Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3403 Algebraic Topology
Fall 2018

Solutions to exercise set 5

This will be a guided tour to the fundamental group and the Hurewicz homomorphism. You can read about this topic in almost every textbook. But you could also take the time to solve the following exercises and enjoy the fun of developing the maths on your own.

Note that we already seen some of the following problems in previous exercises. But feel free to do them again. :)

We fix the following notation:

Let X be a nonempty topological space and let x_0 be a point in X. We write $I = [0, 1]$ for the unit interval and $\partial I = \{0, 1\}$.

We denote by

$$\Omega(X, x_0) := \{\gamma \in C([0, 1], X) : \gamma(0) = x_0 = \gamma(1)\}$$

the set of continuous loops based at x_0.

We are going to call two loops γ_1 and γ_2 based at x_0 are *homotopic relative to* ∂I if there is a continuous map

$$h\colon [0, 1] \times [0, 1] \to X \text{ wich satisfies } \begin{cases} h(s, 0) = \gamma_1(s) \text{ for all } s \\ h(s, 1) = \gamma_2(s) \text{ for all } s \\ h(0, t) = x_0 = h(1, t) \text{ for all } t. \end{cases}$$

On this exercise set, we will always use the word *loop* to denote a loop based at x_0 and say that two loops are *homotopic* when they are homotopic relative to ∂I.

Let γ_1 and γ_2 be two loops based at x_0. We define the loop $\gamma_1 * \gamma_2$ to be the loop given by first walking along γ_1 and then walking along γ_2 with doubled speed, i.e., the map

$$\gamma_1 * \gamma_2 \colon [0, 1] \to X, s \mapsto \begin{cases} \gamma_1(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ \gamma_2(2s - 1) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

Our first goal is to prove the following theorem:

The fundamental group

The set of equivalence classes of loops modulo homotopy

$$\pi_1(X, x_0) := \Omega(X, x_0)/\simeq$$

becomes a group with group operation

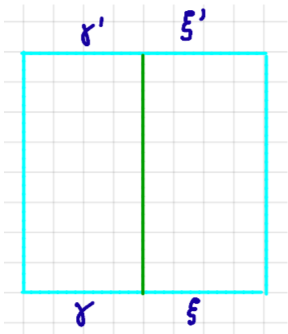
$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 * \gamma_2].$$

The class of the constant loop ϵ_{x_0} is the neutral element. The inverse of $[\gamma]$ is $[\bar{\gamma}]$, where $\bar{\gamma}$ denotes the loop in reverse direction $s \mapsto \gamma(1-s)$.
The group $\pi_1(X, x_0)$ is called the fundamental group of X at x_0.

$\boxed{1}$ In this exercise we are going to show that $\pi_1(X, x_0)$ together with the above described operation is a group.

Let γ, γ', ξ, ξ', ζ denote loops based at x_0. Let ϵ_{x_0} with $\epsilon_{x_0}(t) = x_0$ for all t be the constant loop at x_0. Recall that we use the notation $\gamma \simeq \gamma'$ to say that the two loops γ and γ' are homotopic relative to ∂I.

a) Show that if $\gamma \simeq \gamma'$ and $\xi \simeq \xi'$, then $\gamma * \xi \simeq \gamma' * \xi'$.



Solution: Let h be a homotopy between γ and γ' and j be a homotopy between ξ and ξ'. Then we define the continuous map

$$H\colon [0,1] \times [0,1] \to X, \ (t,s) \mapsto \begin{cases} h(2t, s) & \text{for } 0 \le t \le \frac{1}{2} \\ j(2t-1, s) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

This defines a homotopy between $\gamma * \xi$ and $\gamma' * \xi'$, since

$$H(t,0) = \begin{cases} h(2t,0) = \gamma(2t) = \gamma * \xi(t) & \text{for } 0 \le t \le \frac{1}{2} \\ j(2t-1,0) = \xi(2t-1) = \gamma * \xi(t) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

and

$$H(t,1) = \begin{cases} h(2t,1) = \gamma'(2t) = \gamma' * \xi'(t) & \text{for } 0 \le t \le \frac{1}{2} \\ j(2t-1,1) = \xi'(2t-1) = \gamma' * \xi'(1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Let $\varphi\colon [0,1] \to [0,1]$ be a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$. The composition $\gamma \circ \varphi$ is called a *reparametrization* of γ.

b) Show that φ is homotopic to the identity map of $[0,1]$. Deduce that $\gamma \circ \varphi$ is homotopic to γ.

