thom morphism. As we will see in Proposition 3.17 and 3.18, the behaviors of the cohomology of the groups corresponding to the Lie algebras $\mathfrak{e}_7$ and $\mathfrak{e}_8$ are different.

Remark 3.16. The integral cohomology of the exceptional groups, except for G_2, also have 3-torsion, and we could have used a computation at $p=3$ to show non-surjectivity.

3.4.2. Groups with Lie algebra $\mathfrak{e}_7$.

The free cohomology of the group E_7 is given by

$$H^*_{\mathrm{free}}(E_7;\mathbb{Z})\cong\Lambda(e_3,e_{11},e_{15},e_{19},e_{23},e_{27},e_{35}).$$

The center of E_7 is isomorphic to $\mathbb{Z}/2$ (see [**18**]), and hence there is another group which has the Lie algebra E_7. We will refer to this group as the *centerless* E_7 and denote it by E_7/Γ_2.

Proposition 3.17. *For E_7 and E_7/Γ_2, the generators e_3 and e_{15} in integral cohomology are not in the image of the Thom morphism.*

Proof. We use the $\mathbb{Z}/3$-cohomology of E_7 which is given by

$$H^*(E_7;\mathbb{Z}/3)\cong\ \mathbb{Z}/3[x_8]/(x_8^3)\otimes\Lambda(x_3,x_7,x_{11},x_{15},x_{19},x_{27},x_{35}),$$

with $P^1x_3=x_7$, $P^3x_7=x_{19}$, and $\beta x_7=x_8$. Moreover, we know that the reduction homomorphism $\rho\colon H^3(E_7;\mathbb{Z})\to H^3(E_7;\mathbb{Z}/3)$ sends e_3 to x_3. Let Q_1 denote the first Milnor operation. We then have

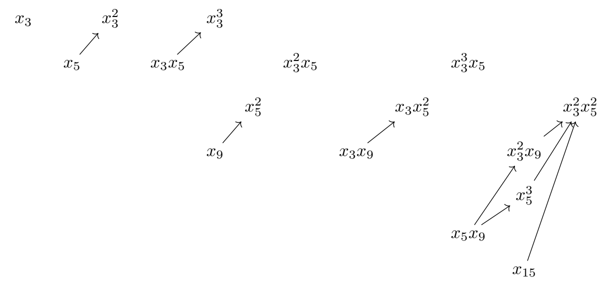
$$Q_1(x_3)=P^1\beta(x_3)-\beta P^1(x_3)=P^1(0)-\beta(x_7)=-x_8,$$

and conclude that e_3 is not in the image of τ by Lemma 2.1. Since dividing out by a subgroup of order 2 changes neither the free cohomology nor the $\mathbb{Z}/3$-cohomology, we deduce that $e_3\in H^3(E_7/\Gamma_2;\mathbb{Z})$ is not in the image of τ either. To see that the generator e_{15} is not in the image of τ we will use 2-torsion. The $\mathbb{Z}/2$-cohomologies of

E_7 and E_7/Γ_2 are given by

$$H^*(E_7;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_3,x_5,x_9]/(x_3^4,x_5^4,x_9^4) \otimes \Lambda(x_{15},x_{17},x_{23},x_{27}),$$
$$H^*(E_7/\Gamma_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,x_5,x_9]/(x_1^4,x_5^4,x_9^4) \otimes \Lambda(x_6,x_{15},x_{17},x_{23},x_{27}).$$

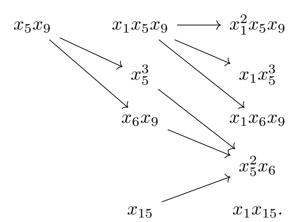
In low degrees we get the following Bockstein diagram for E_7:



The generator $e_{15} \in H^{15}(E_7,\mathbb{Z})$ reduces to $x_{15}+x_3^2x_9$ or $x_{15}+x_5^3$ in $H^{15}(E_7;\mathbb{Z}_2)$. While $\mathrm{Sq}^3(x_{15}) = \mathrm{Sq}^1\mathrm{Sq}^2(x_{15}) = \mathrm{Sq}^1(x_{17}) = x_9^2$, Sq^3 acts trivially on both $x_3^2x_9$ and x_5^3. Hence we get

$$\mathrm{Sq}^3(x_{15}+x_3^2x_9) = \mathrm{Sq}^3(x_{15}+x_5^3) = x_9^2 \neq 0.$$

Thus, $e_{15} \in H^{15}(E_7;\mathbb{Z})$ is not in the image of the Thom morphism. For E_7/Γ_2, the relevant part of the Bockstein diagram is:



It follows that $e_{15} \in H^{15}(E_7/\Gamma_2;\mathbb{Z})$ reduces to either $x_{15}+x_6x_9$ or $x_{15}+x_5^3$ in the group $H^{15}(E_7/\Gamma_2;\mathbb{Z}/2)$. Since Sq^3 acts trivially on x_6x_9 as well, we get again that $\mathrm{Sq}^3(x_{15}+x_6x_9) = \mathrm{Sq}^3(x_{15}+x_5^3) = x_9^2 \neq 0$. This proves the assertion. □

