Norwegian University of Science and Technology
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MA3403 Algebraic Topology Fall 2018

Exercise set 4

1 Let X be a space, $A \subset X$ be a subspace and $j\colon (X,\emptyset) \hookrightarrow (X,A)$ be the inclusion map. Suppose A is contractible.

a) Show that the natural homomorphism $H_n(j)\colon H_n(X) \to H_n(X,A)$ is an isomorphism for all $n \geq 2$.

b) Show that $H_n(j)$ is an isomorphism for all $n \geq 1$ if A and X are path-connected.

c) For $n \geq 1$, let $p \in S^n$ be a point. Show that $S^n \setminus \{p\}$ is contractible.

d) For two distinct points $p_1, p_2 \in S^n$, is $S^n \setminus \{p_1, p_2\}$ contractible?

2 Let $f\colon S^n \to S^n$ be a continuous map. If f is not surjective, then $\deg(f) = 0$.

3 Our goal in this exercise is to construct a surjective map $f\colon S^1 \to S^1$ with $\deg(f) = 0$.

a) Start with a map

$$g\colon S^1 \to S^1,\ e^{is} \mapsto \begin{cases} e^{-is} & \text{if } s \in [0,\pi) \\ e^{is} & \text{if } s \in [\pi, 2\pi). \end{cases}$$

Show that g has degree 0.

b) Compose g with a another map such that the composition becomes a surjective map $f\colon S^1 \to S^1$ of degree 0.

4 Let $f\colon S^n \to S^n$ be a continuous map with $\deg(f) = 0$. Show that there must exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

5 With this exercise we would like to refresh our memory on real projective spaces and connect it to questions on the existence of fixed points.

Recall from Lecture 2 that the real projective space $\mathbb{RP}^k$ is defined to be the quotient of $\mathbb{R}^{k+1} \setminus \{0\}$ under the equivalence relation $x \sim \lambda x$ for $\lambda \in \mathbb{R} \setminus \{0\}$. The topology on $\mathbb{RP}^k$ is the quotient topology.

a) Show that any invertible $\mathbb{R}$-linear map $F\colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ induces a continuous map $f\colon \mathbb{RP}^k \to \mathbb{RP}^k$.

b) Show that for any invertible $\mathbb{R}$-linear map $F\colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ with an eigenvector, the induced map $f\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^k$ has a fixed point.

c) Show that any continuous map $f\colon \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ that is induced by an invertible $\mathbb{R}$-linear map $F\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ has a fixed point.

d) Show that there are continuous maps $f\colon \mathbb{R}\mathrm{P}^{2n-1} \to \mathbb{R}\mathrm{P}^{2n-1}$ without fixed points.

6 Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients in $\mathbb{C}$. The goal of this exercise is to prove the Fundamental Theorem Algebra, i.e., we would like to show that there is a $z \in \mathbb{C}$ with $p(z) = 0$.

We are going to show this as wollows:

Consider p as a map $\mathbb{C} \to \mathbb{C}$. **Assume that p had no root.** Then we can define a new map

$$\hat{p}\colon S^1 \to S^1,\ z \mapsto \frac{p(z)}{|p(z)|}.$$

We are going to show that this assumption leads to a **contradiction**.

a) Show that $\hat{p}$ is homotopic to a constant map. What is the degree of $\hat{p}$?

b) Show that the map

$$H\colon S^1 \times (0,1] \to S^1,\ (z,t) \mapsto \frac{t^n p(\frac{z}{t})}{|t^n p(\frac{z}{t})|}$$

can be continuously extended to a map $S^1 \times [0,1]$, i.e., analyze $H(z,t)$ for $t \to 0$. What is the degree of $\hat{p}$?

c) Deduce that p must have a root, i.e., there must be a $z \in \mathbb{C}$ with $p(z) = 0$.

7 In this exercise we continue our study of the Fundamental Theorem Algebra. Our goal is to connect the degree and the multiplicity of a root of a polynomial.

a) Let $f\colon S^1 \to S^1$ be a continuous map. Show that if f can be extended to a map on D^2, i.e., if there is a continuous map $F\colon D^2 \to S^1$ such that $F_{|S^1} = f$, then $\deg(f) = 0$.

Now let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients in $\mathbb{C}$.

b) Assume that p has no root z with $|z| \leq 1$. Then we can define the map

$$\hat{p}\colon S^1 \to S^1,\ z \mapsto \frac{p(z)}{|p(z)|}.$$

Show that the degree of $\hat{p}$ is 0.

c) Assume that p has exactly one root z_0 with $|z_0| < 1$ and no root z with $|z| = 1$. Then we can define the map

$$\hat{p}\colon S^1 \to S^1,\ z \mapsto \frac{p(z)}{|p(z)|}.$$

Show that the degree of $\hat{p}$ equals the multiplicity of the root z_0, i.e., $\deg(\hat{p}) = m$ where $m \geq 0$ is the unique number such that $p(z) = (z - z_0)^m q(z)$ with $q(z_0) \neq 0$.

Finally, we switch perspectives a bit. We know that the polynomial p satisfies $\lim_{|z\to\infty|} |p(z)| = \infty$. Hence we can extend the map $p\colon \mathbb{C} \to \mathbb{C}$ to a map $p\colon S^2 \to S^2$ where we think of S^2 as the one-point-compactification of $\mathbb{C} \cong \mathbb{R}^2$. We are going to use the following fact: Let $f_m\colon S^2 \to S^2,\ z \mapsto z^m$. The effect of $H_2(f_m)$ as a selfmap of $H_2(S^2)$ and as a selfmap of $H_2(S^2, S^2 \setminus \{0\})$ is given by multiplication by m.

d) Let z_i be a root of p. Show that the local degree $\deg(p|z_i)$ of p at z_i is equal to the multiplicity of z_i as a root of p.