Norwegian University of Science and Technology
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MA3403 Algebraic Topology Fall 2018

Solutions to exercise set 3

Recall the definition from Lecture 7: For $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map. The *degree of* f, denoted by $\deg(f)$, is the integer determined by $H_n(f)([\sigma]) = \deg(f) \cdot [\sigma]$ for a generator $[\sigma] \in H_n(S^n) \cong \mathbb{Z}$.
Since we know $H_{n+1}(D^{n+1}, S^n) \cong \mathbb{Z}$, we can apply the same definition also to selfmaps of the pair (D^{n+1}, S^n): Let $f\colon (D^{n+1}, S^n) \to (D^{n+1}, S^n)$ be a continuous map of pairs. The *degree of* f, again denoted by $\deg(f)$, is the integer determined by $H_{n+1}(f)([\sigma]) = \deg(f) \cdot [\sigma]$ for a generator $[\sigma] \in H_{n+1}(D^{n+1}, S^n) \cong \mathbb{Z}$.

1 Show that the degree has the following properties:

a) The identity has degree 1, i.e., $\deg(\mathrm{id}) = 1$.
Solution: This follows from functoriality as $H_n(\mathrm{id}) = \mathrm{id}_{H_n}$.

b) The degree of a constant map is 0.
Solution: A constant map sends everything through one point. Hence the induced map on homology factors through $H_n(\mathrm{pt}) = 0$ (for $n \geq 1$).

c) If $f, g\colon S^n \to S^n$ are two continuous maps, then $\deg(f \circ g) = \deg(f)\deg(g)$.
Solution: This follows again from functoriality as $H_n(f \circ g) = H_n(f) \circ H_n(g)$ and hence

$$H_n(f \circ g)([\sigma]) = H_n(f)(\deg(g) \cdot [\sigma]) = \deg(f) \cdot \deg(g) \cdot [\sigma].$$

d) If f_0 and f_1 are homotopic, then $\deg(f_0) = \deg(f_1)$.
Solution: This follows from the Homotopy Axiom for homology which says $H_n(f_0) = H_n(f_1)$.

e) If $f\colon S^n \to S^n$ is a homotopy equivalence, then $\deg(f) = \pm 1$.
Solution: If f is a homotopy equivalence, there is a map g such that $f \circ g \simeq \mathrm{id}$. By the two previous points, this implies $\deg(f)\deg(g) = 1$. Since both $\deg(f)$ and $\deg(g)$ are integers, this can only happen if both equal 1 or if both equal -1.

f) For $f\colon (D^{n+1}, S^n) \to (D^{n+1}, S^n)$, let $f_{|S^n}$ denote the restriction of f to S^n. Then $\deg(f) = \deg(f_{|S^n})$.
Solution: This follows from the fact that we have a commutative diagram

where the vertical maps are isomorphisms

$$\begin{array}{ccc} H_{n+1}(D^{n+1}, S^n) & \xrightarrow{H_n(f)} & H_{n+1}(D^{n+1}, S^n) \\ \partial \downarrow \cong & & \cong \downarrow \partial \\ H_n(S^n) & \xrightarrow[H_n(f_{|S^n})]{} & H_n(S^n). \end{array}$$

2 Let $a\colon S^n \to S^n$ be the antipodal map, i.e.,

$$a\colon (x_0, x_1, \ldots, x_n) \mapsto (-x_0, -x_1, \ldots, -x_n).$$

a) Show $\deg(a) = (-1)^{n+1}$.
Solution: Let $r_i\colon S^n \to S^n$ be the reflection map in the ith coordinate

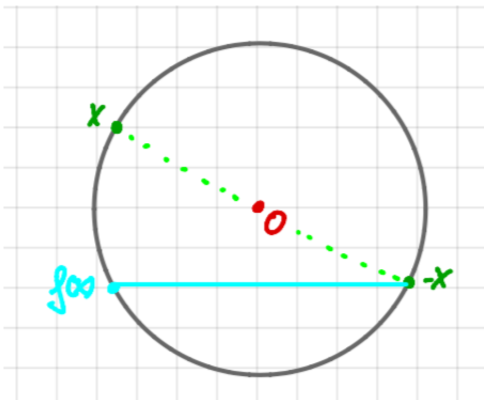
$$r_i\colon (x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

By the symmetry of S^n, the theorem of the lecture also applies to r_i and shows $\deg(r_i) = -1$ for all i. Now the antipodal map a is the composition of the reflections in all coordinates: $a = r_0 \circ r_1 \circ \cdots \circ r_n$. By the previous exercise, this implies

$$\deg(a) = \deg(r_0) \cdots \deg(r_n) = (-1)^{n+1}.$$

b) For n even, show that the antipodal map is not homotopic to the identity on S^n.
Solution: If the antipodal map was homotopic to the identity, then, by the previous exercise, it had the same degree as the identity which is 1. But if n is even, then $\deg(a) = -1$ as we just showed.

3 **a)** If $f\colon S^n \to S^n$ is a continuous map *without fixed points*, i.e., $f(x) \neq x$ for all $x \in S^n$, then $\deg(f) = (-1)^{n+1}$.
Solution: Since $f(x) \neq x$ for all x, the line segment in D^{n+1} between $f(x)$ and $-x$ **does not pass through the origin** in D^{n+1}.



