MA3403 Algebraic Topology
Fall 2018

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Exercise set 2

$\boxed{1}$ Let $f \in C((X, A), (Y, B))$ be a map of pairs.

a) Show that, for every $n \geq 0$, f induces a homomorphism $H_n(X, A) \to H_n(Y, B)$.

b) Show that the connecting homomorphisms fit into a commuative diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "H_n(f)"] \arrow[d, "\partial"'] & H_n(Y,B) \arrow[d, "\partial"] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f_{|A})"'] & H_{n-1}(B).
\end{tikzcd}$$

$\boxed{2}$ Let X be a nonempty topological space. Recall that if ω is a path on X, i.e., a continuous map $\omega\colon [0, 1] \to X$, then we define an associated 1-simplex σ_ω by

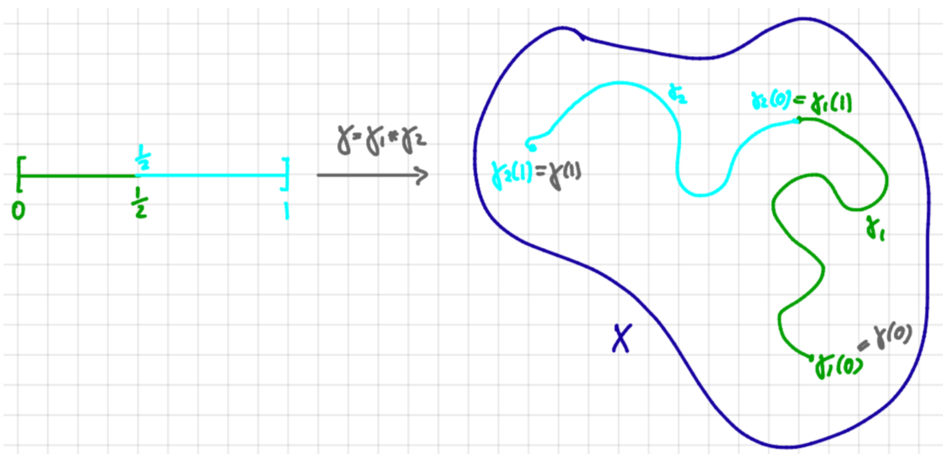
$$\sigma_\omega(t_0, t_1) := \omega(1 - t_0) = \omega(t_1) \text{ for } t_0 + t_1 = 1, 0 \leq t_0, t_1 \leq 1.$$

a) Show that if ω is a constant path, then σ_ω is a boundary.

b) Let γ_1 and γ_2 be paths in X, and let $\gamma := \gamma_1 * \gamma_2$ be the path given by first walking along γ_1 and then walking along γ_2, i.e., the map

$$\gamma = \gamma_1 * \gamma_2\colon [0, 1] \to X, t \mapsto \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that the 1-chain $\sigma_\gamma - \sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary.



For the next two exercises, recall that, given a topological space X and a subspace $A \subset X$, A is called a *retract* of X if there is a retraction $\rho\colon X \to A$, i.e., a continuous map $\rho\colon X \to A$ with $\rho_{|A} = \mathrm{id}_A$. Moreover, we can consider ρ also as a map $X \to X$ via the inclusion $X \xrightarrow{\rho} A \subset X$. If ρ is then in addition homotopic to the identity of X, then A is called a *deformation retract* of X.

$\boxed{3}$ For every $n \geq 2$, show that S^{n-1} is not a deformation retract of the unit disk D^n.

$\boxed{4}$ Show that if A is a retract of X then the map $H_n(i)\colon H_n(A) \to H_n(X)$ induced by the inclusion $i\colon A \subset X$ is injective.

$\boxed{5}$ In this bonus exercise we show that the additivity axiom is needed only for *infinite* disjoint unions:

For two topological spaces X and Y, let $i_X : X \hookrightarrow X \sqcup Y$ and $i_Y : Y \hookrightarrow X \sqcup Y$ be the inclusions into the disjoint union of X and Y. Without referring to the additivity axiom show that the remaining Eilenberg-Steenrod axioms imply that the induced map

$$H_n(i_X) \oplus H_n(i_Y)\colon H_n(X) \oplus H_n(Y) \to H_n(X \sqcup Y)$$

is an isomorphism for every n. (Hint: You may want to apply the long exact sequence and excision with $U = X \subset X \sqcup Y$.)