MA3403 Algebraic Topology
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Exercise set 7

On this exercise set we are going to explore some additional, important topics. We will use them later in the lectures.

We start with an application of the excision property:

1 Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty open subsets. Show that if there is a homeomoprhism $\varphi\colon U \xrightarrow{\approx} V$, then we must have $n = m$.
(Hint: Take a point $x \in U$ and compare $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ and $H_n(U, U - \{x\})$.)

Now we are going to introduce a slight modification of homology:

Definition: Reduced homology

Let X be a nonempty topological space. We define the **reduced homology of X** to be the homology of the **augmented** complex of singular chains

$$\cdots S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\tilde{\epsilon}} \mathbb{Z} \to 0$$

where $\tilde{\epsilon}(\sum_i n_i \sigma_i) := \sum_i n_i$ (and $\mathbb{Z}$ is placed in degree -1). Recall that we checked before (in Lecture 4) that $\tilde{\epsilon} \circ \partial_1 = 0$. Hence the above sequence is a chain complex. Moreover, the construction of the augmented chain complex is **functorial**.
The nth reduced homology group of X is denoted by $\tilde{H}_n(X)$.

Reduced homology does not convey any new information, but is convenient for stating things. It also helps focusing on the important information, since it disregards the contribution in $H_0(X)$ which comes from a single point.

Here are some basic properties of reduced homology:

2 Let X be a nonempty topological space. Show that reduced homology satisfies the following properties:

a) $\tilde{H}_0(X) = \mathrm{Ker}\,(\epsilon\colon H_0(X) \to H_0(\mathrm{pt}))$.

b) $\tilde{H}_n(X) = H_n(X)$ for all $n \geq 1$.

c) If X is path-connected, then $\tilde{H}_0(X) = 0$.

d) For any point $x \in X$, $H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\{x\})$ and $\tilde{H}_n(X) \cong H_n(X, \{x\})$ for all $n \geq 0$.

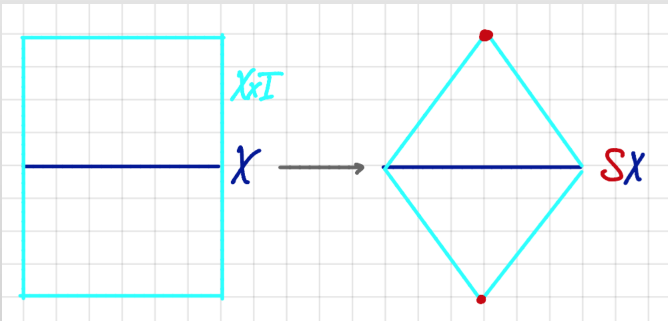
Now we move on towards an important construction on spaces, the suspension. It does not look spectacular, but will prove extremely useful and important later on in our studies of Algebraic Topology:

Definition: Suspension of a space

Let X be a topological space. The **suspension of X** is defined to be the quotient space

$$(X \times [0,1])/((x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X).$$

In other words, SX is constructed by taking a cylinder over X and then collapsing all points $X \times \{0\}$ to a point p_0 and all points $X \times \{1\}$ to a point p_1. The topology on SX is the quotient topology.



For any continuous map $f \colon X \to Y$, there is an induced continuous map

$$S(f) \colon SX \to SY, \; [x, s] \mapsto [f(x), s].$$

3 Our goal in this exercise is to understand SX a bit better and to show that there are isomorphisms

$$\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X) \text{ for all } n \geq 0$$

(Note that reduced homology makes it much easier to state this result. For, without reduced homology we would have to write $H_1(SX) \cong \mathrm{Ker}\,(H_0(X) \to H_0(\mathrm{pt}))$ for $n = 0$.)

a) Show that SX is path-connected and hence $H_0(SX) \cong \mathbb{Z}$.

b) Show that $SX - \{p_1\}$ is contractible.

c) Show that $SX - \{p_0, p_1\}$ is homotopy equivalent to X.

d) Use the Mayer-Vietoris sequence to determine $\tilde{H}_{n+1}(SX)$ for all $n \geq 0$.

A crucial example is the suspension of the sphere.

4

a) Show that the suspension SS^{n-1} of the $(n-1)$-sphere is homeomorphic to the n-sphere.

b) For $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $S(f)\colon SS^n \to SS^n$ be the induced map on suspensions. By using either $SS^n \approx S^{n+1}$ or $H_{n+1}(SS^n) \cong H_n(S^n)$ show that $H_{n+1}(S(f))$ is given by multiplication by an integer which we denote by $\deg(S(f))$. Show that $\deg(S(f)) = \deg(f)$.

c) For $n \geq 1$, show that, for any given $k \in \mathbb{Z}$, there is a map $f\colon S^n \to S^n$ with $\deg(f) = k$.