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MA3403 Algebraic Topology
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Solutions to exercise set 7

On this exercise set we are going to explore some additional, important topics. We will use them later in the lectures.

We start with an application of the excision property:

1 Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty open subsets. Show that if there is a homeomoprhism $\varphi\colon U \xrightarrow{\approx} V$, then we must have $n = m$.

(Hint: Take a point $x \in U$ and compare $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ and $H_n(U, U - \{x\})$.)

Solution: Pick some $x \in U$. Since U is open in $\mathbb{R}^n$, $Z := \mathbb{R}^n - U$ is closed, i.e., $\bar{Z} = Z$. Since $x \in U$, we have $Z \subset \mathbb{R}^n - \{x\}$. Hence we can apply excision and get an isomorphism

$$H_q(\mathbb{R}, \mathbb{R}^n - \{x\}) \xrightarrow{\cong} H_q(U, U - \{x\}) \text{ for all } q.$$

Since $\mathbb{R}^n$ is homotopy equivalent to D^n (both are contractible) and $\mathbb{R}^n - \{x\}$ is homotopy equivalent to S^{n-1}, we know

$$H_q(U, U - \{x\}) \cong H_q(D^n, S^{n-1}) = \begin{cases} 0 & \text{if } q \neq n \\ \mathbb{Z} & \text{if } q = n. \end{cases}$$

But if there was a homeomoprhism $\varphi\colon U \xrightarrow{\approx} V$, it would induce an isomorphism

$$H_q(U, U - \{x\}) \xrightarrow{\cong} H_q(V, V - \{\varphi(x)\}) \text{ for all } q.$$

Since the above argument applies to V and $\mathbb{R}^m$ as well, we see that an isomorphism can only exist if $n = m$.

Now we are going to introduce a slight modification of homology:

Definition: Reduced homology

Let X be a nonempty topological space. We define the **reduced homology of X** to be the homology of the **augmented** complex of singular chains

$$\cdots S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\tilde{\epsilon}} \mathbb{Z} \to 0$$

where $\tilde{\epsilon}(\sum_i n_i\sigma_i) := \sum_i n_i$ (and $\mathbb{Z}$ is placed in degree -1). Recall that we checked before (in Lecture 4) that $\tilde{\epsilon} \circ \partial_1 = 0$. Hence the above sequence is a chain complex.

Moreover, the construction of the augmented chain complex is **functorial**.
The nth reduced homology group of X is denoted by $\tilde{H}_n(X)$.

Reduced homology does not convey any new information, but is convenient for stating things. It also helps focusing on the important information, since it disregards the contribution in $H_0(X)$ which comes from a single point.

Here are some basic properties of reduced homology:

$\boxed{2}$ Let X be a nonempty topological space. Show that reduced homology satisfies the following properties:

a) $\tilde{H}_0(X) = \mathrm{Ker}\,(\epsilon\colon H_0(X) \to H_0(\mathrm{pt}))$.
Solution: By definition $\tilde{H}_0(X) = \mathrm{Ker}\,(\tilde{\epsilon})/(\mathrm{Im}\,\partial_1)$. Since $H_0(x) = Z_0(X)/(\mathrm{Im}\,\partial_1)$, it suffices to observe that the homomorphism $\epsilon\colon H_0(X) \to H_0(\mathrm{pt})$ induced by the canonical map $X \to \mathrm{pt}$ agrees with $\tilde{\epsilon}$.

b) $\tilde{H}_n(X) = H_n(X)$ for all $n \geq 1$.
Solution: This is clear, since the singular chain complex and its augmented version are identical in all degrees $* \geq 0$.

c) If X is path-connected, then $\tilde{H}_0(X) = 0$.
Solution: If X is path-connected, then $H_0(X) \cong \mathbb{Z}$ and therefore $\mathrm{Ker}\,(H_0(X) \to H_0(\mathrm{pt})) = 0$.

d) For any point $x \in X$, $H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\{x\})$ and $\tilde{H}_n(X) \cong H_n(X, \{x\})$ for all $n \geq 0$.
Solution: The composition $\{x\} \hookrightarrow X \xrightarrow{\rho} \{x\}$ is the identity. By our result on weak retracts, this implies $H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\{x\})$. Moreover, the induced map $H_n(\rho)$ splits the long exact sequence of relative homology grops

$$\cdots \to H_n(\{x\}) \to H_n(X) \to H_n(X, \{x\}) \to \cdots$$

Hence

$$H_n(X, \{x\}) \cong \mathrm{Ker}\,(H_0(X) \to H_0(\mathrm{pt})) \cong \tilde{H}_n(X).$$