3.4.3. The group E_8

The free cohomology of the group E_8 is given by

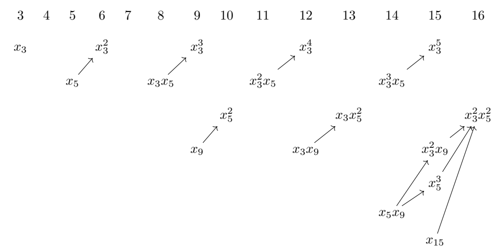
$$H^*_{\mathrm{free}}(E_8;\mathbb{Z}) \cong \Lambda(e_3,e_{15},e_{23},e_{27},e_{35},e_{39},e_{47},e_{59}),$$

while $\mathbb{Z}/2$-cohomology of E_8 is given by the isomorphism

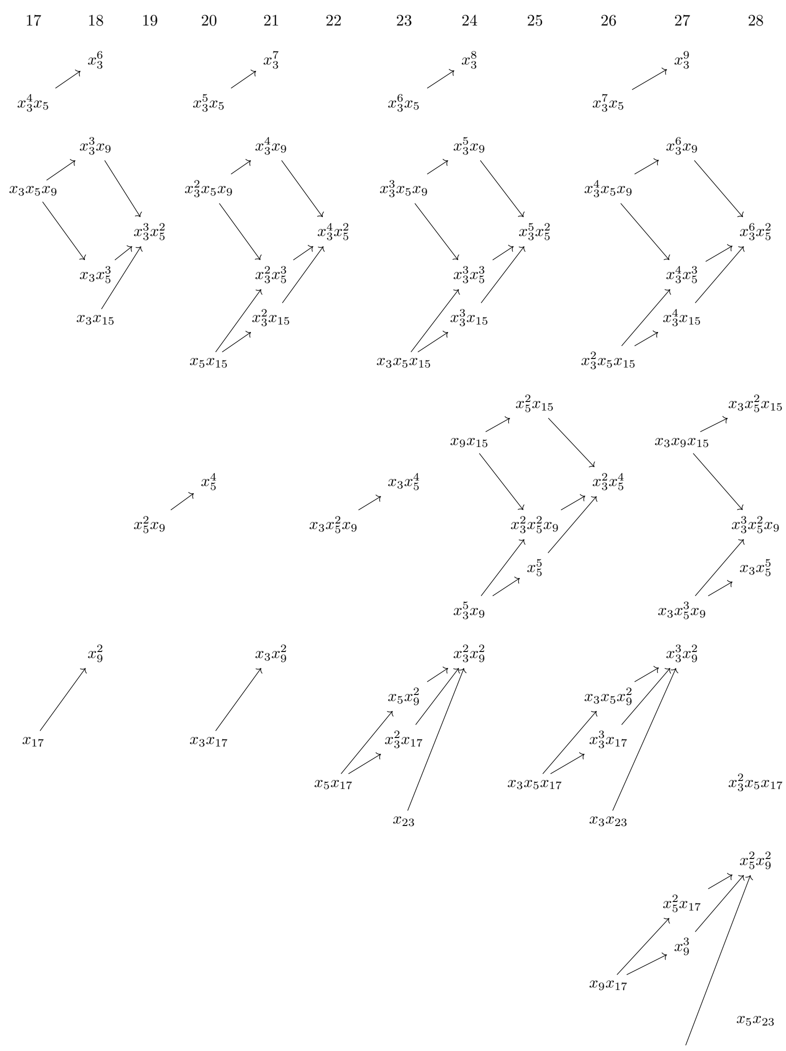
$$H^*(E_8;\mathbb{Z}/2) \cong \ \mathbb{Z}/2[x_3,x_5,x_9,x_{15}]/(x_3^{16},x_5^8,x_9^4,x_{15}^4) \otimes \Lambda(x_{17},x_{23},x_{27},x_{29}).$$

Proposition 3.18. *The generators $e_3,e_{15},e_{23},e_{27} \in H^*_{free}(E_8;\mathbb{Z})$ as well as the sum of any of these generators with a torsion class in the same degree are not in the image of the Thom morphism.*

Proof. In low degrees, we have the following Bockstein diagram for $p = 2$:



The class $e_3 \in H^3_{\text{free}}(E_8; \mathbb{Z})$ reduces to the class $x_3 \in H^3(E_8; \mathbb{Z}/2)$, for which Sq^3 is non-trivial. For the other degrees, we start in degree 27 and work our way down. The Bockstein diagram continues as follows:



(5)

From diagram (5) we see that under the reduction map $H^{27}(E_8;\mathbb{Z}) \to H^{27}(E_8;\mathbb{Z}/2)$ the class e_{27} + (torsion) is sent to either

$$x_{27} + x_5^2 x_{17} + L \quad \text{or} \quad x_{27} + x_9^3 + L, \tag{6}$$

where L is some linear combination of the classes

$$x_3^3 x_9^2,\ x_3^9,\ x_3^6 x_9 + x_3^4 x_5^3,\ \text{and } x_3^4 x_{15} + x_3^4 x_5^3.$$

