Norwegian University of Science
and Technology
Department of Mathematical
Sciences

MA3403 Algebraic Topology
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Solutions to exercise set 8

1 Show that the two different cell structures on S^n we discussed in the lecture lead to cellular chain complexes which have the same homology groups.

Solution: **(a)** The first cell structure of S^n consisted of only one 0-cell and one n-cell. Hence the cellular chain complex looks like

$$0 \to \mathbb{Z} \to 0 \to \ldots \to \mathbb{Z} \to 0$$

with the left-hand copy of $\mathbb{Z}$ in dimension n and the other one in dimension 0. This gives us $H_i(S^n) = \mathbb{Z}$ for $i = 0, n$ and $H_i(S^n) = 0$ for all other i. Note that for $n = 1$, this complex is

$$0 \to \mathbb{Z} \xrightarrow{d=0} \mathbb{Z} \to 0$$

where we know $d = 0$, since $H_0(S^n) = \mathbb{Z}$ as a path-connected space. This implies $H_1(S^1) = \mathbb{Z}$ as well.

(b) The second cell structure of S^n consisted of exactly two i-cells for $i = 0, \ldots, n$. Hence the cellular chain complex looks like

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \oplus \mathbb{Z} \to 0$$

where the nontrivial terms start in dimension n and end in dimension 0.

We learned in the lecture that, in order to determine the differential d_k, we need to understand the effect of the induced diagonal map

$$\begin{array}{ccccc} S^{k-1}_{\alpha_1} \sqcup S^{k-1}_{\alpha_2} & \xrightarrow{f_k} & X_{k-1} = S^{k-1} & \xrightarrow{\text{pinch equator}} & S^{k-1}/S^{k-2} \\ & \searrow^{g} & \downarrow & \nearrow & \\ & & S^{k-1}_{\beta_1} \vee S^{k-1}_{\beta_2}. & & \end{array}$$

But both summands S^{k-1} are attached to $X_{k-1} = S^{k-1}$ via the identity map. Hence the map g is the disjoint union of two homeomorphisms of $k-1$-spheres. Thus the degree of $S^{k-1}_{\alpha_i} \to S^{k-1}_{\beta_j}$ is equal to 1 for each pair (i, j).

Thus the differential d_k is given by

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}, \ (1,0) \mapsto (1,1) \text{ and } (0,1) \mapsto (1,1)$$

Thus both the kernel and the image of d_k are isomorphic to $\mathbb{Z}$. That means that the homology of the second cellular chain complex yields again $H_n(S^n) = \mathbb{Z}$ and $H_0(S^n) = \mathbb{Z}$, whereas for $i \neq 0, n$ the kernels and images cancel out and hence $H_i(S^n) = 0$.

2 Show the statement of the lecture that the isomorphism between the homology of the cellular chain complex is functorial in the following sense: Let $f\colon X \to Y$ be a **cellular** (or filtration-preserving) map between cell complexes, i.e., $f(X_n) \subseteq Y_n$ for all n. Show that f induces a homomorphism of cellular chain complexes $C_*(f)\colon C_*(X) \to C_*(Y)$ which fits into a commutative diagram

$$\begin{array}{ccc} H_*(C_*(X)) & \xrightarrow{H_*(C_*(f))} & H_*(C_*(X)) \\ \cong\downarrow & & \downarrow\cong \\ H_*(X) & \xrightarrow[H_*(f)]{} & H_*(Y). \end{array}$$

Solution: Since f is cellular, it induces a map of pairs $(X_n, X_{n-1}) \to (Y_n, Y_{n-1})$ for each n and hence a map of relative homology groups

$$C_n(X) = H_n(X_n, X_{n-1}) \xrightarrow{H_n(f)} H_n(Y_n, Y_{n-1}) = C_n(Y).$$

This yields the induced map $C_*(f)$.

