Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3403 Algebraic Topology Fall 2018

Solutions to exercise set 2

1 Let $f \in C((X,A),(Y,B))$ be a map of pairs.

a) Show that, for every $n \geq 0$, f induces a homomorphism $H_n(X,A) \to H_n(Y,B)$.
Solution: We would like to define $H_n(f)$ again by sending an n-simplex σ on X to the composite $f \circ \sigma$. To make sure that this map descends to quotients, we need to check that $S_n(A)$ is sent to $S_n(B)$. But this follows immediately from the requirement that $f(A) \subseteq B$:

$$\Delta^n \xrightarrow{\sigma} A \xrightarrow{f} B.$$

Then we extend this definition $\mathbb{Z}$-linearly to become a homomorphism.

b) Show that the connecting homomorphisms fit into a commuative diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{H_n(f)} & H_n(Y,B) \\ \partial \downarrow & & \downarrow \partial \\ H_{n-1}(A) & \xrightarrow[H_{n-1}(f_{|A})]{} & H_{n-1}(B). \end{array}$$

Solution: We just calculate:

$$\begin{aligned} H_{n-1}(f_{|A})(\partial([\sum_j a_j\sigma_j])) &= H_{n-1}(f_{|A})([\partial \sum_j a_j\sigma_j]) \\ &= H_{n-1}(f_{|A})([\sum_j a_j \sum_{i=0}^{n}(-1)^i \sigma_j \circ \phi_i^n]) \\ &= [\sum_j a_j \sum_{i=0}^{n}(-1)^i f_{|A} \circ (\sigma_j \circ \phi_i^n)] \\ &= [\sum_j a_j \sum_{i=0}^{n}(-1)^i (f_{|A} \circ \sigma_j) \circ \phi_i^n] \\ &= [\partial \sum_j a_j (f \circ \sigma_j)] = \partial[\sum_j a_j(f\circ\sigma_j)] \\ &= H_n(f)([\sum_j a_j\sigma_j]). \end{aligned}$$

2 Let X be a nonempty topological space. Recall that if ω is a path on X, i.e., a continuous map $\omega\colon [0,1] \to X$, then we define an associated 1-simplex σ_ω by

$$\sigma_\omega(t_0,t_1) := \omega(1-t_0) = \omega(t_1) \text{ for } t_0+t_1 = 1, 0 \leq t_0, t_1 \leq 1.$$

a) Show that if ω is a constant path, then σ_ω is a boundary.

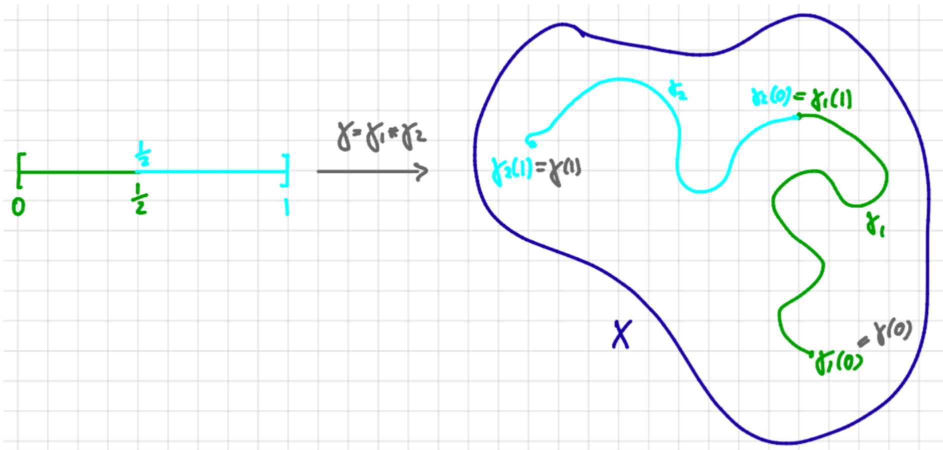
Solution: Let x be the constant value of ω. Let α be the constant 2-simplex with value x. Then

$$\partial(\alpha) = \sigma_\omega - \sigma_\omega + \sigma_\omega = \sigma_\omega.$$

