Norwegian University of Science and Technology
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MA3403 Algebraic Topology
Fall 2018

Exercise set 6

1 In this exercise we give another proof of the exactness of the Mayer-Vietoris sequence. We start with an algebraic lemma which provides good practice in diagram chasing and then we use this result to deduce the MVS from the Excision Axiom.

a) Assume we have a map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & K'_n \arrow[r, "i'_n"] \arrow[d, "f_n"'] & L'_n \arrow[r, "a'_n"] \arrow[d, "g_n"] & M'_n \arrow[r, "b'_n"] \arrow[d, "k_n", "\cong"'] & K'_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & K_n \arrow[r, "i_n"'] & L_n \arrow[r, "a_n"'] & M_n \arrow[r, "b_n"'] & K_{n-1} \arrow[r] & \cdots \end{tikzcd} \tag{1}$$

such that $k_n \colon M'_n \xrightarrow{\cong} M_n$ is an isomorhism for every n. For each n, we define the homomorphism ∂_n to be

$$\partial_n \colon L_n \xrightarrow{a_n} M_n \xrightarrow{k_n^{-1}} M'_n \xrightarrow{b'_n} K'_{n-1}.$$

Show that the sequence

$$\cdots \to K'_n \xrightarrow{\begin{bmatrix} f_n \\ -i'_n \end{bmatrix}} K_n \oplus L'_n \xrightarrow{\begin{bmatrix} i_n & g_n \end{bmatrix}} L_n \xrightarrow{\partial_n} K'_{n-1} \to \cdots$$

is exact.

b) Let $\{A, B\}$ be a cover of X. Apply the previous algebraic observation to the long exact sequences of the pairs (X, A) and $(B, A \cap B)$ and use the excision isomorphism to deduce the Mayer-Vietoris sequence.

2 Let A and B be two disjoint closed subsets of $\mathbb{R}^2$.

a) Show that there is an isomorphism

$$H_1(\mathbb{R}^2 - (A \cup B)) \cong H_1(\mathbb{R}^2 - A) \oplus H_1(\mathbb{R}^2 - B).$$

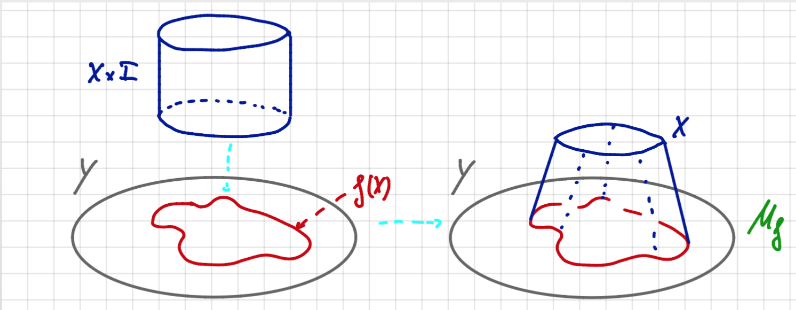
Recall that a path-component of a space X is a maximal path-connected subspace (where the ordering is given by inclusion). For example, if X is path-connected itself, then it has one path-component. If X is the disjoint union of two path-connected spaces U and V, then U and V are the path-components of X.

b) Show that the number of path-components of $\mathbb{R}^2 - (A \cup B)$ is one less than the sum of the numbers of path-components of $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - B$.

Definition: Mapping cylinder

Let $f\colon X \to Y$ be a continuous map. The **mapping cylinder of f** is defined to be te quotient space

$$M_f := (X \times [0,1] \sqcup Y)/((x,0) \sim f(x)).$$



The mapping cylinder fits into a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "f_1"'] & & Y \\
& M_f \arrow[ur, "g"'] &
\end{tikzcd}$$

where f_1 maps x to $(x,1)$ and g maps (x,t) to $f(x)$ for all $x \in X$ and $t \in [0,1]$ and $y \in Y$ to y.

$\boxed{3}$ **a)** Show that the inclusion $i\colon Y \hookrightarrow M_f$ is a deformation retract and g is a deformation retraction.

b) We can construct the **Möbius band** $M := M_f$ as the mapping cylinder of the map

$$f\colon S^1 \to S^1,\ z \mapsto z^2 \ (S^1 \subset \mathbb{C}).$$

Determine the homology of the Möbius band.

c) For $n \geq 1$ and $m \in \mathbb{Z}$, let M_f be the mapping cylinder of a map

$$f\colon S^n \to S^n \text{ with } \deg(f) = m.$$

Show that $H_n(f_1)$ is given by multiplication with m.

d) For $n \geq 1$ and $m \geq 2$, let M_f be the mapping cylinder of a map

$$f\colon S^n \to S^n \text{ with } \deg(f) = m.$$

Show that $X = S^n$ is not a weak retract of M_f.

$\boxed{4}$ We can consider the **real projective plane** $\mathbb{R}\mathrm{P}^2$ as a two dimensional disk D^2 with a Möbius band M attached at its boundary. Writing $A = D^2$ and $B = M$, we have $A \cap B \simeq S^1$. Calculate the homology groups of $\mathbb{R}\mathrm{P}^2$.