MA3403 Algebraic Topology
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Solutions to exercise set 4

1 Let X be a space, $A \subset X$ be a subspace and $j\colon (X, \emptyset) \hookrightarrow (X, A)$ be the inclusion map. Suppose A is contractible.

a) Show that the natural homomorphism $H_n(j)\colon H_n(X) \to H_n(X, A)$ is an isomorphism for all $n \geq 2$.

Solution: Let $i\colon A \hookrightarrow X$ denote the inclusion map. Consider the long exact sequence of pairs

$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(X) \to \cdots$$

Since A is contractible, $H_n(A) = H_{n-1}(A) = 0$ for all $n \geq 2$. Since the sequence is exact, this implies that $H_n(j)$ is an isomorphism.

b) Show that $H_n(j)$ is an isomorphism for all $n \geq 1$ if A and X are path-connected.

Solution: Since A and X are path-connected, the homomorphism

$$\mathbb{Z} \cong H_0(A) \xrightarrow{H_n(i)} H_0(X) \cong \mathbb{Z}$$

is an isomorphism. This implies that $\partial\colon H_1(X, A) \to H_0(A)$ is the zero map. Thus $H_1(j)$ is surjective, since the sequence is exact. Since A is contractible, $H_1(A) = 0$ is trivial which implies that $H_1(j)$ is injective. Hence $H_1(j)$ is an isomorphism.

c) For $n \geq 1$, let $p \in S^n$ be a point. Show that $S^n \setminus \{p\}$ is contractible.

Solution: Stereographic projection from p provides a homeomorphism between $S^n \setminus \{p\}$ and $\mathbb{R}^n$.

We could also argue that S^n is a cell complex with one 0-cell and one n-cell. Taking p to be the 0-cell, we see that $S^n \setminus \{p\}$ is homeomorphic to D^n.

d) For two distinct points $p_1, p_2 \in S^n$, is $S^n \setminus \{p_1, p_2\}$ contractible?

Solution: We just learned that $S^n \setminus \{p_1\}$ is homeomorphic to D^n. That implies that $S^n \setminus \{p_1, p_2\}$ is homeomorphic to $D^n \setminus \{0\}$ and hence not contractible. We could also just refer to the lemma of Lecture 8 where we calculated the corresponding homology.

2 Let $f\colon S^n \to S^n$ be a continuous map. If f is not surjective, then $\deg(f) = 0$.

Solution: If f is not surjective, then we can find a point $x \in S^n$ which is not in the image of f. Then f factors as

$$f\colon S^n \to S^n \setminus \{x\} \hookrightarrow S^n.$$

Hence applying the homology functor H_n yields a commutative diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{H_n(f)} & H_n(S^n) \\ & \searrow \quad \nearrow & \\ & H_n(S^n \setminus \{x\}). & \end{array}$$

But $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$ via the stereographic projection (from the point x). In particular, $S^n \setminus \{x\}$ is contractible and $H_n(S^n \setminus \{x\}) = 0$ for $n \geq 1$. Hence $H_n(f)$ must be the zero map and $\deg(f) = 0$.

3 Our goal in this exercise is to construct a surjective map $f\colon S^1 \to S^1$ with $\deg(f) = 0$.

a) Start with a map

$$g\colon S^1 \to S^1,\ e^{is} \mapsto \begin{cases} e^{-is} & \text{if } s \in [0, \pi) \\ e^{is} & \text{if } s \in [\pi, 2\pi). \end{cases}$$

Show that g has degree 0.

Solution: It suffices to find a homotopy between g and a constant map. For example, the map

$$h\colon S^1 \times [0,1] \to S^1,\ (s,t) \mapsto g((1-t)s)$$

is a homotopy between g and the constant map at $g(0) = 1$. By homotopy invariance of the degree, this implies $\deg(g) = 0$, since any constant map is of degree 0.

b) Compose g with a another map such that the composition becomes a surjective map $f\colon S^1 \to S^1$ of degree 0.

Solution: It suffices to compose g with the map $d\colon S^1 \to S^1$ that doubles the speed, i.e., $d(e^{is}) = e^{i2s}$. For, the image of

$$[\pi, 2\pi) \to S^1,\ s \mapsto e^{i2s}$$

is all of S^1. Hence we define $f\colon S^1 \to S^1$ to be the map $d \circ g$.
It remains to check $\deg(f) = 0$. This follows from the fact that the degree is multiplicative, i.e., $\deg(d \circ g) = \deg(d)\deg(g)$. So whatever the degree of d is (it is 2 by the way as d is $z \mapsto z^2$), $\deg(g) = 0$ forces $\deg(f) = 0$.