Solution: We can choose a straight line homotopy

$$h\colon [0,1]\times[0,1] \to [0,1],\ (s,t) \mapsto (1-t)\varphi(s) + ts.$$

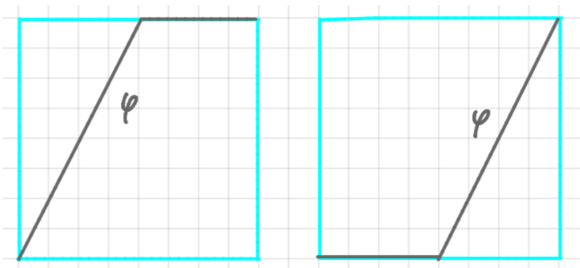
Note that $h(s,t)$ is between the two numbers s and $\varphi(s)$ for all t. This shows that $h(s,t)$ is always contained in $[0,1]$ and hence h is a homotopy between $h(-,0) = \varphi$ and $h(-,1) = \mathrm{id}_{[0,1]}$.

Now we can define a continuous map

$$H\colon [0,1]\times[0,1] \to X,\ (s,t) \mapsto \gamma((1-t)\varphi(s) + ts).$$

Again, since $(1-t)\varphi(s) + ts$ is between s and $\varphi(s)$, we can apply γ to this number. We observe that H is a homotopy between $H(-,0) = \gamma \circ \varphi$ and $H(-,1) = \gamma \circ \mathrm{id}_{[0,1]} = \gamma$.

c) Choose appropriate reparametrizations of the paths involved to show $\epsilon_{x_0} * \gamma \simeq \gamma \simeq \gamma * \epsilon_{x_0}$.



Solution: Considering the definition of composing paths we observe that we have

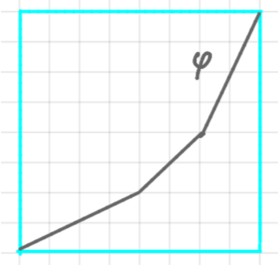
$$\epsilon_{x_0} * \gamma = \gamma \circ \varphi \text{ for } \varphi(s) = \begin{cases} 0 & \text{for } 0 \le s \le \frac{1}{2} \\ 2s-1 & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

and

$$\gamma * \epsilon_{x_0} = \gamma \circ \varphi \text{ for } \varphi(s) = \begin{cases} 2s & \text{for } 0 \le s \le \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

Hence both paths can be viewed as reparametrizations. By the previous point, this proves the desired homotopy relations.

d) Choose a φ corresponding to the following picture to show that $\gamma * (\xi * \zeta)$ is a reparametrization of $(\gamma * \xi) * \zeta$ by φ. Conclude that $(\gamma * \xi) * \zeta \simeq \gamma * (\xi * \zeta)$.



Solution: We choose φ to be

$$\varphi\colon I \to I,\ \varphi(s) = \begin{cases} \frac{1}{2}s & \text{for } 0 \le s \le \frac{1}{2} \\ s - \frac{1}{4} & \text{for } \frac{1}{2} \le s \le \frac{3}{4} \\ 2s & \text{for } \frac{3}{4} \le s \le 1. \end{cases}$$

By checking the values at the overlaps we see that φ is continuous. Then we observe

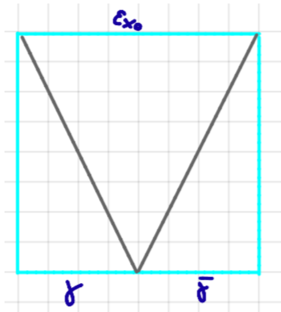
$$((\gamma * \xi) * \zeta)(s) = \begin{cases} \gamma(4s) & \text{for } 0 \le s \le \frac{1}{4} \\ \xi(4s-1) & \text{for } \frac{1}{4} \le s \le \frac{1}{2} \\ \zeta(2s-1) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

