Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3403 Algebraic Topology
Fall 2018

Solutions to exercise set 1

$\boxed{1}$ Let $f\colon X \to Y$ be a continuous map between topological spaces X and Y.

a) Let $K \subseteq X$ be compact. Show that $f(K) \subseteq Y$ is compact.

Solution: Let $\{V_i\}_{i\in I}$ be a collection of open sets in Y which cover $f(K)$. Hence the collection $\{f{-}1(V_i)\}_{i\in I}$ of open subsets in X covers K. Since K is compact, finitely many of the $f^{-1}(V_i)$ suffice to cover K, say

$$K \subset f^{-1}(V_{i_1}) \cup \cdots \cup f^{-1}(V_{i_n}).$$

But that implies

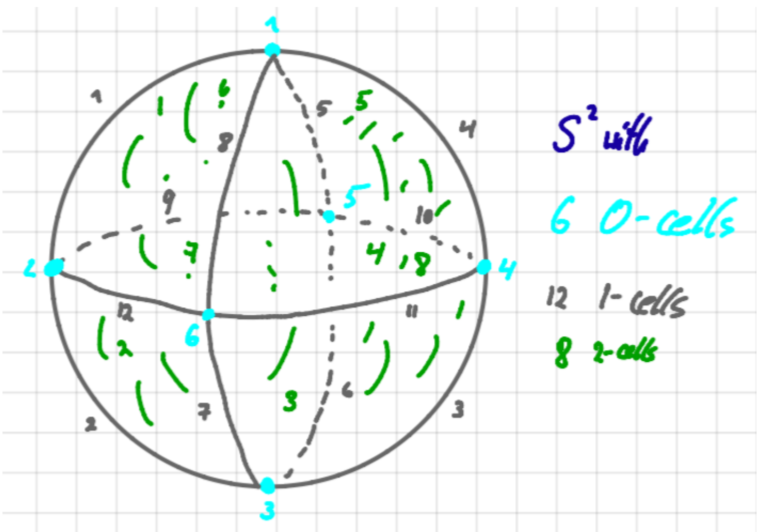
$$f(K) \subset f(f^{-1}(V_{i_1}) \cup \cdots \cup f^{-1}(V_{i_n})) = V_{i_1} \cup \cdots V_{i_n}.$$

Hence $f(K)$ is compact.

b) Give an example of a map f and a compact subset $K \subseteq Y$ such that $f^{-1}(K) \subseteq X$ is not compact.

Solution: There are of course many different types of examples. A simple one is $f\colon (0,1) \hookrightarrow [0,1]$ with the subspace topology induced from $\mathbb{R}$ and $K = [0,1]$. Then $f^{-1}([0,1]) = (0,1)$ is not compact.

$\boxed{2}$ Draw a picture of S^2 as a cell complex with six 0-cells, twelve 1-cells and eight 2-cells.



$\boxed{3}$ Show that the stereographic projection

$$\phi\colon S^1 \to \mathbb{R} \cup \{\infty\},\ (x,y) \mapsto \begin{cases} \frac{x}{1-y} & \text{if } y \neq 1 \\ \infty & \text{if } y = 1 \end{cases}$$

defines a homeomorphism from S^1 to the one-point compactification $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$.

Solution: First of all, ϕ is a bijection, since the map

$$\mathbb{R} \cup \{\infty\} \to S^1,\ \begin{cases} x \mapsto \frac{1}{1+x^2}(2x, |x|^2 - 1) & x \neq \infty \\ \infty \mapsto (0,1) & x = \infty. \end{cases}$$

is an inverse. It remains to show that ϕ is continuous. Since S^1 is compact and $\mathbb{R} \cup \{\infty\}$ is Hausdorff (check that you understand why it is Hausdorff), this will prove that ϕ is a homeomorphism.

By definition, there are two types of open sets in $\hat{\mathbb{R}}$. First, let U be an open subset of $\mathbb{R}$. Since the map $\mathbb{R}^2 \setminus \{(0,1)\} \to \mathbb{R},\ (x,y) \mapsto \frac{x}{1-y}$ is continuous, its restriction to S^1 is also continuous (by definition of the subspace topology). Thus $\phi^{-1}(U)$ is open in S^1.

Second, let $U_K = \mathbb{R} \setminus K \cup \{\infty\}$ for a compact subset $K \subset \mathbb{R}$. Since K is compact, it is closed and bounded. By the above argument, we know $\phi^{-1}(K)$ is closed in S^1. Hence

$$\phi^{-1}(\mathbb{R} \setminus K \cup \{\infty\}) = \phi^{-1}(\mathbb{R} \cup \{\infty\}) \setminus \phi^{-1}(K) = S^1 \setminus \phi^{-1}(K)$$

is open in S^1.

$\boxed{4}$ **a)** Let X and Y be topological spaces. Show that homotopy defines an equivalence relation on the set $C(X,Y)$ of continuous maps $X \to Y$.

Solution: We need to check that the homotopy relation $\simeq$ is reflexive, symmetric, and transitive:

Reflexivity is clear as every map is homotopic to itself via the homotopy $h(x,t) = f(x)$ for all t and x.

For symmetry, suppose $f \simeq g$ and let h be a homotopy. Then the map defined by $(x,t) \mapsto h(x, 1-t)$ is a homotopy from g to f. Hence $g \simeq f$ as well.