The square Sq^3 acts on x_{27} by $\mathrm{Sq}^3(x_{27}) = \mathrm{Sq}^1\mathrm{Sq}^2(x_{27}) = \mathrm{Sq}^1(x_{29}) = x_{15}^2$, while Sq^3 applied to any of the other terms in (6) does not yield an x_{15}^2-term. Thus, x_{15}^2 cannot get cancelled out, and it follows that e_7 and e_7 plus torsion are not in the image of the Thom morphism. For the class e_{23}, we get four alternatives for its reduction to $\mathbb{Z}/2$-cohomology:

$$x_{23} + x_5 x_9^2,\ x_{23} + x_5 x_9^2 + x_3 x_5^4,\ x_{23} + x_3^2 x_{17},\ \text{and } x_{23} + x_3^2 x_{17} + x_3 x_5^4. \tag{7}$$

The square Sq^4 acts on these classes by

$$\mathrm{Sq}^4(x_{23} + x_5 x_9^2) = x_{27} + x_9^3,\ \mathrm{Sq}^4(x_{23} + x_5 x_9^2 + x_3 x_5^4) = x_{27} + x_9^3 + x_3^9,$$
$$\mathrm{Sq}^4(x_{23} + x_3^2 x_{17}) = x_{27} + x_5^2 x_{17},\ \mathrm{Sq}^4(x_{23} + x_3^2 x_{17} + x_3 x_5^4) = x_{27} + x_5^2 x_{17} + x_3^9.$$

We see that each cohomology class in (7) is mapped to a class in (6) by Sq^4. It follows that e_{23} reduces to a class on which $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4$ is nonzero. This shows that e_{23} plus any torsion is not in the image of the Thom morphism. Finally, there are four possibilities for the mod-2 reduction of e_{15} plus torsion, given by

$$x_{15} + x_3^2 x_9,\ x_{15} + x_3^2 x_9 + x_3^5,\ x_{15} + x_5^3,\ \text{and } x_{15} + x_5^3 + x_3^5. \tag{8}$$

The square Sq^8 acts on these classes by

$$\mathrm{Sq}^8(x_{15} + x_3^2 x_9) = x_{23} + x_3^2 x_{17}, \quad \mathrm{Sq}^8(x_{15} + x_3^2 x_9 + x_3^5) = x_{23} + x_3^2 x_{17} + x_3 x_5^4,$$
$$\mathrm{Sq}^8(x_{15} + x_5^3) = x_{23} + x_5 x_9^2, \quad \mathrm{Sq}^8(x_{15} + x_5^3 + x_3^5) = x_{23} + x_5 x_9^2 + x_3 x_5^4.$$

Hence all classes in (8) have a nonzero image under $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^8$. This shows that e_{15} plus torsion is not in the image of the Thom morphism. □

4. Geometric examples for special orthogonal groups

In this section we switch perspectives and give a concrete and geometric construction of certain cobordism elements for special orthogonal groups. We begin with $SO(5)$ and will then generalise to higher dimensional $SO(n)$.

4.1. A geometric cobordism class on $SO(5)$

Recall from Proposition 3.1 that the generator $e_3 \in H^*_{\text{free}}(SO(5);\mathbb{Z}) \cong \Lambda(e_3, e_7)$ is not in the image of τ. However, we will now show that the element $2e_3$ is in the image of the Thom morphism. We will prove this by geometrically constructing an element of $MU^3(SO(5))$ which is mapped to $2e_3 \in H^3(SO(5);\mathbb{Z})$ under τ. To do so we make use of the fact that $SO(5)$ is a 10-dimensional compact manifold. Let $2\tilde{e}_3 \in H_7(SO(5);\mathbb{Z})$ denote the image of $2e_3$ under the isomorphism $H^3(SO(5);\mathbb{Z}) \cong H_7(SO(5);\mathbb{Z})$ defined by Poincaré duality. By [**23**, Proposition 1.2], elements in $MU^3(SO(5))$ can be represented by proper complex-oriented maps of the form $M \to SO(5)$ where M is a

7-dimensional manifold. Thus, in order to show that $2e_3$ is in the image of τ, it suffices to find a proper complex-oriented map $g\colon M \to SO(5)$ such that $g_*[M] = 2\tilde{e}_3$ where $[M]$ denotes the fundamental class of M in $H_7(SO(5);\mathbb{Z})$.