Composition with φ yields

$$((\gamma * \xi) * \zeta)(\varphi((s)) = \begin{cases} \gamma(4\varphi(s)) = \gamma(2s) & \text{for } 0 \le \varphi(s) \le \frac{1}{4} \iff 0 \le s \le \frac{1}{2} \\ \xi(4\varphi(s)-1) = \xi(4s-1) & \text{for } \frac{1}{4} \le \varphi(s) \le \frac{1}{2} \iff \frac{1}{2} \le s \le \frac{3}{4} \\ \zeta(2\varphi(s)-1) = \zeta(4s-3) & \text{for } \frac{1}{2} \le \varphi(s) \le 1 \iff \frac{3}{4} \le s \le 1. \end{cases}$$

Now we observe that the latter is exactly the path $\gamma * (\xi * \zeta)$. Hence we have shown $((\gamma * \xi) * \zeta) \circ \varphi = \gamma * (\xi * \zeta)$ which implies the desired homotopy by the previous point.

e) Show that $\gamma * \bar{\gamma} \simeq \epsilon_{x_0} \simeq \bar{\gamma} * \gamma$ by writing down precise formulae for the following picture.



Solution: We can write down a homotoopy $h\colon I \times I \to X$ by

$$h(s,t) = \begin{cases} \gamma(2s) & \text{for } 0 \le s \le \frac{1}{2} \text{ and } 0 \le t \le 1-2s \\ \gamma(1-t) & \text{for } 0 \le s \le \frac{1}{2} \text{ and } 1-2s \le t \le 1 \\ \bar{\gamma}(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \text{ and } 0 \le t \le 2s-1 \\ \bar{\gamma}(t) & \text{for } \frac{1}{2} \le s \le 1 \text{ and } 2s-1 \le t \le 1. \end{cases}$$

We convince ourselves that this map is continuous by checking that the values at the overlaps agree. Hence h is a homotopy between $h(s,0) = \gamma * \bar{\gamma}(s)$ and $h(s,1) = \epsilon_{x_0}(s)$.
By switching the roles of γ and $\bar{\gamma}$ in the formulae for h we can construct a homotopy $\epsilon_{x_0} \simeq \bar{\gamma} * \gamma$.

Our next goal is to construct a homomorphim $\pi_1(X, x_0) \to H_1(X)$.

$\boxed{2}$ Recall that if γ is a loop on X we define an associated 1-simplex σ_γ by

$$\sigma_\gamma(1-t,t) := \gamma(t) \text{ for } 0 \leq t \leq 1.$$

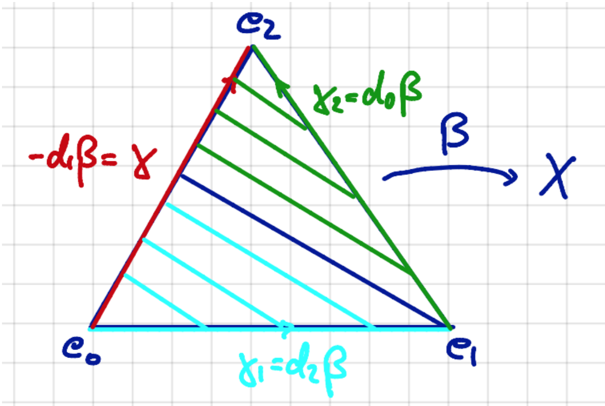
Note that if γ is a loop, then σ_γ is a 1-cycle.

A brief **reminder** before we start. For solving the following problems remember that if you want to construct a 2-simplex with a certain boundary, you need to define a map on all of Δ^2 and not just its boundary. Omitting to describe the map on all of Δ^2 would make the tasks trivial. Now let us get to work:

a) Show that if $\gamma = \epsilon_{x_0}$ is the constant loop at x_0, then σ_γ is a boundary.
Solution: Let $\alpha\colon \Delta^2 \to X$ be the constant 2-simplex with value x_0. Then

$$\partial(\alpha) = \sigma_\gamma - \sigma_\gamma + \sigma_\gamma = \sigma_\gamma.$$

b) Show that the 1-chain $\sigma_{\gamma_1} + \sigma_{\gamma_2} - \sigma_{\gamma_1 * \gamma_2}$ is a boundary.
Solution: We define a 2-simplex $\beta\colon \Delta^2 \to X$ to be equal γ_1 on the edge from e_0 to e_1, to be equal γ_2 on the edge from e_1 to e_2, and to be constant on the lines perpendicular to the edge from e_0 to e_2. That implies that β equals γ on the edge from e_0 to e_2:



