MA3403 Algebraic Topology
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Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 8

1 Show that the two different cell structures on S^n we discussed in the lecture lead to cellular chain complexes which have the same homology groups.

2 Show the statement of the lecture that the isomorphism between the homology of the cellular chain complex is functorial in the following sense: Let $f\colon X \to Y$ be a **cellular** (or filtration-preserving) map between cell complexes, i.e., $f(X_n) \subseteq Y_n$ for all n. Show that f induces a homomorphism of cellular chain complexes $C_*(f)\colon C_*(X) \to C_*(Y)$ which fits into a commutative diagram

$$\begin{array}{ccc} H_*(C_*(X)) & \xrightarrow{H_*(C_*(f))} & H_*(C_*(X)) \\ \cong\downarrow & & \downarrow\cong \\ H_*(X) & \xrightarrow[H_*(f)]{} & H_*(Y). \end{array}$$

3 Let X be a cell complex and A a subcomplex. Show that the quotient X/A inherits a cell structure such that the quotient map $q\colon X \to X/A$ is cellular.

4 Consider S^1 with its standard cell structure, i.e. one 0-cell e^0 and one 1-cell e^1. Let X be a cell complex obtained from S^1 by attaching two 2-cells e_1^2 and e_2^2 to S^1 by maps f_2 and f_3 of degree 2 and 3, respectively. We may express this construction as

$$X = S^1 \cup_{f_2} e_1^2 \cup_{f_3} e_2^2.$$

a) Determine all the subcomplexes of X.

b) Determine the cellular chain complex of X and compute the homology of X.

c) For each subcomplex Y of X, compute the homology of Y and of the quotient space X/Y.

d) As a more challenging task show that the only subcomplex Y of X for which $X \xrightarrow{q} X/Y$ is a homotopy equivalence is the trivial subcomplex consisting only of the 0-cell.
(Hint: Study the effect of $H_2(q)$.)
Note that one can nevertheless show that X is homotopy equivalent to S^2. But we are lacking some results in homotopy theory to prove this.

For the next exercise, note that if X and Y are cell complexes, then $X \times Y$ is a cell complex with cells the products $e^n_{\alpha,X} \times e^m_{\beta,Y}$ where $e^n_{\alpha,X}$ ranges over the cells of X and $e^m_{\beta,Y}$ ranges over the cells of Y.

5 Show that the Euler characteristic has the following properties:

a) If X and Y are finite cell complexes, then

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

b) Assume the finite cell complex X is the union of the two union of two subcomplexes A and B. Then

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$