To compute the homology group $H_7(SO(5);\mathbb{Z})$, we recall the cell structure of special orthogonal groups using maps from products of real projective spaces from [**8**, Proposition 3D.1]. Let v be a nonzero vector in $\mathbb{R}^n$. We define the linear transformation $r(v)\colon \mathbb{R}^n \to \mathbb{R}^n$ to be the reflection across the orthogonal complement of v. We may use this map to define an embedding from $\mathbb{RP}^{k-1}$ to $SO(n)$ for $k \leqslant n$ as follows. Representing elements of $\mathbb{RP}^{k-1}$ by vectors in $\mathbb{R}^k$ and embedding into $\mathbb{R}^n$ in the canonical way if $k < n$, we define the map

$$\mathbb{RP}^{k-1} \longrightarrow SO(n),\ [v] \longmapsto r(v)\cdot r(e_1)$$

where $e_1, \ldots, e_n$ are the standard basis vectors. We extend this to a map defined on products of real projective spaces by taking compositions, i.e.,

$$\begin{aligned} f_{i_1,\ldots,i_m}\colon \mathbb{RP}^{i_1} \times \ldots \times \mathbb{RP}^{i_m} &\longrightarrow SO(n) \\ ([v_1],\ldots,[v_m]) &\longmapsto r(v_1)\cdot r(e_1)\cdots r(v_m)\cdot r(e_1). \end{aligned}$$

For $SO(n)$, there is a k-cell for each sequence $(i_1,\ldots,i_m)$ which satisfies both $n > i_1 > \ldots > i_m > 0$ and $i_1 + \ldots + i_m = k$. The characteristic map is given by

$$D^k \xrightarrow{\ \cong\ } D^{i_1} \times \ldots \times D^{i_m} \longrightarrow \mathbb{RP}^{i_1} \times \ldots \times \mathbb{RP}^{i_m} \longrightarrow SO(n),$$

where the second map is the product of the characteristic maps for the top cells of each real projective space. There is a single 0-cell, namely the identity of $SO(n)$. This gives us all the information we need to construct the cellular chain complex of $SO(5)$. The differentials are determined by the differentials in the cellular chain complexes of real projective spaces, as well as the product formula

$$d(e^i \times e^j) = d(e^i) \times e^j + (-1)^i e^i \times d(e^j).$$

The cell structure yields that $H_7(SO(5);\mathbb{Z}) \cong \mathbb{Z}$ where the generator is induced from the cell $(4,3)$ with characteristic map

$$f_{4,3}\colon \mathbb{RP}^4 \times \mathbb{RP}^3 \to SO(5).$$

However, since $\mathbb{RP}^4$ is not orientable, the map $f_{4,3}$ does not represent a bordism class. In fact, we know from the algebraic obstruction that the image of e_3 under $H_7(SO(5))$ cannot be hit by a bordism class on $SO(5)$. We will now show that we can replace $\mathbb{RP}^4 \times \mathbb{RP}^3$ with an orientable smooth manifold M and the map $f_{4,3}$ with a smooth map $g\colon M \to SO(5)$ with the same image as $f_{4,3}$.

To do so, we first observe that the cell decomposition implies that every element of $SO(5)$ can be expressed as a composition of reflections in $\mathbb{R}^5$, where every pair of reflections leaves a 3-dimensional subspace fixed and performs a rotation in the remaining 2-plane. Let $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ be the Grassmann manifold of *oriented* 2-dimensional planes in $\mathbb{R}^5$. We will write elements of $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ in the form (L,σ), where L is a plane and σ is an orientation of L. We then define

$$g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \longrightarrow SO(5)$$

to be the map that sends $((L,\sigma), e^{it})$ to the element of $SO(5)$ which rotates the plane L by the angle t according to the orientation σ. More precisely, given a point

$((L,\sigma),e^{it}) \in \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$, let $r_{L,\sigma,t}$ be the rotation of L by the angle t along σ. Let $L^\perp$ denote the orthogonal complement of L in $\mathbb{R}^5$ with the respect to the standard inner product. Then we can write $v \in \mathbb{R}^5$ in a unique way as $v = v_1 + v_2$ such that $v_1 \in L$ and $v_2 \in L^\perp$. The transformation $g((L,\sigma),e^{it}) \in SO(5)$ is then defined by

$$g((L,\sigma),e^{it})(v) = r_{L,\sigma,t}(v_1) + v_2.$$

Lemma 4.1. *The map g admits a complex orientation. In particular, g is a proper complex-oriented smooth map and represents an element in $MU^3(SO(5))$.*

Proof. The element $g((L,\sigma),e^{it})$ varies smoothly with (L,σ) and t in $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$. The fact that g admits a complex orientation follows from the facts that S^1 is stably almost complex, $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \cong SO(5)/(SO(2) \times SO(3))$ is almost complex, and $SO(5)$ is a compact Lie group. □

Lemma 4.2. *The images of the maps $f_{4,3}$ and g in $SO(5)$ are equal, i.e., the image of the map $g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \to SO(5)$ is the cell $(4,3)$.*

Proof. To simplify the notation we write $f = f_{4,3}$. We begin with showing that we have $\mathrm{Im}\, f \subseteq \mathrm{Im}\, g$. Let $(u,v) \in \mathbb{RP}^4 \times \mathbb{RP}^3$. We will show that there exist two elements in $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$ which map to the element $f(u,v) \in SO(5)$. When showing this we will assume that u and v are both different from $\pm e_1$. However, otherwise the argument is similar. The 4-planes $e_1^\perp$ and $u^\perp$ intersect on a 3-dimensional subspace of $\mathbb{R}^5$ which remains fixed under the map $r(u) \cdot r(e_1)$. We will call this subspace M_u. Likewise, we let M_v denote the 3-dimensional subspace fixed by $r(v) \cdot r(e_1)$. We observe that both M_u and M_v are contained in $\mathrm{Span}\{e_2,e_3,e_4,e_5\}$.