The boundary of β is then given by

$$\partial(\beta) = d_0(\beta) - d_1(\beta) + d_2(\beta) = \sigma_{\gamma_2} - \sigma_{\gamma_1 * \gamma_2} + \sigma_{\gamma_1}.$$

c) Show that the 1-chain $\sigma_\gamma + \sigma_{\bar{\gamma}}$ is a boundary.
Solution: We define a 2-simplex $\beta\colon \Delta^2 \to X$ to be equal γ on the edge from e_0 to e_1 to be constant on the lines parallel to the edge (e_0, e_2). Then β is $\bar{\gamma}$ on the edge (e_1, e_2). The boundary of β is then given by

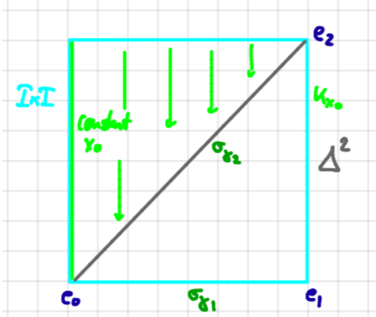
$$\partial(\beta) = d_0(\beta) - d_1(\beta) + d_2(\beta) = \sigma_{\bar{\gamma}} - \kappa_{x_0} + \sigma_\gamma$$

where κ_{x_0} denotes the constant 1-simplex at x_0. Since κ_{x_0} is a boundary, this shows that $\sigma_\gamma + \sigma_{\bar{\gamma}}$ is a boundary.

d) Show that if γ_1 and γ_2 are homotopic loops, then $\sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary.
Solution: Let $h\colon I \times I \to X$ be a homotopy between γ_1 and γ_2. We present two ways of constructing an appropriate 2-chain:
• Since h is a homotopy relative to ∂I, h is constant on $\{0\} \times I$. Hence h factors as

$$h = \sigma \circ q\colon I \times I \xrightarrow{q} (I \times I)/(\{0\} \times I) \cong \Delta^2 \xrightarrow{\sigma} X$$

where q is the map which collapses the edge $\{0\} \times I$ to the vertex e_0.



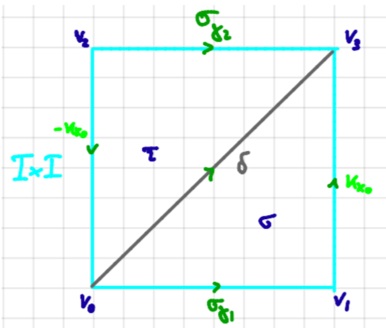
The induced map $\Delta^2 \to X$ is a singular simplex σ which agrees with γ_1 on the edge (e_0, e_1), is constant with value x_0 on the edge $e_1, e_2)$ and agrees with γ_2 on the edge (e_0, e_2). The boundary of σ is

$$\partial(\sigma) = d_0\sigma - d_1\sigma + d_2\sigma = \sigma_{\gamma_1} - \sigma_{\gamma_2} + \kappa_{x_0}$$

where κ_{x_0} denotes the constant 1-simplex at x_0. Since κ_{x_0} is a boundary, this shows that $\sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary.

• For another argument, we think of $I \times I$ as a square with vertices v_0, v_1, v_2, v_3 in counter-clockwise order. Then we see that h equals γ_1 along the lower horizontal edge and equals γ_2 along the upper horizontal edge. Along the left and right vertical edges it is constant with value x_0.

Now we divide $I \times I$ along the diagonal points to get two 2-simplices σ and τ as indicated in the picture: σ is the simplex with vertices v_0, v_1, v_3 and τ is the simplex with vertices v_0, v_2, v_3.



