MA3403 Algebraic Topology
Fall 2018

Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 6

1 In this exercise we give another proof of the exactness of the Mayer-Vietoris sequence. We start with an algebraic lemma which provides good practice in diagram chasing and then we use this result to deduce the MVS from the Excision Axiom.

a) Assume we have a map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & K'_n \arrow[r, "i'_n"] \arrow[d, "f_n"'] & L'_n \arrow[r, "a'_n"] \arrow[d, "g_n"] & M'_n \arrow[r, "b'_n"] \arrow[d, "k_n", "\cong"'] & K'_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & K_n \arrow[r, "i_n"'] & L_n \arrow[r, "a_n"'] & M_n \arrow[r, "b_n"'] & K_{n-1} \arrow[r] & \cdots \end{tikzcd} \tag{1}$$

such that $k_n \colon M'_n \xrightarrow{\cong} M_n$ is an isomorhism for every n. For each n, we define the homomorphism ∂_n to be

$$\partial_n \colon L_n \xrightarrow{a_n} M_n \xrightarrow{k_n^{-1}} M'_n \xrightarrow{b'_n} K'_{n-1}.$$

Show that the sequence

$$\cdots \to K'_n \xrightarrow{\begin{bmatrix} f_n \\ -i'_n \end{bmatrix}} K_n \oplus L'_n \xrightarrow{\begin{bmatrix} i_n & g_n \end{bmatrix}} L_n \xrightarrow{\partial_n} K'_{n-1} \to \cdots$$

is exact.

Solution: We check exactness at each spot:

- at $K_n \oplus L'_n$: we calculate

$$\begin{bmatrix} i_n & g_n \end{bmatrix} \cdot \begin{bmatrix} f_n \\ -i'_n \end{bmatrix} = i_n \circ f_n - g_n \circ i'_n = 0$$

since the squares in (1) commute. Thus $\operatorname{Im}\left(\begin{bmatrix} f_n \\ -i'_n \end{bmatrix}\right) \subseteq \operatorname{Ker}\left(\begin{bmatrix} i_n & g_n \end{bmatrix}\right)$.

Now let $(x_n, y'_n) \in K_n \oplus L'_n$ such that

$$\begin{bmatrix} i_n & g_n \end{bmatrix} \cdot \begin{pmatrix} x_n \\ y'_n \end{pmatrix} = i_n(x_n) + g_n(y'_n) = 0.$$

Hence $g_n(y'_n) = -i_n(x_n)$ and therefore $g_n(y'_n)$, being in the image of i_n, is mapped to 0 in M_n by exactness of the rows in (1). Since k_n is an isomorphism, this implies that y'_n as mapped to 0 in M'_n as well. By exactness of the rows in (1), this implies that there is an element $\tilde{x}'_n \in K'_n$ with $i'_n(\tilde{x}'_n) = y'_n$. The difference $x_n - f_n(\tilde{x}'_n)$ is sent to 0 by i_n. By exactness of the rows in (1), there is a $z_{n+1} \in M_{n+1}$ which is mapped to $x_n - f_n(\tilde{x}'_n)$. Since k_{n+1} is an isomorhpism, there is a unique z'_{n+1} mapping to $z_{n+1} \in M_{n+1}$. Moreover, $i'_n(b'_{n+1}(z'n+1)) = 0$. Hence we can replace $\tilde{x}'_n$ by $x'_n := b'_{n+1}(z'_{n+1}) + \tilde{x}'_n$ to get an element x'_n in K'_n which maps to y'_n in L'_n and x_n in K_n. Thus $\mathrm{Ker}\left(\begin{bmatrix} i_n & g_n \end{bmatrix}\right) \subseteq \mathrm{Im}\left(\begin{bmatrix} f_n \\ -i'_n \end{bmatrix}\right)$.