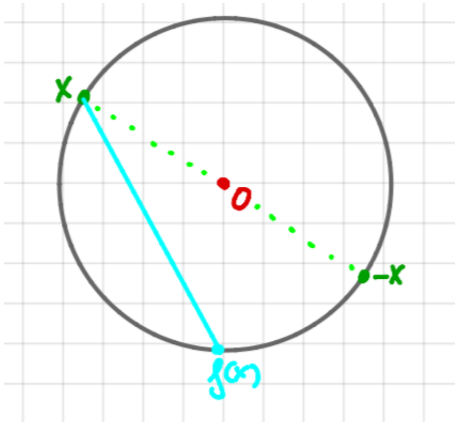
Hence we can define the continuous map

$$h(x,t)\colon S^n \times [0,1] \to S^n, (x,t) \mapsto \frac{f(x)(1-t)-tx}{|f(x)(1-t)-tx|}.$$

Since $h(x,0) = f(x)$ and $h(x,1) = -x = a(x)$, h defines a homotopy between f and the antipodal map a. By a previous exercise, this implies $\deg(f) = \deg(a) = (-1)^{n+1}$.

b) If $f\colon S^n \to S^n$ is a continuous map *without an antipodal point*, i.e., $f(x) \neq -x$ for all $x \in S^n$, then $\deg(f) = 1$.

Solution: Since $f(x) \neq -x$ for all x, the line segment in D^{n+1} between x and $f(x)$ **does not pass through the origin** in D^{n+1}.



Hence we can define the continuous map

$$h(x,t)\colon S^n \times [0,1] \to S^n, (x,t) \mapsto \frac{f(x)(1-t)+tx}{|f(x)(1-t)+tx|}.$$

Since $h(x,0) = f(x)$ and $h(x,1) = x$, h defines a homotopy between f and the identity map. By a previous exercise, this implies $\deg(f) = \deg(\mathrm{id}) = 1$.

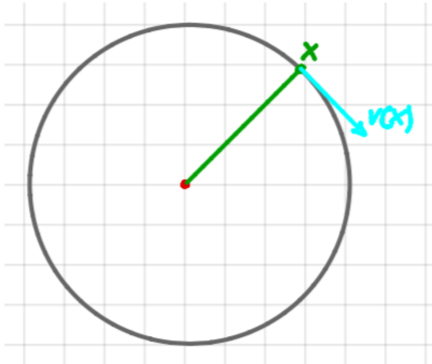
c) If n is even and $f\colon S^n \to S^n$ is any continuous map, show that there is a point $x \in S^n$ with $f(x) = \pm x$.

Solution: It f had neither a fixed point nor an antipodal point, then the previous points would imply

$$1 = \deg(f) = (-1)^{n+1}.$$

Hence n would have to be odd. This contradicts the assumption that n is even.

A *vector field* on S^n is a continuous map $v\colon S^n \to \mathbb{R}^{n+1}$ with $x \perp v(x)$ for all $x \in S^n$ (x and $v(x)$ are orthogonal to each other).



4 Prove the following theorem: The n-dimensional sphere S^n admits a vector field v *without zeros*, i.e., $v(x) \neq 0$ for all $x \in S^n$, if and only if n is odd.

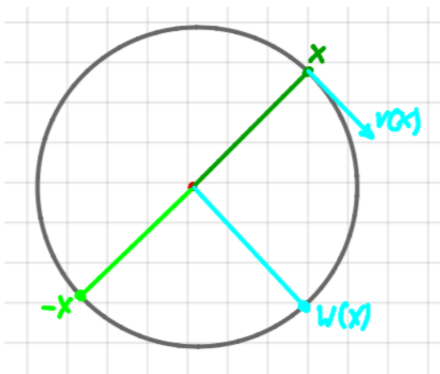
Solution: If n is odd, say $n = 2k-1$, then $S^{2k-1} \subset C^k = \mathbb{R}^{2k}$ and the map $z \mapsto iz$ induces a nonzero vector field. More explicitely, we can define a vector field by

$$v\colon S^n \to \mathbb{R}^{n+1}, (x_0, x_1, \ldots, x_n) \mapsto (-x_1, x_0, -x_2, x_3, \ldots, -x_n, x_{n-1}).$$

Since each x is on S^n and hence not the origin, it follows $v(x) \neq 0$. Moreover, the scalar product $x \cdot v(x)$ on $\mathbb{R}^{n+1}$ is 0 which implies $x \perp v(x)$.

If n even, assume there was a vector field v without zeros. Then we can define a new continuous map

$$w\colon S^n \to S^n,\ x \mapsto \frac{v(x)}{|v(x)|}.$$



Then we could define a continuous map

$$h\colon S^n \times [0,1] \to S^n,\ (x,t) \mapsto \cos(\pi t)x + \sin(\pi t)w(x).$$

Note that $h(x,t) \in S^n$, for (where $\cdot$ denotes the scalar product on $\mathbb{R}^{n+1}$)

$$\begin{aligned} h(x,t) \cdot h(x,t) &= (\cos(\pi t)x + \sin(\pi t)w(x))^2 \\ &= \cos^2(\pi t)(x \cdot x) + \sin^2(\pi t)(w(x) \cdot w(x)) + 2\cos(\pi t)\sin(\pi t)(x \cdot w(x)) \\ &= \cos^2(\pi t) + \sin^2(\pi t) + 0 \text{ since } x \cdot x = 1 = w(x) \cdot w(x),\ x \cdot w(x) = 0 \\ &= 1. \end{aligned}$$

Since $h(x, 0) = x$ and $h(x, 1) = -x$, h would be a homotopy between the identity and the antipodal map on S^n. By a previous exercise, this would imply

$$1 = \deg(\mathrm{id}) = \deg(a).$$

But this would contradict another result we proved: $\deg(a) = (-1)^{n+1} = -1$, since n is even.