For calculating the boundary of $\sigma - \tau$ we write δ for the diagonal viewed as a 1-simplex. Then we get

$$\begin{aligned}\partial(\sigma - \tau) &= \kappa_{x_0} - \delta + \sigma_{\gamma_1} - (\sigma_{\gamma_2} - \delta - \kappa_{x_0}) \\ &= \sigma_{\gamma_1} - \sigma_{\gamma_2} + 2\kappa_{x_0}.\end{aligned}$$

Since the constant 1-simplex is a bpundary, we can disregard $2\kappa_{x_0}$ and see that $\sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary.

e) Conclude that the map

$$\phi\colon \pi_1(X, x_0) \to H_1(X),\ [\gamma] \mapsto [\sigma_\gamma]$$

is a homomorphism of groups. It is called the *Hurewicz homomorphism.*

Solution: We have shown that in $H_1(X)$ we have

$$[\sigma_{\epsilon_{x_0}}] = 0,\ [\sigma_{\gamma_1 * \gamma_2}] = [\sigma_{\gamma_1}] + [\sigma_{\gamma_2}],\ [\sigma_{\bar{\gamma}}] = -[\sigma_\gamma],\ \text{and } \gamma \simeq \gamma' \Rightarrow [\sigma_\gamma] = [\sigma_{\gamma'}].$$

This is all we need to conclude that ϕ is a homomorphism of groups.

The fundamental group $\pi_1(X, x_0)$ is in general not abelian. Since $H_1(X)$ is by definition an abelian group, the homomorphism φ factors through the maximal abelian quotient of $\pi_1(X, x_0)$. Reall that this quotient is defined as follows:

For a group G, the commutator subgroup $[G, G]$ of G is the smallest subgroup of G containing all commutators $[g, h] = ghg^{-1}h^{-1}$ for all $g, h \in G$. Note that $[G, G]$ is a normal subgroup. The quotient $G_{\mathrm{ab}} := G/[G, G]$ is the maximal abelian quotient of G and is called the *abelianization* of G.

The abelianization has the following universal property: Let $q\colon G \to G/[G, G]$ be the quotient map. If H is an abelian and $\eta G \to H$ a homomorphism of groups, then there is a unique homomorphism of abelian groups $\eta_{\mathrm{ab}}\colon G_{\mathrm{ab}} \to H$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[d, "q"'] & H \\ G_{\mathrm{ab}}. \arrow[ur, "\eta_{\mathrm{ab}}"'] & \end{tikzcd}$$

$\boxed{3}$ Assume that X is path-connected. We are going to show that the induced homomorphism

$$\phi_{\mathrm{ab}}\colon \pi_1(X, x_0)_{\mathrm{ab}} \to H_1(X),$$

which is also called the *Hurewicz homomorphism*, is an isomorphism.

We are going to construct an inverse ψ of ϕ_{ab} as follows:

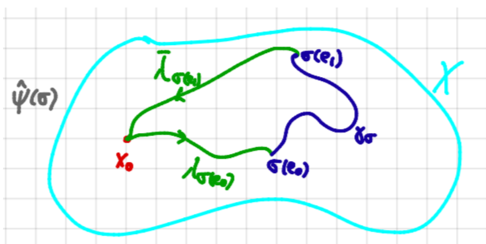
For any $x \in X$, we choose a continuous path λ_x from x_0 to x. If $x = x_0$, then we choose λ_{x_0} to be the constant path at x_0.

Let $\sigma\colon \Delta^1 \to X$ be a 1-chain in X. Denote the associated path in X by

$$\gamma_\sigma\colon [0, 1] \to X,\ t \mapsto \sigma(1 - t, t).$$

Then we define a loop

$$\hat{\psi}(\sigma)\colon [0, 1] \to X,\ t \mapsto \lambda_{\sigma(e_0)} * \gamma_\sigma * \bar{\lambda}_{\sigma(e_1)}.$$

