Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3403 Algebraic Topology Fall 2018

Exercise set 3

Recall the definition from Lecture 7: For $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map. The *degree of* f, denoted by $\deg(f)$, is the integer determined by $H_n(f)([\sigma]) = \deg(f) \cdot [\sigma]$ for a generator $[\sigma] \in H_n(S^n) \cong \mathbb{Z}$.
Since we know $H_{n+1}(D^{n+1}, S^n) \cong \mathbb{Z}$, we can apply the same definition also to selfmaps of the pair (D^{n+1}, S^n): Let $f\colon (D^{n+1}, S^n) \to (D^{n+1}, S^n)$ be a continuous map of pairs. The *degree of* f, again denoted by $\deg(f)$, is the integer determined by $H_{n+1}(f)([\sigma]) = \deg(f) \cdot [\sigma]$ for a generator $[\sigma] \in H_{n+1}(D^{n+1}, S^n) \cong \mathbb{Z}$.

$\boxed{1}$ Show that the degree has the following properties:

a) The identity has degree 1, i.e., $\deg(\mathrm{id}) = 1$.

b) The degree of a constant map is 0.

c) If $f, g\colon S^n \to S^n$ are two continuous maps, then $\deg(f \circ g) = \deg(f)\deg(g)$.

d) If f_0 and f_1 are homotopic, then $\deg(f_0) = \deg(f_1)$.

e) If $f\colon S^n \to S^n$ is a homotopy equivalence, then $\deg(f) = \pm 1$.

f) For $f\colon (D^{n+1}, S^n) \to (D^{n+1}, S^n)$, let $f_{|S^n}$ denote the restriction of f to S^n. Then $\deg(f) = \deg(f_{|S^n})$.

$\boxed{2}$ Let $a\colon S^n \to S^n$ be the antipodal map, i.e.,

$$a\colon (x_0, x_1, \dots, x_n) \mapsto (-x_0, -x_1, \dots, -x_n).$$

a) Show $\deg(a) = (-1)^{n+1}$.
(Hint: Use the result from Lecture 7 on the degree of a reflection.)

b) For n even, show that the antipodal map is not homotopic to the identity on S^n.
(Hint: Use what you have just learned in the previous exercises.)

$\boxed{3}$ **a)** If $f\colon S^n \to S^n$ is a continuous map *without fixed points*, i.e., $f(x) \neq x$ for all $x \in S^n$, then $\deg(f) = (-1)^{n+1}$.
(Hint: Show that f is homotopic to the antipodal map.)

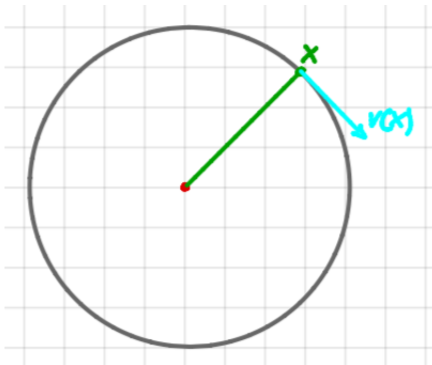
b) If $f\colon S^n \to S^n$ is a continuous map *without an antipodal point*, i.e., $f(x) \neq -x$ for all $x \in S^n$, then $\deg(f) = 1$.

(Hint: Show that f is homotopic to the identity map.)

c) If n is even and $f\colon S^n \to S^n$ is any continuous map, show that there is a point $x \in S^n$ with $f(x) = \pm x$.

(Hint: Apply the previous observations.)

A *vector field* on S^n is a continuous map $v\colon S^n \to \mathbb{R}^{n+1}$ with $x \perp v(x)$ for all $x \in S^n$ (x and $v(x)$ are orthogonal to each other).



$\boxed{4}$ Prove the following theorem: The n-dimensional sphere S^n admits a vector field v *without zeros*, i.e., $v(x) \neq 0$ for all $x \in S^n$, if and only if n is odd.

In particular, every vector field on S^2 must have a zero. This is often rephrased as: you cannot comb a hairy ball without leaving a bald spot.

(Hint: If n even, show that the assumption $v(x) \neq 0$ would allow to define a homotopy between the identity and the antipodal map. When you write down the homotopy maje sure the image lies on S^n.)