We first consider the case where $u = v$. Then $M_u = M_v$, and it follows that $f(u,v)$ is a rotation in the plane $L = M_u^\perp$. For each of the two possible orientations of L, there is precisely one angle in S^1 which gives the rotation corresponding to $f(u,v)$, which shows that there are two elements of $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$ which map to $f(u,v)$. On the other hand, if $u \neq v$, we get that $M_u \cap M_v$ is a 2-dimensional subspace of $\mathrm{Span}\{e_2,e_3,e_4,e_5\}$. Let $N = (M_u \cap M_v)^\perp$. We observe that $f(u,v) \in SO(5)$ maps N to N. Since N is 3-dimensional, a 1-dimensional subspace of N is left fixed by $f(u,v)$, and we call this line T. We have now seen that $f(u,v)$ leaves $T \oplus (M_u \cap M_v)$ fixed. The remaining 2-dimensional subspace of N is then our choice of L, in other words

$$L := (T \oplus (M_u \cap M_v))^\perp.$$

Having found the plane where the rotation takes place, we may combine orientations σ and elements of S^1 as in the case $u = v$ to get the desired element of $SO(5)$. This proves that $\mathrm{Im}\, f \subseteq \mathrm{Im}\, g$.

We now show that $\mathrm{Im}\, g \subseteq \mathrm{Im}\, f$. Let

$$((L,\sigma),e^{it}) \in \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1.$$

Our goal is to find vectors u and v such that $f(u,v) = g((L,\sigma),e^{it})$. We can first observe that $L \cap \mathbb{R}^4$ is at least 1-dimensional. We then choose v' to be any unit vector in this intersection and note that v' represents a point in $\mathbb{RP}^3$. Next, we need to find a suitable $u \in \mathbb{RP}^4$ such that

$$r(u) \cdot r(v') = g((L,\sigma),e^{it}).$$

To ensure that $L^\perp$ is fixed by both $r(u)$ and $r(v')$ we need that u is in L. Furthermore, the angle between u and v' is uniquely determined by t to yield the desired rotation. In fact, the angle must be $t/2$ or $t/2+\pi$, depending on which representative we choose for the point in $\mathbb{RP}^4$. With the exception of the cases $t=0$ and $t=\pi$, this leaves two options for u, which we choose between by making sure the rotation $r(u)\cdot r(v')$ goes in the right direction according to the orientation σ. The composition $r(v')\cdot r(e_1)$ has a unique inverse, which is given by $r(v)\cdot r(e_1)$ for some vector v in $\mathbb{RP}^3$. We then have

$$\begin{aligned} f(u,v) &= r(u)\cdot r(e_1)\cdot r(v)\cdot r(e_1) = r(u)\cdot r(e_1)\cdot [r(v')\cdot r(e_1)]^{-1} \\ &= r(u)\cdot r(e_1)\cdot r(e_1)\cdot r(v') = r(u)\cdot r(v') \\ &= g((L,\sigma),e^{it}). \end{aligned}$$

This proves that $\operatorname{Im} f = \operatorname{Im} g$. □

We can now show the main result of this section.

Theorem 4.3. *The cobordism class represented by g maps to $2e_3 \in H^3(SO(5);\mathbb{Z})$ under the Thom morphism.*

Proof. By Poincaré duality it suffices to show that the fundamental class of the manifold $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)\times S^1$ in homology is mapped to two times the generator $\tilde{e}_3$ in $H_7(SO(5);\mathbb{Z})$. By Lemma 4.2, the image of g is the cell $(4,3)$. Hence we can consider g as a map

$$\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)\times S^1 \longrightarrow (4,3).$$

Let $q\colon (4,3)\longrightarrow (4,3)/(4,3)_6$ be the map that collapses the 6-skeleton of the cell $(4,3)$. We then get the following commutative diagram in homology

$$\begin{array}{ccc} H_7(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)\times S^1;\mathbb{Z}) & \xrightarrow{\;g_*\;} & H_7((4,3);\mathbb{Z}) \\ & \searrow^{(q\circ g)_*} & \cong\big\downarrow q_* \\ & & H_7((4,3)/(4,3)_6;\mathbb{Z}). \end{array}$$

Using the homology long exact sequence of the pair $((4,3)_6,(4,3))$, it is straightforward to see that the map q_* is an isomorphism. The quotient $(4,3)/(4,3)_6$ is homeomorphic to S^7. Hence by choosing an orientation we can assume that $q\circ g$ is a map between compact, oriented topological manifolds. Proving that g_* is a multiplication by ± 2 is hence reduced to the claim that the map $(q\circ g)_*$ has degree ± 2. We will show this claim by computing the local degree of $q\circ g$ at two points in $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)\times S^1$.