- at L_n: we calculate

$$\begin{aligned}\partial_n(i_n(x_n) + g_n(y'_n)) &= (b'_n \circ k_n^{-1} \circ a_n \circ i_n)(x_n) + (b'_n \circ k_n^{-1} \circ a_n \circ g_n)(y'_n) \\ &= (b'_n \circ k_n^{-1}) \circ (a_n \circ i_n)(x_n) + (b'_n \circ a'_n)(y'_n) \\ &= 0 + 0 = 0,\end{aligned}$$

since $a_n \circ i_n = 0$ and $k_n^{-1} \circ a_n \circ g_n = a'_n$.
Now let $y_n \in L_n$ with $\partial_n(y_n) = 0$. By definition of ∂_n, this means $(b'_n \circ k_n^{-1} \circ a_n)(y_n) = 0$. Hence $k_n^{-1} \circ a_n)(y_n)$ is in the kernel of b'_n. Therefore there is a $y'_n \in L'_n$ with $a'_n(y'_n) = k_n^{-1} \circ a_n)(y_n)$. Now consider

$$\begin{aligned}a_n(y_n - g_n(y'_n)) &= a_n(y_n) - (a_n \circ g_n)(y'_n) \\ &= a_n(y_n) - (k_n \circ a'_n)(y'_n) = 0\end{aligned}$$

by the choice of y'_n. Hence, by exactness of the rows in (1), there is an $x_n \in K_n$ with $i_n(x_n) = y_n - g_n(y'_n)$. Thus we have found an (x_n, y'_n) which is mapped to y_n under $\begin{bmatrix} i_n & g_n \end{bmatrix}$.

- at K'_{n-1}: we calculate

$$\begin{bmatrix} f_{n-1} \\ -i'_{n-1} \end{bmatrix} \circ \partial_n = \begin{bmatrix} f_{n-1} \circ b'_n \circ k_n^{-1} \circ a_n \\ -i'_{n-1} \circ b'_n \circ k_n^{-1} \circ a_n \end{bmatrix} = \begin{bmatrix} b_n \circ a_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since $i'_{n-1} \circ b'_n = 0$, $f_{n-1} \circ b'_n \circ k_n^{-1} = b_n$ and $b_n \circ a_n = 0$.
Now let $x'_{n-1} \in K'_{n-1}$ with $f_{n-1}(x'_{n-1}) = 0$ and $i'_{n-1}(x'_{n-1}) = 0$. The latter equation implies that there is a $z'_n \in M'_n$ with $b'_n(z'_n) = x'_{n-1})$. Since

$$b_n(k_n(z'_n)) = f_{n-1}(b'_n(z'_n)) = f_{n-1}(x'_{n-1}) = 0$$

there is an element $y_n \in L_n$ with $a_n(y_n) = k_n(z'_n)$. By definition of ∂_n, this shows

$$\begin{aligned}\partial_n(y_n) &= (b'_n \circ k_n^{-1} \circ a_n)(y_n) \\ &= (b'_n \circ k_n^{-1})(k_n(z'_n)) = b'_n(z'_n) = x'_{n-1}.\end{aligned}$$

b) Let $\{A, B\}$ be a cover of X. Apply the previous algebraic observation to the long exact sequences of the pairs (X, A) and $(B, A \cap B)$ and use the excision isomorphism to deduce the Mayer-Vietoris sequence.

Solution: By definition of $\{A, B\}$ being a cover of X, we know $A^\circ \cup B^\circ = X$. If we set $Z := X - B$, this implies

$$\bar{Z} = X - B^\circ \subset A^\circ.$$

Moreover, since $A - Z = A - (X - B) = A \cap B$, we can apply excision to obtain an isomorphism

$$H_n(k)\colon H_n(X, A) \xrightarrow{\cong} H_n(B, A \cap B)$$

induced by the inclusion of pairs $k\colon (B, A\cap B) \hookrightarrow (X, A)$.
The obvious inclusions induce a map of long exact sequences for pairs of spaces

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A\cap B) \arrow[r, "H_n(j_B)"] \arrow[d, "H_n(j_A)"'] & H_n(B) \arrow[r] \arrow[d, "H_n(i_B)"] & H_n(B, A\cap B) \arrow[r] \arrow[d, "H_n(k)", "\cong"'] & H_{n-1}(A\cap B) \arrow[r] \arrow[d, "H_{n-1}(j_A)"] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(i_A)"'] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Now it suffices to apply the above algebraic result to this map of exact sequences to obtain the MVS.

