Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 5

This will be a guided tour to the fundamental group and the Hurewicz homomorphism. You can read about this topic in almost every textbook. But you could also take the time to solve the following exercises and enjoy the fun of developing the maths on your own.

Note that we already seen some of the following problems in previous exercises. But feel free to do them again. :)

We fix the following notation:

Let X be a nonempty topological space and let x_0 be a point in X. We write $I = [0, 1]$ for the unit interval and $\partial I = \{0, 1\}$.

We denote by

$$\Omega(X, x_0) := \{\gamma \in C([0, 1], X) : \gamma(0) = x_0 = \gamma(1)\}$$

the set of continuous loops based at x_0.

We are going to call two loops γ_1 and γ_2 based at x_0 are *homotopic relative to ∂I* if there is a continuous map

$$h\colon [0, 1] \times [0, 1] \to X \text{ wich satisfies } \begin{cases} h(s, 0) = \gamma_1(s) \text{ for all } s \\ h(s, 1) = \gamma_2(s) \text{ for all } s \\ h(0, t) = x_0 = h(1, t) \text{ for all } t. \end{cases}$$

On this exercise set, we will always use the word *loop* to denote a loop based at x_0 and say that two loops are *homotopic* when they are homotopic relative to ∂I.

Let γ_1 and γ_2 be two loops based at x_0. We define the loop $\gamma_1 * \gamma_2$ to be the loop given by first walking along γ_1 and then walking along γ_2 with doubled speed, i.e., the map

$$\gamma_1 * \gamma_2 \colon [0, 1] \to X, s \mapsto \begin{cases} \gamma_1(2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ \gamma_2(2s - 1) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Our first goal is to prove the following theorem:

The fundamental group

The set of equivalence classes of loops modulo homotopy

$$\pi_1(X, x_0) := \Omega(X, x_0)/\simeq$$

becomes a group with group operation

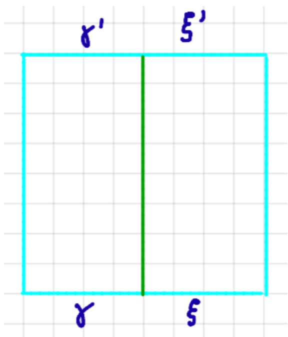
$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 * \gamma_2].$$

The class of the constant loop ϵ_{x_0} is the neutral element. The inverse of $[\gamma]$ is $[\bar{\gamma}]$, where $\bar{\gamma}$ denotes the loop in reverse direction $s \mapsto \gamma(1-s)$.
The group $\pi_1(X, x_0)$ is called the fundamental group of X at x_0.

$\boxed{1}$ In this exercise we are going to show that $\pi_1(X, x_0)$ together with the above described operation is a group.

Let γ, γ', ξ, ξ', ζ denote loops based at x_0. Let ϵ_{x_0} with $\epsilon_{x_0}(t) = x_0$ for all t be the constant loop at x_0. Recall that we use the notation $\gamma \simeq \gamma'$ to say that the two loops γ and γ' are homotopic relative to ∂I.

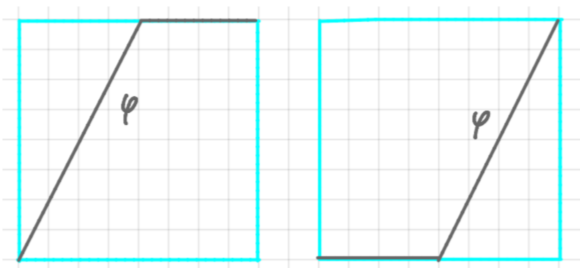
a) Show that if $\gamma \simeq \gamma'$ and $\xi \simeq \xi'$, then $\gamma * \xi \simeq \gamma' * \xi'$.



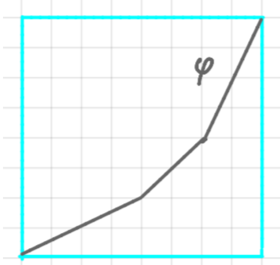
Let $\varphi\colon [0,1] \to [0,1]$ be a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$. The composition $\gamma \circ \varphi$ is called a *reparametrization* of γ.

b) Show that φ is homotopic to the identity map of $[0,1]$. Deduce that $\gamma \circ \varphi$ is homotopic to γ.

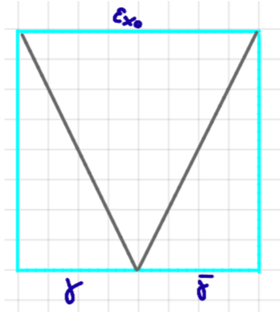
c) Choose appropriate reparametrizations of the paths involved to show $\epsilon_{x_0} * \gamma \simeq \gamma \simeq \gamma * \epsilon_{x_0}$.



d) Choose a φ corresponding to the following picture to show that $\gamma * (\xi * \zeta)$ is a reparametrization of $(\gamma * \xi) * \zeta$ by φ. Conclude that $(\gamma * \xi) * \zeta \simeq \gamma * (\xi * \zeta)$.



e) Show that $\gamma * \bar{\gamma} \simeq \epsilon_{x_0} \simeq \bar{\gamma} * \gamma$ by writing down precise formulae for the following picture.



Our next goal is to construct a homomorphim $\pi_1(X, x_0) \to H_1(X)$.

$\boxed{2}$ Recall that if γ is a loop on X we define an associated 1-simplex σ_γ by

$$\sigma_\gamma(1-t, t) := \gamma(t) \text{ for } 0 \leq t \leq 1.$$

Note that if γ is a loop, then σ_γ is a 1-cycle.