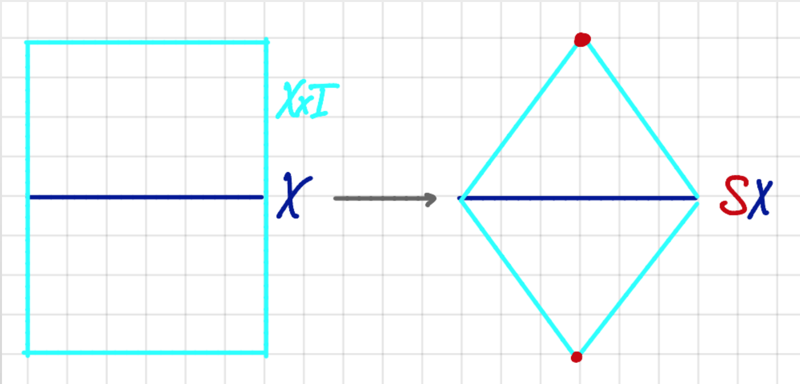
Now we move on towards an important construction on spaces, the suspension. It does not look spectacular, but will prove extremely useful and important later on in our studies of Algebraic Topology:

Definition: Suspension of a space

Let X be a topological space. The **suspension of X** is defined to be the quotient space

$$(X \times [0,1])/((x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X).$$

In other words, SX is constructed by taking a cylinder over X and then collapsing all points $X \times \{0\}$ to a point p_0 and all points $X \times \{1\}$ to a point p_1. The topology on SX is the quotient topology.



For any continuous map $f\colon X \to Y$, there is an induced continuous map

$$S(f)\colon SX \to SY,\ [x,s] \mapsto [f(x),s].$$

$\boxed{3}$ Our goal in this exercise is to understand SX a bit better and to show that there are isomorphisms

$$\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X) \text{ for all } n \geq 0$$

(Note that reduced homology makes it much easier to state this result. For, without reduced homology we would have to write $H_1(SX) \cong \mathrm{Ker}\,(H_0(X) \to H_0(\mathrm{pt}))$ for $n = 0$.)

a) Show that SX is path-connected and hence $H_0(SX) \cong \mathbb{Z}$.
Solution: Let $[x,s]$ be an arbitrary point in SX. Then we can define the continuous map $[0,1] \to SX,\ t \mapsto [x,st]$. This is a path from p_0 to $[x,s]$. Hence we can connect every point in SX via a path with p_0. Going back and forth shows that we can connect any two points in SX via a path. In other words, SX is path-connected and $H_0(SX) \cong \mathbb{Z}$.

b) Show that $SX - \{p_1\}$ is contractible.
Solution: We need to prove that the identity map of $SX - \{p_1\}$ is homotopy equivalent to the constant map $SX - \{p_1\} \to \{p_0\}$. To do this we just write down a homotopy:

$$h\colon SX - \{p_1\} \times [0,1] \to SX - \{p_1\},\ ([x,s],t) \mapsto [x,st].$$

For $t = 0$, we have $h([x,s],0) = [x,0] = p_0$ and, for $t = 1$, we have $h([x,s],1) = [x,s]$.

c) Show that $SX - \{p_0,p_1\}$ is homotopy equivalent to X.
Solution: Actually, the inclusion

$$i\colon X \times \{1/2\} \hookrightarrow SX - \{p_0,p_1\}$$

is a deformation retract. For, we can define a map

$$\rho\colon SX - \{p_0, p_1\} \to X \times \{1/2\},\ [x, s] \mapsto (x, 1/2)$$

such that $\rho \circ i = \mathrm{id}_{X\times\{1/2\}}$.
It remains to check $i \circ \rho \simeq \mathrm{id}_{SX-\{p_0,p_1\}}$. To show this we define a homotopy

$$h\colon SX - \{p_0, p_1\} \times [0, 1] \to SX - \{p_0, p_1\},\ ([x, s], t) \mapsto [x, 1/2 + t(s - 1/2)].$$

For $t = 0$, we get $h([x, s], 0) = [x, 1/2] = i(\rho([x, s])$ and, for $t = 1$, we get $h([x, s], 1) = [x, s]$.

d) Use the Mayer-Vietoris sequence to determine $\tilde{H}_{n+1}(SX)$ for all $n \geq 0$.
Solution: We can cover SX by $A = SX - \{p_0\}$ and $B = SX - \{p_1\}$ with $A \cap B = SX - \{p_0, p_1\}$. The Mayer-Vietoris sequence looks like

$$\cdots \to H_{n+1}(A) \oplus H_{n+1}(B) \to H_{n+1}(SX) \xrightarrow{\partial^{MV}_{n+1}} H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to \cdots$$