4 Let $f\colon S^n \to S^n$ be a continuous map with $\deg(f) = 0$. Show that there must exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

Solution: We learned in a previous exercise that if f has no fixed point, then f is homotopic to the antipodal map. But that would imply $\deg(f) = (-1)^{n+1} \neq 0$. And we also learned that if there was no point y with $f(y) = -y$, then f would be homotopic to the identity. That would imply $\deg(f) = 1 \neq 0$. Hence both assumptions lead to a contradiction to $\deg(f) = 0$. This shows there must be points x and y on S^n with $f(x) = x$ and $f(y) = -y$.

5 With this exercise we would like to refresh our memory on real projective spaces and connect it to questions on the existence of fixed points.

Recall from Lecture 2 that the real projective space $\mathbb{R}\mathrm{P}^k$ is defined to be the quotient of $\mathbb{R}^{k+1} \setminus \{0\}$ under the equivalence relation $x \sim \lambda x$ for $\lambda \in \mathbb{R} \setminus \{0\}$. The topology on $\mathbb{R}\mathrm{P}^k$ is the quotient topology.

a) Show that any invertible $\mathbb{R}$-linear map $F\colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ induces a continuous map $f\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^k$.

Solution: Since F is linear, it satisfies $F(\lambda x) = \lambda F(x)$. Hence $x \sim y$ implies $F(x) \sim F(y)$. This allows us to define f by $[x] \mapsto [F(x)]$ where $[x]$ denote the equivalence class of x.

To show that$f$ is continuous, let $\pi\colon \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{R}\mathrm{P}^k$ be the quotient map $\pi(x) = [x]$. Let $V \subset \mathbb{R}\mathrm{P}^k$ be an open subset. By definition of the quotient topolgy, this means that there is an open subset $U \subset \mathbb{R}^{k+1} \setminus \{0\}$ such that $\pi^{-1}(V) = U$. Since F is linear, it is continuous. Hence $F^{-1}(U)$ is open in $\mathbb{R}^{k+1} \setminus \{0\}$. Since the diagram

$$\begin{array}{ccc} \mathbb{R}^{k+1} \setminus \{0\} & \xrightarrow{F} & \mathbb{R}^{k+1} \setminus \{0\} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{R}\mathrm{P}^k & \xrightarrow[f]{} & \mathbb{R}\mathrm{P}^k \end{array}$$

commutes, we have $\pi^{-1}(f^{-1}(V)) = F^{-1}(\pi^{-1}(V)) = F^{-1}(U)$. By definition of the quotient topology, this shows $f^{-1}(V)$ is open and f is continuous.

b) Show that for any invertible $\mathbb{R}$-linear map $F\colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ with an eigenvector, the induced map $f\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^k$ has a fixed point.

Solution: Let $x \in \mathbb{R}^{k+1}$ be an eigenvector of F. That means $x \neq 0$ and $F(x) = \lambda x$. Since F is invertible, we must have $\lambda \neq 0$. Hence $[x]$ is a fixed point of f.

c) Show that any continuous map $f\colon \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ that is induced by an invertible $\mathbb{R}$-linear map $F\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ has a fixed point.

Solution: The characteristic polynomial $\det(F - \lambda \mathrm{Id})$ has degree $2n + 1$. In particular, it is a polynomial of odd degree. The Intermediate Value Theorem in Calculus tells us that this polynomial must have a zero. In other words, F must have an eigenvalue. Since F is invertible, this eigenvalue must be $\neq 0$. By the previous point, this implies that f has a fixed point.

d) Show that there are continuous maps $f\colon \mathbb{R}\mathrm{P}^{2n-1} \to \mathbb{R}\mathrm{P}^{2n-1}$ without fixed points.

Solution: From what we just learned we know that we need to find an invertible linear map $F\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors. The induced map f will then not have a fixed point. For example, we could choose

$$F\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ (x_1, x_2, \dots, x_{2n}) \mapsto (-x_2, x_1, -x_4, x_3, \dots, -x_{2n}, x_{2n-1}).$$

6 Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients in $\mathbb{C}$. The goal of this exercise is to prove the Fundamental Theorem Algebra, i.e., we would like to show that there is a $z \in \mathbb{C}$ with $p(z) = 0$.

We are going to show this as wollows:

Consider p as a map $\mathbb{C} \to \mathbb{C}$. **Assume that p had no root.** Then we can define a new map

$$\hat{p}\colon S^1 \to S^1,\ z \mapsto \frac{p(z)}{|p(z)|}.$$

We are going to show that this assumption leads to a **contradiction**.

a) Show that $\hat{p}$ is homotopic to a constant map. What is the degree of $\hat{p}$?