$\boxed{2}$ Let A and B be two disjoint closed subsets of $\mathbb{R}^2$.

a) Show that there is an isomorphism

$$H_1(\mathbb{R}^2 - (A\cup B)) \cong H_1(\mathbb{R}^2 - A) \oplus H_1(\mathbb{R}^2 - B).$$

Solution: We set $U = \mathbb{R}^2 - A$ and $\mathbb{R}^2 - B$. Since A and B are closed, U and V are open. Moreover, $\mathbb{R}^2 - (A \cup B) = U \cap V$. The MVS looks like

$$H_2(\mathbb{R}^2) \to H_1(U\cap V) \to H_1(U) \oplus H_1(V) \to H_1(\mathbb{R}^2).$$

Since R^2 is contractible, the groups $H_2(\mathbb{R}^2)$ and $H_1(\mathbb{R}^2)$ vanish. Since the sequence is exact, this implies that $H_1(U\cap V) \to H_1(U) \oplus H_1(V)$ is an isomorphism.

Recall that a path-component of a space X is a maximal path-connected subspace (where the ordering is given by inclusion). For example, if X is path-connected itself, then it has one path-component. If X is the disjoint union of two path-connected spaces U and V, then U and V are the path-components of X.

b) Show that the number of path-components of $\mathbb{R}^2 - (A\cup B)$ is one less than the sum of the numbers of path-components of $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - B$.

Solution: The other part of the MVS looks like

$$H_1(\mathbb{R}^2) \to H_0(U\cap V) \to H_0(U) \oplus H_0(V) \to H_0(\mathbb{R}^2).$$

We know $H_1(\mathbb{R}^2) = 0$ and $H_0(\mathbb{R}^2) = \mathbb{Z}$, since $\mathbb{R}^2$ is path-connected. Let $\pi_0(U)$, $\pi_0(V)$, $\pi_0(U\cap V)$ denote the respective sets of path-components. Then our previous results on H_0 imply that the above sequence becomes

$$0 \to \bigoplus_{\pi_0(U\cap V)} \mathbb{Z} \to (\bigoplus_{\pi_0(U)} \mathbb{Z}) \oplus (\bigoplus_{\pi_0(V)} \mathbb{Z}) \to \mathbb{Z}.$$

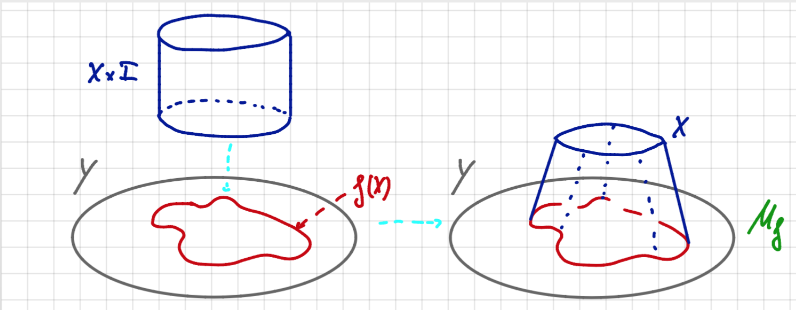
Since the sequence is exact, this implies

$$\#\pi_0(U\cap V) = \#\pi_0(U) + \#\pi_0(V) - 1.$$

Definition: Mapping cylinder

Let $f\colon X \to Y$ be a continuous map. The **mapping cylinder of f** is defined to be te quotient space

$$M_f := (X \times [0,1] \sqcup Y)/((x,0) \sim f(x)).$$



The mapping cylinder fits into a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_1"'] & & Y \\ & M_f \arrow[ur, "g"'] & \end{tikzcd}$$

where f_1 maps x to $(x,1)$ and g maps (x,t) to $f(x)$ for all $x \in X$ and $t \in [0,1]$ and $y \in Y$ to y.

$\boxed{3}$ **a)** Show that the inclusion $i\colon Y \hookrightarrow M_f$ is a deformation retract and g is a deformation retraction.

Solution: We observe that g leaves Y fixed and that there is a homotopy h between id_{M_f} and $i \circ g$ defined by

$$h\colon M_f \times [0,1] \to M_f, \begin{cases} ((x,t),s) \mapsto (x,st) & \text{for } 0 < s \leq 1 \\ (y,0) \mapsto y & \text{for } s = 0. \end{cases}$$

Note that this map is well-defined, since $(x,0)$ is identified with $f(x) \in Y$.

b) We can construct the **Möbius band** $M := M_f$ as the mapping cylinder of the map

$$f\colon S^1 \to S^1,\ z \mapsto z^2\ (S^1 \subset \mathbb{C}).$$

Determine the homology of the Möbius band.