For transitivity, assume $f \simeq g$ and $g \simeq k$. Let h_1 and h_2 be corresponding homotopies. Then we can define a homotopy h between f and k by

$$h(x,t) = \begin{cases} h_1(x, 2t) & \text{for } 0 \leq t \leq 1/2 \\ h_2(x, 2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

(Convince yourself that h is well-defined.) Hence we have $f \simeq k$.

b) Show that *being homotopy equivalent* defines an equivalence relation on topological spaces.

Solution: Reflexivity is immediate, since the identity map is a homotopy equivalence.

Symmetry is also clear, since if X is homotopy equivalent to Y with a homotopy equivalence $f\colon X \to Y$, there is by definition a map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The map g shows that Y is also homotopy equivalent to X.

For transitivity, assume $X \simeq Y$ and $Y \simeq Z$ with homotopy equivalences $f_1\colon X \to Y$ and $f_2\colon Y \to Z$. Let $g_1\colon Y \to X$ and $g_2\colon Z \to Y$ be homotopy inverses. Let h_1 be a homotopy between $g_1 \circ f_1$ and id_X, h_2 be a homotopy between $f_1 \circ g_1$ and id_Y, h_2 be a homotopy between id_Y and $g_2 \circ f_2$, and h_4 be a homotopy between $f_2 \circ g_2$ and id_Z. Define $f := f_2 \circ f_1\colon X \to Z$ and $g := g_1 \circ g_2\colon Z \to X$. Then we can construct a homotopy H_1 between $f \circ g$ and id_Z by

$$H_1\colon Z \times [0,1] \to Z,\ (z,t) \mapsto \begin{cases} (f_2 \circ h_2 \circ g_1)(z, 2t) & \text{for } 0 \le t \le 1/2 \\ h_4(z, 2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Similarly, we can construct a homotopy H_2 between $g \circ f$ and id_X by

$$H_2\colon X \times [0,1] \to X,\ (x,t) \mapsto \begin{cases} (g_1 \circ h_3 \circ f_1)(x, 2t) & \text{for } 0 \le t \le 1/2 \\ h_1(x, 2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

$\boxed{5}$ **a)** Show that S^1 is a strong deformation retract of $D^2 \setminus \{0\}$.

Solution: A strong deformation retraction is given by

$$\rho\colon D^2 \setminus \{0\} \to S^1,\ p \mapsto \frac{xp}{|p|}.$$

Note that ρ is well-defined, since $p \neq 0$ and leaves points on S^1 fixed. The map

$$h\colon D^2 \setminus \{0\} \to D^2 \setminus \{0\},\ tp + (1-t)\frac{p}{|p|}$$

is a homotopy between id_X and $D^2 \setminus \{0\} \xrightarrow{\rho} S^1 \subset D^2 \setminus \{0\}$ which leaves S^1 fixed at all times.

b) Show that $D^2 \setminus \{0\}$ is not contractible.

Solution: We will learn more about this later, and there are much better conceptual arguments which explain this fact. Here is an elementary argument: We need to show that the constant map $c\colon D^2 \setminus \{0\} \to D^2 \setminus \{0\},\ p \mapsto (1,0)$ and the identity map $\mathrm{id}\colon D^2 \setminus \{0\} \to D^2 \setminus \{0\},\ p \mapsto p$ are not homotopic. (Note that the choice of the point $(1,0)$ is just for convenience. The same argument works for any other point in $D^2 \setminus \{0\}$.)

Assume there was a homotopy $h\colon D^2 \setminus \{0\} \times [0,1] \to D^2 \setminus \{0\}$ from c to id. For every fixed point $p \in S^1 \subset D^2 \setminus \{0\}$, $h(p,t)$ defines a path from p to $(1,0)$ in $D^2 \setminus \{0\}$. Let Z be the subspace of S^1 of points with negative x-coordinate:

$$Z := \{p = (x,y) \in S^1 : x \le 0\}.$$

Then by the **Intermediate Value Theorem**, for every $p \in Z$, there is t such that the x-coordinate of $h(p,t)$ is 0. Since $[0,1]$ is **compact**, there is in fact a **minimal** such t for each $p \in Z$. We denote this minimum by $t_0(p)$ and write

$$h(p, t_0(p)) = (0, y_0(p)).$$

As $(0,0)$ is not a point of $D^2 \setminus \{0\}$, for each p, we have either $y_0(p) > 0$ or $y_0(p) < 0$.
Since h is continuous in both variables, $y_0(p)$ depends continuously on p as well. Thus, if $y_0(p) > 0$ for some p, then there is an open neighborhood $U \subset S^1$ around p such that $y_0(q) > 0$ for all $q \in U$. In other words, the subset

$$U_{>0} := \{p = (x,y) \in Z : y_0(p) > 0\} \text{ is open in } Z.$$

Similarly, the subset

$$U_{<0} := \{p = (x,y) \in Z : y_0(p) < 0\} \text{ is open in } Z.$$

Both spaces are nonempty, since $(0,1) \in U_{>0}$ and $(0,-1) \in U_{<0}$. Moreover, they are disjoint and mutual complements of each other in Z, i.e.

$$U_{>0} = Z \setminus U_{<0} \text{ and } U_{<0} = Z \setminus U_{>0}.$$

Thus, Z is the disjoint union of the two nonempty and both open and closed subsets $U_{>0}$ and $U_{<0}$. Since Z is connected (being the continuous image of a closed interval), this would imply either $Z = U_{>0}$ or $Z = U_{<0}$. But this is impossible. Thus the homotopy h cannot exist.