Let $y\in SO(5)$ be the point corresponding to the matrix

$$y=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then y defines a rotation by the angle π in the plane $\mathrm{Span}\{\mathrm{e}_4,\mathrm{e}_5\}$. If we define the plane $L = \mathrm{Span}\{\mathrm{e}_4, \mathrm{e}_5\}$, and let $\pm\sigma$ be the two possible orientations of L, we get

$$g((L,\sigma), e^{i\pi}) = g((L,-\sigma), e^{i\pi}) = y,$$

and these are the only two points that are sent to y under g. Since $\mathrm{Im}\, g = \mathrm{Im}\, f$ by Lemma 4.2, it follows that $y \in (4,3)$. We now define open neighborhoods of the points $((L,\pm\sigma), e^{i\pi})$ that are mapped homeomorphically to an open neighborhood of y. Let

$$\mathcal{U} = \{(M,\sigma) \in \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \mid M \cap \mathrm{Span}\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\} = 0\}.$$

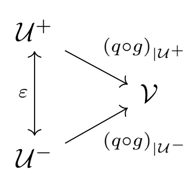
Since $\mathcal{U}$ consists of two open path-components, so does the product

$$\mathcal{U} \times (S^1 \setminus \{e^0\}).$$

We denote these two path-components of $\mathcal{U} \times (S^1 \setminus \{e^0\})$ by $\mathcal{U}^+$ and $\mathcal{U}^-$. We then have $((L,\sigma), e^{i\pi}) \in \mathcal{U}^+$ and $((L,-\sigma), e^{i\pi}) \in \mathcal{U}^-$. We let $\mathcal{V} \subset (4,3)/(4,3)_6$ be the interior of the cell $(4,3)$. We observe that $\mathcal{V}$ consists of the rotations of $\mathbb{R}^5$ that do not leave any nonzero vector in $\mathrm{Span}\{\mathrm{e}_4, \mathrm{e}_5\}$ fixed, which corresponds to the planes in $\mathcal{U}$. From the proof of Lemma 4.2 we deduce that g maps precisely two points of $\mathcal{U} \times (S^1 \setminus \{e^0\})$ to every point in $\mathcal{V}$. This implies that g sends $\mathcal{U}^+$ and $\mathcal{U}^-$ homeomorphically to $\mathcal{V}$. Thus, at each of the points $((L,\pm\sigma), e^{i\pi})$, the map $q \circ g$ has local degree $+1$ or -1. The points $((L,\sigma), e^{it})$ and $((L,-\sigma), e^{-it})$ have the same image in $SO(5)$ under g. The map $S^1 \to S^1$, $e^{it} \mapsto e^{-it}$, reverses the orientation. We recall that $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ is a double cover of the unoriented Grassmannian $\mathrm{Gr}_2(\mathbb{R}^5)$, and that $\mathrm{Gr}_2(\mathbb{R}^5)$ is not orientable. By [**17**, Theorem 15.36], these observations imply that the map $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \to \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ which sends (L,σ) to $(L,-\sigma)$ is not orientation-preserving, since this map is the only non-trivial automorphism of $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ compatible with the projection to $\mathrm{Gr}_2(\mathbb{R}^5)$. Hence, the map

$$\begin{aligned} \varepsilon\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 &\longrightarrow \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \\ ((L,\sigma), e^{it}) &\longmapsto ((L,-\sigma), e^{-it}) \end{aligned}$$

is a product of two maps which reverse the orientation. Thus, ε preserves the orientation. By the construction of ε, the diagram



commutes. Since ε preserves the orientation, we know that either both $(q \circ g)_{|\mathcal{U}^+}$ and $(q \circ g)_{|\mathcal{U}^-}$ preserve the orientation, or they both reverse it. Thus, $q \circ g$ has the same local degree at the points $((L,\pm\sigma), e^{i\pi})$, and we conclude that g_* is given by multiplication by ± 2, which completes the proof. □

By Lemma 2.2, Theorem 4.3 implies the following result:

Corollary 4.4. *The class* $[g] \otimes \frac{1}{2}$ *is a non-trivial element in the kernel of*

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}}\colon MU^3(SO(5)) \otimes_{MU^*} \mathbb{R}/\mathbb{Z} \longrightarrow H^3(SO(5);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

4.2. Generalization to higher dimensions

We will now explain how the method to prove Theorem 4.3 can be generalised to special orthogonal groups of higher dimensions. We first show which cells provide the generators for cohomology groups of $SO(n)$ and then make a generalised geometric construction. Let $k \geqslant 0$ and $n \geqslant 2k+3$. Then there is a non-torsion generator e_{4k+3} in $H^{4k+3}(SO(n);\mathbb{Z})$. We can use the cell structure of $SO(n)$ to describe e_{4k+3} as follows. Let $(i_1,\ldots,i_m)$ be a sequence of integers with $n-1 \geqslant i_1 > i_2 > \ldots > i_m \geqslant 1$. We let $(\widehat{i_1,\ldots,i_m})$ denote the image of the map