Solution: It is actually quite easy to get such a homotopy. We define a map

$$h\colon S^1 \times [0,1] \to S^1,\ (z,t) \mapsto \frac{p(tz)}{|p(tz)|} = \hat{p}(tz).$$

Since we assumed p had no root, h is defined and continuous for all t. Hence it defines a homotopy between the constant map $z \mapsto h(z,0) = a_0/|a_0|$ and $z \mapsto h(z,1) = \hat{p}$. Since the degree of a constant map is 0 and the degree is invariant under homotopy, this shows $\deg(\hat{p}) = 0$.

b) Show that the map

$$H\colon S^1 \times (0,1] \to S^1,\ (z,t) \mapsto \frac{t^n p(\frac{z}{t})}{|t^n p(\frac{z}{t})|}$$

can be continuously extended to a map $S^1 \times [0,1]$, i.e., analyze $H(z,t)$ for $t \to 0$. What is the degree of $\hat{p}$?

Solution: The map H is defined and continuous for (z,t) with $t > 0$. It remains to check what happens when $t \to 0$. Therefor we observe

$$\begin{aligned} t^n p(\frac{z}{t}) &= t^n \left(\frac{z}{t}\right)^n + t^n a_{n-1} \left(\frac{z}{t}\right)^{n-1} + \cdots + t^n a_1 \frac{z}{t} + t^n a_0 \\ &= z^n + t(t^{n-2} a_{n-1} z^{n-1} + \cdots + t^{n-2} a_1 z + t^{n-1} a_0). \end{aligned}$$

This shows

$$\lim_{t\to 0} t^n p(\frac{z}{t}) = z^n, \text{ and hence } \lim_{t\to 0} H(z,t) = \frac{z^n}{|z^n|} \text{ for all } z.$$

Hence we can extend H continuously by defining $H(z,0) := z^n$ for all $z \in S^1$ (which satisfy $|z| = 1$).

With this definition at hand, the map H defines a homotopy between the maps

$$z \mapsto H(z,0) = z^n = f_n(z) \text{ and } z \mapsto H(z,1) = \hat{p}.$$

Since the degree is homotopy invariant and $\deg(f_n) = n$, thos shows $\deg(\hat{p}) = n$.

c) Deduce that p must have a root.

Solution: Since $n > 0$, $\deg(\hat{p})$ cannot be equal to 0 and equal to n. Hence the assumption that such a map $\hat{p}$ exists leads to a contradiction. Therefore, our assumption that p had no root must be false.

Since we started with an arbitrary non-constant polynomial, this proves the Fundamental Theorem of Algebra.

7 In this exercise we continue our study of the Fundamental Theorem Algebra. Our goal is to connect the degree and the multiplicity of a root of a polynomial.

a) Let $f\colon S^1 \to S^1$ be a continuous map. Show that if f can be extended to a map on D^2, i.e., if there is a continuous map $F\colon D^2 \to S^1$ such that $F_{|S^1} = f$, then $\deg(f) = 0$.

Solution: If such an F exists, then we f factors as the composition

$$f = F \circ i\colon S^1 \xhookrightarrow{i} D^2 \xrightarrow{F} S^1$$

where i denotes the inclusion. By applying the homology functor H_1, we get $H_1(f)$ factors as

$$H_1(f) = H_1(F) \circ H_1(i)\colon H_1(S^1) \xrightarrow{H_1(i)} H_1(D^2) \xrightarrow{H_1(F)} H_1(S^1).$$

Since $H_1(D^2) = 0$, this shows that $H_1(f)$ must be the zero map. Thus $\deg(f) = 0$.

Now let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients in $\mathbb{C}$.

b) Assume that p has no root z with $|z| \leq 1$. Then we can define the map

$$\hat{p}\colon S^1 \to S^1,\ z \mapsto \frac{p(z)}{|p(z)|}.$$

Show that the degree of $\hat{p}$ is 0.

Solution: Since p has no root in $D^2 \subset \mathbb{C}$, we can extend the map $\hat{p}$ to a continuous map on $D^2 \to S^1$

$$\hat{P}\colon D^2 \to S^1,\ z \mapsto \frac{p(z)}{|p(z)|}.$$

which agrees with $\hat{p}$ on S^1. By the previous point, this implies $\deg(\hat{p}) = 0$.

c) Assume that p has exactly one root z_0 with $|z_0| < 1$ and no root z with $|z| = 1$. Then we can define the map

$$\hat{p}\colon S^1 \to S^1,\ z \mapsto \frac{p(z)}{|p(z)|}.$$

Show that the degree of $\hat{p}$ equals the multiplicity of the root z_0, i.e., $\deg(\hat{p}) = m$ where $m \geq 0$ is the unique number such that $p(z) = (z - z_0)^m q(z)$ with $q(z_0) \neq 0$.