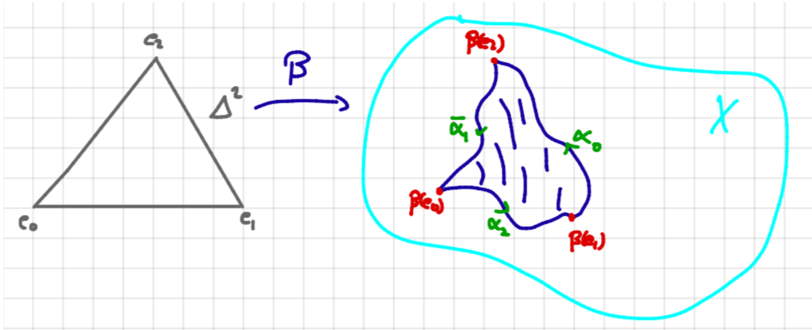


We extend this definition $\mathbb{Z}$-linearly to obtain homomorphism of abelian groups

$$\hat{\psi}\colon S_1(X) \to \pi_1(X, x_0)_{\mathrm{ab}}.$$

a) As a preparation show the following: Let $\beta\colon \Delta^2 \to X$ be a 2-simplex. Let α be the path corresponding to the ith face $\beta\circ\phi_i$ of β. Show that the loop $\alpha_2 * \alpha_0 * \bar{\alpha}_1$ based at $y_0 := \beta(e_0)$ is homotopic to the constant loop at y_0.

Solution: When you picture an example of β the assertion should seem obvious to you. To prove it we observe that we can consider β as a homotopy between the two loops. There are many different ways to do this.



• For example, remembering that β is defined for the barycentric coordinates (t_0, t_1, t_2), we could first contract the t_2-coordinate, i.e., we shrink the image $\beta(\Delta^2)$ continuously to the the image of the edge (e_0, e_1) under β. Then we contract the t_1-coordinate, i.e., we shrink the image of the edge (e_0, e_1) continuously to the point $\beta(e_0)$. What we are left with is the constant loop at $\beta(e_0)$.

• More conretely, we could use that Δ^2 is convex, i.e., the linesegment between any two points p_1 and p_2 of Δ^2 is contained in Δ^2. This implies that $\beta(tp_1 + (1-t)p_2)$ is defined for all $t \in [0,1]$.

The paths corresponding to walking along the edge of β can be described as

$$s \mapsto \begin{cases} \alpha_2((1-3s)e_0 + 3se_1) = \beta((1-3s)e_0 + 3se_1) & \text{for } s \in [0, \frac{1}{3}] \\ \alpha_0((1-3s)e_1 + 3se_2) = \beta((1-3s)e_1 + 3se_2) & \text{for } s \in [\frac{1}{3}, \frac{2}{3}] \\ \bar{\alpha}_1(3se_0 + (1-3s)e_2) = \beta(3se_0 + (1-3s)e_2) & \text{for } s \in [\frac{2}{3}, 1]. \end{cases}$$

Now we define a map $h\colon I \times I \to X$ by

$$h(s,t) = \begin{cases} \beta(te_0 + (1-t)((1-3s)e_0 + 3se_1)) & \text{for } s \in [0, \frac{1}{3}] \\ \beta((te_0 + (1-t)(1-3s)e_1 + 3se_2)) & \text{for } s \in [\frac{1}{3}, \frac{2}{3}] \\ \beta(tr_0 + (1-t)(3se_0 + (1-3s)e_2)) & \text{for } s \in [\frac{2}{3}, 1]. \end{cases}$$

As we just observed, $h(s,t)$ is well-defined for all $(s,t) \in I \times I$. Hence it defines a homotopy between $s \mapsto h(-,1)$ which is the constant path at $\beta(e_0)$ and $h(-,0)$ which is the loop $\alpha_2 * \alpha_0 * \bar{\alpha}_1$.

b) Show that $\hat{\psi}$ sends the group $B_1(X)$ of 1-boundaries to the neutral element $1 \in \pi_1(X, x_0)_{\mathrm{ab}}$.

Solution: Let $\beta\colon \Delta^2 \to X$ be a 2-simplex. To simplify the notation, we write $\beta(e_i) = y_i$ and α_i for the ith face $\beta \circ \phi_i$. Then

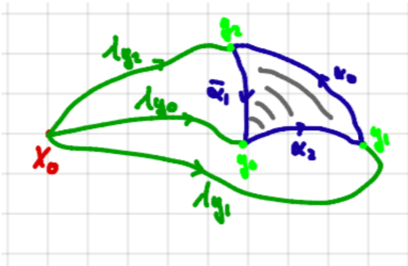
$$\begin{aligned} \hat{\psi}(\partial\beta) &= \hat{\psi}(\alpha_0 - \alpha_1 + \alpha_2) \\ &= \hat{\psi}(\alpha_0)\hat{\psi}(\alpha_1)^{-1}\hat{\psi}(\alpha_2) \\ &= [\lambda_{y_1} * \alpha_0 * \bar{\lambda}_{y_2}][\lambda_{y_0} * \alpha_1 * \bar{\lambda}_{y_2}]^{-1}[\lambda_{y_0} * \alpha_2 * \bar{\lambda}_{y_1}]. \end{aligned}$$