$$f_{j_1,\ldots,j_s}\colon \mathbb{R}P^{j_1}\times\ldots\times\mathbb{R}P^{j_s} \longrightarrow SO(n),$$

where $(j_1,\ldots,j_s)$ is the sequence obtained by removing the numbers $i_1,\ldots,i_m$ from the sequence $(n-1,n-2,\ldots,1)$. We deduce from the cell structure that e_{4k+3} is induced by the cell $(\widehat{2k+2,2k+1})$. Using the methods of section 3.1.1 we can show that for every k, the generator e_{4k+3} is not in the image of the Thom morphism for sufficiently large n. Determining a minimal such n is more difficult, and we have been unable to find a more efficient method than to study the Bockstein diagrams on a case by case basis. However, we will now show how a multiple of e_{4k+3} can always be constructed geometrically, whether or not e_{4k+3} itself is in the image of the Thom morphism. Let $n \geqslant 3$ be odd. We then define the map

$$g_n\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^n)\times S^1 \longrightarrow SO(n)$$

in the same way as the map $g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)\times S^1 \to SO(5)$ in section 4.1. For $m > n$, we will also denote by g_n the composition $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^n)\times S^1 \to SO(n) \hookrightarrow SO(m)$ with the canonical embedding of $SO(n)$ into $SO(m)$.

Lemma 4.5. *For every $n \geqslant 3$ odd and every $m \geqslant n$, the map g_n admits a complex orientation. In particular, g_n is a proper complex-oriented smooth map and represents an element in $MU^*(SO(m))$.*

Proof. This follows again from the facts that S^1 is stably almost complex, $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^n)$ is almost complex, and $SO(m)$ is a compact Lie group. □

We define the map

$$\begin{aligned}\varepsilon_n\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^n)\times S^1 &\longrightarrow \widetilde{\mathrm{Gr}_2}(\mathbb{R}^n)\times S^1\\ ((L,\sigma),e^{it}) &\longmapsto ((L,-\sigma),e^{-it}).\end{aligned}$$

Lemma 4.6. *The maps τ_n, g_n and $f_{i_1,\ldots,i_m}$ have the properties*

(i) $\operatorname{Im} g_n = \operatorname{Im} f_{n-1,n-2}$
(ii) $g_n \circ \varepsilon_n = g_n$
(iii) ε_n *is orientation-preserving.*

Proof. This follows from similar arguments as in the proofs of Lemma 4.2 and Theorem 4.3. □

We now construct the cobordism class which maps to a multiple of the element $e_{4k+3} \in H^{4k+3}(SO(m))$. The construction depends on whether m is even or odd, and we start with $m = 2n+1$ odd. For k and m with $2k+1 < m$, let i denote the canonical embedding $SO(2k+1) \to SO(m)$. We can now define the map

$$h_{2n+1,k} := g_{2n+1} \times g_{2n-1} \times \cdots \times g_{2k+5} \times i\colon \qquad (9)$$
$$\prod_{l=k+2}^{n} \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^{2l+1}) \times S^1\right) \times SO(2k+1) \longrightarrow SO(2n+1).$$

It follows from Lemma 4.6 that the image of this map is the cell

$$(2n, 2n-1, \ldots, 2k+3, 2k, \ldots, 2, 1) = (\widehat{2k+2}, \widehat{2k+1}).$$

If $m = 2n$ is even, then we need to define one more map. Given the map

$$f_k\colon \mathbb{RP}^k \to SO(m),$$

let f'_k be the composite map

$$S^k \longrightarrow \mathbb{RP}^k \xrightarrow{f_k} SO(m),$$

where the first map is the canonical double cover. We can then define

$$h_{2n,k} := f'_{2n-1} \times g_{2n-1} \times g_{2n-3} \times \cdots \times g_{2k+5} \times i\colon \qquad (10)$$
$$S^{2n-1} \times \prod_{l=k+2}^{n-1} \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^{2l+1}) \times S^1\right) \times SO(2k+1) \longrightarrow SO(2n).$$

Again, it follows from Lemma 4.6 that the image of this map is the cell

$$(2n-1, 2n-2, \ldots, 2k+3, 2k, \ldots, 2, 1) = (\widehat{2k+2}, \widehat{2k+1}).$$

Theorem 4.7. *Let $k \geqslant 0$ and $n \geqslant k+1$. Then the Thom morphism sends the cobordism class represented by the map $h_{2n+1,k}$ in $MU^{4k+3}(SO(2n+1))$ to $2^{n-k-1}e_{4k+3} \in H^{4k+3}(SO(2n+1);\mathbb{Z})$. If $n \geqslant k+2$, then the Thom morphism sends the cobordism class represented by the map $h_{2n,k}$ in $MU^{4k+3}(SO(2n))$ to the element $2^{n-k-1}e_{4k+3} \in H^{4k+3}(SO(2n);\mathbb{Z})$.*

Proof. The assertion follows as in the proof of Theorem 4.3 from the fact that each factor $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^{2l+1}) \times S^1$ is wrapped twice around the cell $(2l, 2l-1)$. □