Solution: We can again define the map

$$h\colon S^1 \times [0,1] \to S^1,\ (z,t) \mapsto \frac{(z - z_0)^m q(tz + (1-t)z_0)}{|(z - z_0)^m q(tz + (1-t)z_0)|}.$$

Then h is a homotopy between $h(z,1) = \hat{p}$ and

$$h(z,0) = \frac{(z - z_0)^m q(z_0)}{|(z - z_0)^m q(z_0))|} = \frac{(z - z_0)^m}{|(z - z_0)^m|} \cdot \frac{q(z_0)}{|q(z_0)|}.$$

Note that $\frac{q(z_0)}{|q(z_0)|}$ is a well defined number, since $q(z_0) \neq 0$. Note that multiplying by a constant number does not change the degree.

Let f_{m,z_0} denote the map

$$f_{m,z_0}\colon S^1 \to S^1,\ z \mapsto \frac{(z-z_0)^m}{|(z-z_0)^m|}.$$

By homotopy invariance of degrees, we then have

$$\deg(\hat{p}) = \deg(f_{m,z_0}).$$

Now it suffices to observe that the map

$$h_{z_0}\colon S^1 \times [0,1] \to S^1,\ (z,t) \mapsto \frac{(z-tz_0)^m}{|(z-tz_0)^m|}$$

is well-defined, since $|z_0| < 1$, and hence defines a homotopy between $h_{z_0}(z,1) = f_{m,z_0}$ and $h_{z_0}(z,0) = \frac{z^m}{|z^m|}$. Thus $\deg(f_{m,z_0}) = m$.

Finally, we switch perspectives a bit. We know that the polynomial p satisfies $\lim_{|z\to\infty|} |p(z)| = \infty$. Hence we can extend the map $p\colon \mathbb{C} \to \mathbb{C}$ to a map $p\colon S^2 \to S^2$ where we think of S^2 as the one-point-compactification of $\mathbb{C} \cong \mathbb{R}^2$. We are going to use the following fact: Let $f_m\colon S^2 \to S^2$, $z \mapsto z^m$. The effect of $H_2(f_m)$ as a selfmap of $H_2(S^2)$ and as a selfmap of $H_2(S^2, S^2 \setminus \{0\})$ is given by multiplication by m.

d) Let z_i be a root of p. Show that the local degree $\deg(p|z_i)$ of p at z_i is equal to the multiplicity of z_i as a root of p.

Solution: We choose a small open subset U_i around z_i in S^2 which is homeomorphic to an open disk such that U_i does not contain any other roots of p. Then we can choose an open subset V around 0 in S^2 such that $p(U_i \setminus \{z_i\}) \subset V \setminus \{0\}$. The local degree of p at z_i is by definition the integer which describes the effect of the homomorphism

$$\mathbb{Z} \cong H_2(U_i, U_i \setminus \{z_i\}) \xrightarrow{H_2(p_{|U_i})} H_2(V, V \setminus \{0\}) \cong \mathbb{Z}.$$

We can write $p(z)$ as $p(z) = (z-z_0)^m q(z)$ with $q(z) \neq 0$ for all z. We define a map

$$j\colon U_i \times [0,1] \to V,\ (z,t) \mapsto (z-z_0)^m q(tz + (1-t)z_0).$$

Note that, since z_i is the only root of p in U_i, we know $j(z,t) \neq 0$ for all t and $z \in U_i \setminus \{z_i\}$. Thus

$$j(U_i \setminus \{z_i\} \times [0,1]) \subset V \setminus \{0\}.$$

Hence j defines a relative homotopy between

$$j(z,0) = (z-z_0)^m q(z_0) \text{ and } j(z,1) = (z-z_0)^m q(z) = p(z)$$

as maps of pairs $(U_i, U_i \setminus \{z_i\}) \to (V, V \setminus \{0\})$.
Since multiplication with a constant gives homotopic maps and since homology is invariant under homotopy, this shows that

$$H_2(p_{|U_i}) = H_2(j-,1)) = H_2(j(-,0))\colon H_2(U_i, U_i \setminus \{z_i\}) \to H_2(V, V \setminus \{0\}).$$

Finally, the commutative diagram

$$\begin{array}{ccc} (U_i, U_i \setminus \{z_i\}) & \xrightarrow{j(-,0)} & (V, V \setminus \{0\}) \\ \downarrow & & \downarrow \\ (S^2, S^2 \setminus \{0\}) & \xrightarrow[f_m]{} & (S^2, S^2 \setminus \{0\}) \end{array}$$

where the vertical maps are inclusions of pairs induces a commutative diagram

$$\begin{array}{ccc} H_2(U_i, U_i \setminus \{z_i\}) & \xrightarrow{H_2(j(-,0))} & H_2(V, V \setminus \{0\}) \\ \cong \downarrow & & \downarrow \cong \\ H_2(S^2, S^2 \setminus \{0\}) & \xrightarrow[H_2(f_m)]{} & H_2(S^2, S^2 \setminus \{0\}). \end{array}$$

Since $H_2(f_m)$ is given by multiplication by m, this shows

$$\deg(p|z_i) = \deg(j(-,0) = \deg(f_m) = m.$$