A brief **reminder** before we start. For solving the following problems remember that if you want to construct a 2-simplex with a certain boundary, you need to define a map on all of Δ^2 and not just its boundary. Omitting to describe the map on all of Δ^2 would make the tasks trivial. Now let us get to work:

a) Show that if $\gamma = \epsilon_{x_0}$ is the constant loop at x_0, then σ_γ is a boundary.

b) Show that the 1-chain $\sigma_{\gamma_1} + \sigma_{\gamma_2} - \sigma_{\gamma_1 * \gamma_2}$ is a boundary.

c) Show that if γ_1 and γ_2 are homotopic loops, then $\sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary. (Hint: For a homotopy h between γ_1 and γ_2, think of $I \times I$ as a square. Then you can either collapse it to a triangle or divide it along the diagonal to get two triangles. This will give you a way to construct 2-simplices out of h.)

d) Show that the 1-chain $\sigma_\gamma + \sigma_{\bar{\gamma}}$ is a boundary.

e) Conclude that the map

$$\phi\colon \pi_1(X, x_0) \to H_1(X), \; [\gamma] \mapsto [\sigma_\gamma]$$

is a homomorphism of groups. It is called the *Hurewicz homomorphism*.

The fundamental group $\pi_1(X, x_0)$ is in general not abelian. Since $H_1(X)$ is by definition an abelian group, the homomorphism φ factors through the maximal abelian quotient of $\pi_1(X, x_0)$. Reall that this quotient is defined as follows:

For a group G, the commutator subgroup $[G, G]$ of G is the smallest subgroup of G containing all commutators $[g, h] = [ghg^{-1}h^{-1}]$ for all $g, h \in G$. Note that $[G, G]$ is a normal subgroup. The quotient $G_{\mathrm{ab}} := G/[G, G]$ is the maximal abelian quotient of G and is called the *abelianization* of G.

The abelianization has the following universal property: Let $q\colon G \to G/[G, G]$ be the quotient map. If H is an abelian and $\eta G \to H$ a homomorphism of groups, then there is a unique homomorphism of abelian groups $\eta_{\mathrm{ab}}\colon G_{\mathrm{ab}} \to H$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[d, "q"'] & H \\ G_{\mathrm{ab}}. \arrow[ur, "\eta_{\mathrm{ab}}"'] & \end{tikzcd}$$

$\boxed{3}$ Assume that X is path-connected. We are going to show that the induced homomorphism

$$\phi_{\mathrm{ab}}\colon \pi_1(X, x_0)_{\mathrm{ab}} \to H_1(X),$$

which is also called the *Hurewicz homomorphism*, is an isomorphism.

We are going to construct an inverse ψ of ϕ_{ab} as follows:

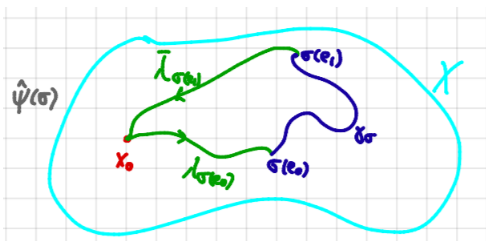
For any $x \in X$, we choose a continuous path λ_x from x_0 to x. If $x = x_0$, then we choose λ_{x_0} to be the constant path at x_0.

Let $\sigma\colon \Delta^1 \to X$ be a 1-chain in X. Denote the associated path in X by

$$\gamma_\sigma\colon [0,1] \to X,\ t \mapsto \sigma(1-t, t).$$

Then we define a loop

$$\hat{\psi}(\sigma)\colon [0,1] \to X,\ t \mapsto \lambda_{\sigma(e_0)} * \gamma_\sigma * \bar{\lambda}_{\sigma(e_1)}.$$



We extend this definition $\mathbb{Z}$-linearly to obtain homomorphism of abelian groups

$$\hat{\psi}\colon S_1(X) \to \pi_1(X, x_0)_{\mathrm{ab}}.$$

a) As a preparation show the following: Let $\beta\colon \Delta^2 \to X$ be a 2-simplex. Let α_i be the path corresponding to the ith face $\beta \circ \phi_i$ of β. Show that the loop $\alpha_2 * \alpha_0 * \bar{\alpha}_1$ based at $y_0 := \beta(e_0)$ is homotopic to the constant loop ϵ_{y_0} at y_0.

Note: If you do not want to derive formulae, draw a picture to convince yourself that the statement makes sense and describe in words why it is true.

b) Show that $\hat{\psi}$ sends the group $B_1(X)$ of 1-boundaries to the neutral element $1 \in \pi_1(X, x_0)_{\mathrm{ab}}$.
(Hint: Use that $\pi_1(X, x_0)_{\mathrm{ab}}$ is abelian and that the loop given by walking along the boundary of a 2-simplex is homotopic to the constant loop.)

c) Conclude that $\hat{\psi}$ induces a homomorphism of abelian groups

$$\psi\colon H_1(X) \to \pi_1(X, x_0)_{\mathrm{ab}}.$$

d) Show that if γ is a loop, then $\psi(\phi_{\mathrm{ab}}([\gamma])) = [\gamma]$.

e) Let σ be a 1-simplex. Show that $\phi_{\mathrm{ab}}(\psi([\sigma])) = [\sigma + \kappa_{\sigma(e_0)} - \kappa_{\sigma(e_1)}]$, where κ_y denotes the constant 1-simplex with value y.

f) Show that, if c is a 1-cycle, then $\phi_{\mathrm{ab}}(\psi([c])) = [c]$.

g) Conclude that ψ is an inverse of ϕ_{ab} and hence that ϕ_{ab} is an isomorphism.