We have proved that A is contractible. The argument shows B is contractible. Hence $H_q(A) = H_q(B) = 0$ for all $q \geq 1$. Thus for $n \geq 1$, the exact sequence reduces to

$$0 \to H_{n+1}(SX) \xrightarrow{\partial^{MV}_{n+1}} H_n(A \cap B) \to 0$$

This shows that ∂^{MV}_{n+1} is an isomorphism. Since we also know that $H_n(i)\colon H_n(A\cap B) \xrightarrow{\cong} H_n(X)$ is an isomorphism as well, we get the desired isomorphism $H_{n+1}(SX) \xrightarrow{\cong} H_n(X)$ for all $n \geq 1$.
For $n = 0$, the sequence is

$$H_1(A) \oplus H_1(B) \to H_1(SX) \xrightarrow{\partial^{MV}_1} H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(SX).$$

Using what we laready know, this becomes

$$0 \to H_1(SX) \xrightarrow{\partial^{MV}_1} H_0(X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

where the last map is defined by $(a, b) \mapsto a + b$ (we have done this before). The kernel of this map is isomorphic to $\mathbb{Z}$. Hence the image of the homomorphism $H_0(X) \to \mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $\mathbb{Z}$. This implies that $H_1(SX)$ is the kernel of the homomorphism $H_0(X) \to H_0(\mathrm{pt})$ induced by the canonical map $X \to \mathrm{pt}$.
Note that we could rephrase this result as $H_1(SX) \oplus \mathbb{Z} \cong H_0(X)$.

A crucial example is the suspension of the sphere.

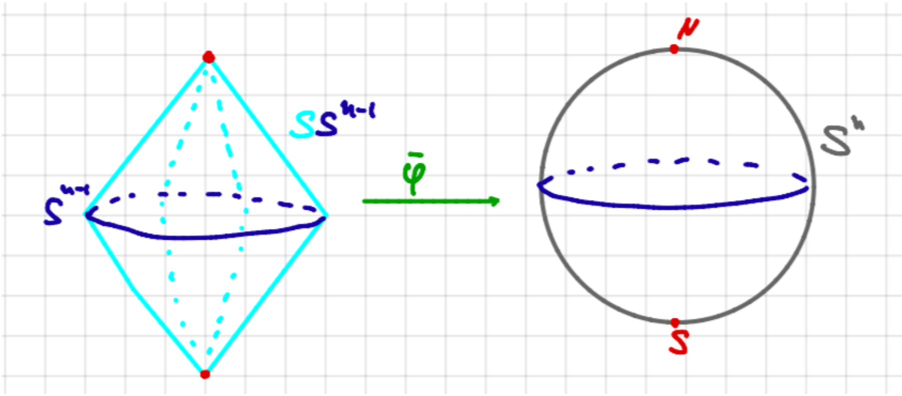
$\boxed{4}$ **a)** Show that the suspension SS^{n-1} of the $(n-1)$-sphere is homeomorphic to the n-sphere.
Solution: Let $N = (0, \ldots, 0, 1)$ be the north- and $S = (0, \ldots, 0, -1)$ be the south-pole of S^n. We can define a map

$$\varphi\colon S^{n-1} \times [0, 1] \to S^n,\ (x, s) \mapsto (x\sqrt{1 - (2s - 1)^2}, 2s - 1)$$

By picturing this map, we convince ourselves that it is continuous.Since it is constant on the equivalence classes which define SS^{n-1}, it descends to a continuous map

$$\bar{\varphi}\colon SS^{n-1} \to S^n.$$

This map is easily checked to be bijective. Since $S^{n-1} \times I$ is compact, its continuous image SS^{n-1} is also compact. Hence $\bar{\varphi}$ is a homeomorphism, since it is a continuous bijective map between compact spaces.



b) For $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $S(f)\colon SS^n \to SS^n$ be the induced map on suspensions. By using either $SS^n \approx S^{n+1}$ or $H_{n+1}(SS^n) \cong H_n(S^n)$ show that $H_{n+1}(S(f))$ is given by multiplication by an integer which we denote by $\deg(S(f))$. Show that $\deg(S(f)) = \deg(f)$.

Solution: We obtained the isomorphism $H_{n+1}(SS^n) \cong H_n(S^n)$ as the connecting homomorphism in a Mayer-Vietoris sequence. This construction is functorial so that we get a commutative diagram

$$\begin{tikzcd}
H_{n+1}(SS^n) \arrow[r, "H_n(S(f))"] \arrow[d, "\partial"', "\cong"] & H_{n+1}(SS^n) \arrow[d, "\cong"', "\partial"] \\
H_n(S^n) \arrow[r, "H_n(f)"'] & H_n(S^n).
\end{tikzcd}$$

Hence if $H_n(f)$ is multiplication by d, then so is $H_n(S(f))$.

c) For $n \geq 1$, show that, for any given $k \in \mathbb{Z}$, there is a map $f\colon S^n \to S^n$ with $\deg(f) = k$.

Solution: We know this fact for $n = 1$, since the map $f_k\colon S^1 \to S^1$, $z \mapsto z^k$ has degree k. By the previous point, it suffices to take the successive suspension of f_k to obtain a map $S^n \to S^n$ with degree k.