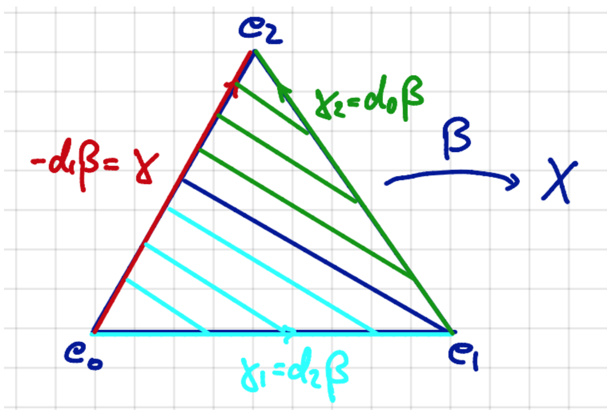
b) Let γ_1 and γ_2 be paths in X, and let $\gamma ;= \gamma_1 * \gamma_2$ be the path given by first walking along γ_1 and then walking along γ_2, i.e., the map

$$\gamma = \gamma_1 * \gamma_2 \colon [0,1] \to X, t \mapsto \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that the 1-chain $\sigma_\gamma - \sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary.



Solution: We define a 2-simplex $\beta \colon \Delta^2 \to X$ to be equal γ_1 on the edge from e_0 to e_1, to be equal γ_2 on the edge from e_1 to e_2, and to be constant on the lines perpendicular to the edge from e_0 to e_2. That implies that β equals γ on the edge from e_0 to e_2:



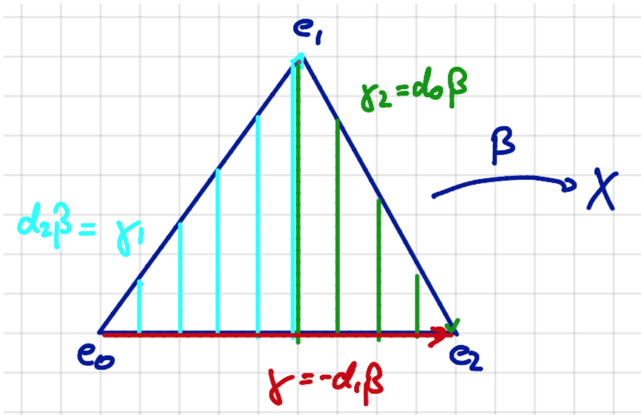
The boundary of β is then given by

$$\partial(\beta) = d_0(\beta) - d_1(\beta) + d_2(\beta) = \sigma_{\gamma_2} - \sigma_\gamma + \sigma_{\gamma_1}.$$

It is a bit tedious to write down the correct formula for β with barycentric coordinates. (At least I was to lazy to do it.) But if you insist on formulae, let us simplify the task and assume that we place Δ^2 into $\mathbb{R}^2$ with vertices

$e_0 = (0,0)$, $e_1 = (\frac{1}{2}, 1)$ and $e_2 = (0,1)$. With this special placement of Δ^2 in the plane, we can write β as

$$\beta(t_0, t_1) = \begin{cases} \gamma_1(2t_0) & \text{for } 0 \leq t_0 \leq \frac{1}{2} \\ \gamma_2(2t_0 - 1) & \text{for } \frac{1}{2} \leq t_0 \leq 1. \end{cases}$$



3 For every $n \geq 2$, show that S^{n-1} is not a deformation retract of the unit disk D^n.

Solution: Let $i\colon S^{n-1} \hookrightarrow D^n$ be the inclusion map. If S^{n-1} was a deformation retract of D^n, then there would be a map $\rho\colon D^n \to S^{n-1}$ with $\rho \circ i = \mathrm{id}_{S^{n-1}}$. By functoriality, this would imply that we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z} \cong H_{n-1}(S^{n-1}) & & \xrightarrow{\mathrm{id}_{H_{n-1}(S^{n-1})}} & & H_{n-1}(S^{n-1}) \cong \mathbb{Z} \\ & \searrow^{H_n(i)} & & \nearrow_{H_n(\rho)} & \\ & & H_{n-1}(D^n) = 0. & & \end{array}$$

But the middle group $H_{n-1}(D^n)$ is the zero group and the identity homomorphism of $\mathbb{Z}$ cannot factor through a trivial group. This contradicts the existence of ρ.

4 Show that if A is a retract of X then the map $H_n(i)\colon H_n(A) \to H_n(X)$ induced by the inclusion $i\colon A \subset X$ is injective.