Corollary 4.8. *For every $k \geqslant 0$, there is a sufficiently large integer m such that the class $[h_{m,k}] \otimes \frac{1}{2^{n-k-1}}$ with $n = \lfloor \frac{m}{2} \rfloor$ is a non-trivial element in the kernel of*

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}}\colon MU^*(SO(m)) \otimes_{MU^*} \mathbb{R}/\mathbb{Z} \longrightarrow H^*(SO(m);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Remark 4.9. There is a notable case where the construction of map (10) can be simplified. For $SO(8)$, the factor S^7 can be replaced by $\mathbb{RP}^7$ since this space is parallelizable. Thus, the map

$$f_7 \times g_7 \times g_5\colon\ \mathbb{RP}^7 \times \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1\right) \times \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1\right) \longrightarrow SO(8)$$

represents an element of $MU^3(SO(8))$ which is mapped to $4e_3 \in H^3(SO(8);\mathbb{Z})$.

References

[1] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics. University of Chicago Press, 1974.

[2] M. Atiyah, F. Hirzebruch, Analytic cycles on complex manifolds, Topology **1** (1962), 25–45.

[3] P. F. Baum, W. Browder, The cohomology of quotients of classical groups, Topology **3** (1965), 305–336.

[4] O. Benoist, J.C. Ottem, Two coniveau filtrations, Duke Math. J. **170** (2021), 2719–2753.

[5] H. Cartan, Sur les groupes d'Eilenberg-MacLane II, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 704–707.

[6] H. Cartan, Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1954/1955. Algèbres d'Eilenberg-MacLane et homotopie, 1955.

[7] D. Grady, H. Sati, Spectral sequences in smooth generalized cohomology, Algebr. Geom. Topol. **17** (2017), no. 4, 2357–2412.

[8] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[9] K. B. Haus, G. Quick, Geometric pushforward in Hodge filtered complex cobordism and secondary invariants, Doc. Math. 29 (2024), 457–509.

[10] L. Hodgkin, On the K-theory of Lie groups, Topology **6** (1967), 1–36.

[11] M. J. Hopkins, G. Quick, Hodge filtered complex bordism, J. Topology **8** (2015), 147–183.

[12] M. J. Hopkins, I. M. Singer, Quadratic functions in geometry, topology, and M-theory, J. Differential Geom. **70** (2005), 329–452.

[13] J. Hunton, M. Mimura, T. Nishimoto, B. Schuster, Higher v_n torsion in Lie groups, J. Math. Soc. Japan **50** (1998), no. 4, 801–818.

[14] M. Kameko, On the integral Tate conjecture for finite fields, Math. Proc. Camb. Phil. Soc. **158** (2015), 531–546.

[15] M. Kameko, Non-torsion non-algebraic classes in the Brown–Peterson tower, Math. Proc. Cambridge Philos. Soc. **171** (2021), no.1, 113–132.

[16] E. Kaspersen, G. Quick, A note on the Thom morphism for the classifying space of certain Lie groups and gauge groups, preprint, arXiv:2405.12717.

[17] J. M. Lee, *Introduction to smooth manifolds*, Second edition, Springer, 2013.

[18] M. Mimura, H. Toda, Topology of Lie groups. I and II, Translations of Mathematical Monographs, 91, American Mathematical Society, Providence, RI, 1991.

[19] R. E. Mosher, M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, Publishers, New York–London, 1968.

[20] T. Nishimoto, Higher torsion in the Morava K-theory of SO(m) and Spin(m), J. Math. Soc. Japan **53** (2001), no. 2, 383–394.

[21] C. Procesi, *Lie Groups*, Springer, 2007.

[22] G. Quick, Examples of non-algebraic classes in the Brown–Peterson tower, Math. Z. **293** (2019), no.1-2, 25–37.

[23] D. G. Quillen, Some elementary proofs of some results of complex cobordism theory using Steenrod operations, Adv. Math. **7** (1971), 29–56.

[24] H. Tamanoi, Spectra of BP-linear relations, v_n-series, and BP-cohomology of Eilenberg–Mac Lane spaces, Trans. Amer. Math. Soc. **352** (2000), no. 11, 5139–5178.

[25] R. Thom, Quelques propriétés globales des variétés diffŕentiables, Comment. Math. Helv. **28** (1954), 17–86.

[26] B. Totaro, Torsion algebraic cycles and complex cobordism, J. Amer. Math. Soc. **10** (1997), no. 2, 467–493.

[27] W. S. Wilson, The Ω-spectrum for Brown–Peterson cohomology, part II, Am. J. Math. **97** (1975), 101–123.

[28] N. Yagita, Brown–Peterson cohomology groups of exceptional Lie groups, J. Pure Appl. Algebra **17** (1980), no. 2, 223–226.

[29] N. Yagita, On mod odd prime Brown–Peterson cohomology groups of exceptional Lie groups., J. Math. Soc. Japan **34** (1982), no. 2, 293–305.

[30] N. Yagita, On relations between Brown–Peterson cohomology and the ordinary mod p cohomology theory, Kodai Math. J. **7** (1984), no. 2, 273–285.

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