Solution: Since $S^1 = Y \hookrightarrow M_f = M$ is a deformation retract, we know $H_n(M) = H_n(S^1)$. Thus $H_0(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}$ and $H_n(M) = 0$ for all $n \geq 2$.

c) For $n \geq 1$ and $m \in \mathbb{Z}$, let M_f be the mapping cylinder of a map

$$f\colon S^n \to S^n \text{ with } \deg(f) = m.$$

Show that $H_n(f_1)$ is given by multiplication with m.

Solution: Since $M_f \simeq Y = S^n$, we know $H_q(M_f) = H_q(S^n)$ for all q. Hence the map

$$H_n(f_1)\colon H_n(S^n) \cong \mathbb{Z} \to \mathbb{Z} \cong H_n(M_f)$$

is given by multiplication by an integer. In order to determine this integer, we use that the induced diagram in homology

$$\begin{tikzcd} H_n(S^n) \arrow[rr, "H_n(f)"] \arrow[dr, "H_n(f_1)"'] & & H_n(S^n) \\ & H_n(M_f) \arrow[ur, "H_n(g)"'] & \end{tikzcd}$$

commutes. Since $i\colon Y = S^n \hookrightarrow M_f$ is a deformation retract, , we know $H_n(i) = 1$ and therefore $H_n(g) = 1$. Since $H_n(f) = m$ by assumption, we must have $H_n(f_1) = m$ (by which we mean multiplication by m).

d) For $n \geq 1$ and $m \geq 2$, let M_f be the mapping cylinder of a map

$$f\colon S^n \to S^n \text{ with } \deg(f) = m.$$

Show that $X = S^n$ is not a weak retract of M_f.

Solution: The long exact sequence associated to the pair (M_f, X) looks like

$$0 \to H_n(S^n) \to H_n(M_f) \to H_n(M_f, S^n) \to 0.$$

Since f has degree m, this sequence is isomoprhic to

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0.$$

In particular, we know $H_n(M_f, S^n) \cong \mathbb{Z}/m$.

Now if $j\colon X = S^n \hookrightarrow M_f$ was a weak retract, then there would be a map $\rho\colon M_f \to S^n$ such that $\rho \circ j \simeq \mathrm{id}_{S^n}$. Consequently, the sequence would split. But $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/m$. Hence there cannot be such a map ρ.

$\boxed{4}$ We can consider the **real projective plane** $\mathbb{RP}^2$ as a two dimensional disk D^2 with a Möbius band M attached at its boundary. Writing $A = D^2$ and $B = M$, we have $A \cap B \simeq S^1$. Calculate the homology groups of $\mathbb{RP}^2$.

Solution: As a quotient of a path-connected space, $\mathbb{RP}^2$ is path-connected. Thus $H_0(\mathbb{RP}^2) = \mathbb{Z}$. From the homology of the circle and the Möbius band, the MVS implies that $H_n(\mathbb{RP}^2) = 0$ for all $n \geq 3$.

Now we look at the remaining MVS:

$$\begin{aligned} \cdots \to H_2(D^2) \oplus H_2(M) \to H_2(\mathbb{RP}^2) &\xrightarrow{\partial_2^{MV}} H_1(S^1) \to \\ \to H_1(D^2) \oplus H_1(M) \to H_1(\mathbb{RP}^2) &\xrightarrow{\partial_1^{MV}} H_0(S^1) \to \\ \to H_0(D^2) \oplus H_0(M) \to H_0(\mathbb{RP}^2) &\to 0. \end{aligned}$$

Throwing in information we have on the other homology groups, the sequence becomes

$$0 \to H_2(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_2^{MV}} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to H_1(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_1^{MV}} \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

The map $H_1(S^1) \cong \mathbb{Z} \to \mathbb{Z}H_1(M)$ corresponds to wrapping around the circle twice along the boundary of the Möbius strip. Hence the effect on homology is to send the generator in $H_1(S^1)$ to twice the generator in $H_1(M)$. In other words, the map is multiplication by 2. In paricular, the map is injective. Thus $H_2(\mathbb{R}\mathrm{P}^2) = 0$.

The map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is injective. Hence the map $H_1(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial_1^{MV}} \mathbb{Z}$ must be the zero map. Hence we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to H_1(\mathbb{R}\mathrm{P}^2) \to 0.$$

Thus $H_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2$.