Solution: If A is a retract of X, there is continuous map $\rho\colon X \to A$ such that $\rho \circ i = \mathrm{id}_A$. Taking homology and functoriality yield

$$H_n(\rho) \circ H_n(i) = H_n(\rho \circ i) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)}.$$

Hence $H_n(i)$ must be injective and $H_n(\rho)$ must be surjective.

5 In this bonus exercise we show that the additivity axiom is needed only for *infinite* disjoint unions:

For two topological spaces X and Y, let $i_X\colon X \hookrightarrow X \sqcup Y$ and $i_Y\colon Y \hookrightarrow X \sqcup Y$ be the inclusions into the disjoint union of X and Y. Without referring to the additivity axiom show that the remaining Eilenberg-Steenrod axioms imply that the induced map

$$H_n(i_X) \oplus H_n(i_Y)\colon H_n(X) \oplus H_n(Y) \to H_n(X \sqcup Y)$$

is an isomorphism for every n. (Hint: You may want to apply the long exact sequence and excision with $U = X \subset X \sqcup Y$.)

Solution: The axioms tell us that the pair $(X \sqcup Y, X)$ is equipped with a long exact sequence

$$\cdots \xrightarrow{\partial} H_n(X) \xrightarrow{H_n(i_X)} H_n(X \sqcup Y) \xrightarrow{H_n(j)} H_n(X \sqcup Y, X) \xrightarrow{\partial} H_{n-1}(X) \xrightarrow{\partial} \cdots$$

The inclusion map $k_Y\colon (Y, \emptyset) \hookrightarrow (X \sqcup Y, X)$ is an excision map (with $U = X$). By the excision axiom, k_Y induces an isomorphism

$$H_n(k_Y)\colon H_n(X) \xrightarrow{\cong} H_n(X \sqcup Y, X).$$

But we also know $k_Y = j \circ i_Y$:

$$\begin{array}{ccc} (Y,\emptyset) & \xrightarrow{\quad k_Y \quad} & (X \sqcup Y, X) \\ & {}_{i_Y}\searrow \quad \nearrow_{j} & \\ & (X \sqcup Y, \emptyset). & \end{array}$$

By functoriality, this implies $H_n(k_Y) = H_n(j) \circ H_n(i_Y)$. Since $H_n(k)$ is an isomorphism, it has an inverse which fits into the sequence as

$$\begin{array}{ccccccccc} \cdots \xrightarrow{\partial} H_n(X) & \xrightarrow{H_n(i_X)} & H_n(X \sqcup Y) & \xrightarrow{H_n(j)} & H_n(X \sqcup Y, X) & \xrightarrow{\partial} & H_{n-1}(X) & \xrightarrow{H_{n-1}(i_X)} & \cdots \\ & & & {}^{H_n(i_Y)}\nwarrow & \downarrow {\scriptstyle H_n(k_Y)^{-1}} & & & & \\ & & & & H_n(Y) & & & & \end{array}$$

And we know

$$H_n(j) \circ (H_n(i_Y) \circ H_n(k_Y)^{-1}) = \mathrm{id}_{H_n(X \sqcup Y, X)}$$

Thus $H_n(j)$ is surjective and $\partial\colon H_n(X \sqcup Y, X) \to H_{n-1}(X)$ is the zero map for every n. By exactness of the sequence, this implies that $H_n(i)$ is injective for every n.

Moreover, since the sequence is exact, it simplifies into a *split* short exact sequence

$$0 \to H_n(X) \xrightarrow{H_n(i_X)} H_n(X \sqcup Y) \xrightarrow{H_n(j)} H_n(X \sqcup Y, X) \to 0$$

with

$$\begin{aligned} H_n(X \sqcup Y) &\cong H_n(i_X)(H_n(X)) \oplus (H_n(i_Y) \circ H_n(k_Y)^{-1})(H_n(X \sqcup Y, X)) \\ &\cong H_n(i_X)(H_n(X)) \oplus H_n(i_Y)(H_n(Y)). \end{aligned}$$

(Note that we used $H_n(k_Y)^{-1}(H_n(X \sqcup Y, X)) = H_n(i_Y)(H_n(Y))$ for the last step.) Hence $H_n(i_X) \oplus H_n(i_Y)$ is an isomorphism.