Now we use that the image of $\hat{\psi}$ is in the abelian group $\pi_1(X, x_0)_{\mathrm{ab}}$. For then we can regroup the paths and use the definition of multiplication:

$$= [\lambda_{y_0} * \alpha_2 * \bar{\lambda}_{y_1} * \lambda_{y_1} * \alpha_0 * \bar{\lambda}_{y_2} * \lambda_{y_2} * \bar{\alpha}_1 * \bar{\lambda}_{y_0}].$$

Now we can simplify the paths $\bar{\lambda}_{y_1} * \lambda_{y_1}$ and $\bar{\lambda}_{y_2} * \lambda_{y_2}$ which are homotopic to constant paths:

$$= [\lambda_{y_0} * \alpha_2 * \alpha_0 * \bar{\alpha}_1 * \bar{\lambda}_{y_0}].$$



The path $\alpha_2 * \alpha_0 * \bar{\alpha}_1$ is the boundary of β. Thus it is homotopic to a constant path, since we can contract the whole image of Δ^2 to a point. Thus we get

$$\hat{\psi}(\partial\beta) = [\epsilon_{x_0}] = 1 \in \pi_1(X, x_0)_{\mathrm{ab}}.$$

c) Conclude that $\hat{\psi}$ induces a homomorphism of abelian groups

$$\psi\colon H_1(X) \to \pi_1(X, x_0)_{\mathrm{ab}}.$$

Solution: By the previous point, $\hat{\psi}$ sends $B_1(X)$ to the identity element. Hence $\hat{\psi}$ factors through the quotient $S_1(X)/B_1(X)$ and the same is true for its restriction to $Z_1(X)/B_1(X) = H_1(X)$.

d) Show that if γ is a loop, then $\psi(\phi_{\mathrm{ab}}([\gamma])) = [\gamma]$.
Solution: Using the definition of the map $\hat{\psi}$ we get

$$\psi(\phi_{\mathrm{ab}}([\gamma])) = \psi([\sigma_\gamma]) = [\lambda_{x_0} * \sigma_\gamma * \bar{\lambda}_{x_0}].$$

Since we chose λ_{x_0} to be the constant loop, we get the class of γ back.

e) Let σ be a 1-simplex. Show that $\phi_{\mathrm{ab}}(\psi([\sigma])) = [\sigma + \kappa_{\sigma(e_0)} - \kappa_{\sigma(e_1)}]$, where κ_y denotes the constant 1-simplex with value y.
Solution: We compute:

$$\begin{aligned}\phi_{\mathrm{ab}}(\psi([\sigma])) &= \phi_{\mathrm{ab}}(\lambda_{\sigma(e_0)} * \sigma * \bar{\lambda}_{\sigma(e_1)}]) \\ &= [\kappa_{\sigma(e_0)} + \sigma_\gamma - \kappa_{\sigma(e_1)}].\end{aligned}$$

f) Show that, if c is a 1-cycle, then $\phi_{\mathrm{ab}}(\psi([c])) = [c]$.
Solution: If $c = \sum_j n_j \sigma_j$ is a 1-cycle, then $\sum_j n_j(\sigma_j(e_0) - \sigma_j(e_1) = 0$. By the previous point, this shows $\phi_{\mathrm{ab}}(\psi([c])) = [c]$.

g) Conclude that ψ is an inverse of ϕ_{ab} and hence that ϕ_{ab} is an isomorphism.
Solution: We just proved $\phi_{\mathrm{ab}} \circ \psi = \mathrm{id}_{H_1(X)}$ and had already shown $\psi \circ \phi_{\mathrm{ab}} = \mathrm{id}_{\pi_1(X,x_0)_{\mathrm{ab}}}$. Hence ϕ_{ab} is an isomorphism with inverse ψ.