This map is a chain homomorphism, since we have a commutative diagram

$$\begin{array}{ccccc} H_n(X_n, X_{n-1}) & \xrightarrow{\partial_n^X} & H_{n-1}(X_{n-1}) & \xrightarrow{j_{n-1}^X} & H_{n-1}(X_{n-1}, X_{n-2}) \\ {\scriptstyle H_n(f_{|X_n})}\downarrow & & {\scriptstyle H_n(f_{|X_{n-1}})}\downarrow & & \downarrow{\scriptstyle H_n(f_{|X_{n-1}})} \\ H_n(Y_n, Y_{n-1}) & \xrightarrow{\partial_n^Y} & H_{n-1}(Y_{n-1}) & \xrightarrow{j_{n-1}^Y} & H_{n-1}(Y_{n-1}, Y_{n-2}). \end{array}$$

To check the compatibility with the isomorphism to the singular homologies of X and Y, we recall the construction of this isomorphism. Since f preserves the filtration by skeleta, we get the commutative diagram

$$\begin{array}{ccc} H_n(C_*(X)) & \xrightarrow{H_n(C_*(f))} & H_n(C_*(X)) \\ \cong\downarrow & & \downarrow\cong \\ H_n(X_n)/\mathrm{Im}\,(\partial_n^X) & \xrightarrow{H_n(f_{|X_n})} & H_n(Y_n)/\mathrm{Im}\,(\partial_n^Y) \\ \cong\downarrow & & \downarrow\cong \\ H_n(X_{n+1}) & \xrightarrow{H_n(f_{|X_{n+1}})} & H_n(Y_{n+1}) \\ \cong\downarrow & & \downarrow\cong \\ H_n(X) & \xrightarrow[H_n(f)]{} & H_n(Y). \end{array}$$

3 Let X be a cell complex and A a subcomplex. Show that the quotient X/A inherits a cell structure such that the quotient map $q\colon X \to X/A$ is cellular.

Solution: The cells of the quotients are the cells of X which lie in the complement $X - A$ plus one new 0-cell which is the image of $A \to X/A$. Note that, since A is a subcomplex, the cells of X are either in A or in $X - A$. If $f_\alpha\colon S_\alpha^{n-1} \to X_{n-1}$ is an

attaching map of an n- cell in $X - A$, then the attaching map of the corresponding n-cell in X/A is the composite

$$S^{n-1}_\alpha \xrightarrow{f_\alpha} X_{n-1} \to X_{n-1}/A_{n-1}$$

where we use that the $(n-1)$-skeleton A_{n-1} of A is a subspace of X_{n-1}, since A is a subcomplex.

The quotient satisfies $q(X_n) \subset X_n/A_n = (X/A)_n$. Hence it is cellular.

4 Consider S^1 with its standard cell structure, i.e. one 0-cell e^0 and one 1-cell e^1. Let X be a cell complex obtained from S^1 by attaching two 2-cells e^2_1 and e^2_2 to S^1 by maps f_2 and f_3 of degree 2 and 3, respectively. We may express this construction as

$$X = S^1 \cup_{f_2} e^2_1 \cup_{f_3} e^2_2.$$

a) Determine all the subcomplexes of X.

Solution: We have a trivial subcomplex $*$ consisting just of the 0-cell. By construction of X, S^1 is also a subcomplex. Then we have two subcomplexes with just one 2-cell attached to S^1. A complex $A = S^1 \cup_{f_2} e^2_1$ with attaching map f_2 and a complex $B = S^1 \cup_{f_3} e^2_2$ one with attaching map f_3. Their skeleta are $* = \mathrm{Sk}_0 A \subset S^1 = \mathrm{Sk}_1 A \subset \mathrm{Sk}_2 A = A$ and $* = \mathrm{Sk}_0 B \subset S^1 = \mathrm{Sk}_1 B \subset \mathrm{Sk}_2 B = B$. (There is no other subcomplex, since there are no other attaching maps in X.)

b) Determine the cellular chain complex of X and compute the homology of X.

Solution: The cellular chain complex is

$$\mathbb{Z}[e^2_1] \oplus \mathbb{Z}[e^2_2] \xrightarrow{d_2=(2,3)} \mathbb{Z}[e^1] \xrightarrow{d_1=0} \mathbb{Z}[e^0] \to 0,$$

where d_2 is the map $(a, b) \mapsto 2a + 3b$.

Since 2 and 3 are relatively prime, we know

$$2a + 3b = 0 \iff (a, b) = m(3, -2) \text{ for some } m \in \mathbb{Z}.$$

Hence the kernel of d_2 is

$$\mathrm{Ker}\,(d_2) = \{(3m, -2m) \in \mathbb{Z} \oplus \mathbb{Z} : m \in \mathbb{Z}\}.$$

On the other hand, 2 and 3 being relatively prime also implies that there are a and b in $\mathbb{Z}$ such that $2a + 3b = 1$. Thus d_2 is surjective. This shows that the homology of X is given by

$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 2 \\ 0 & n = 1 \\ \mathbb{Z} & n = 0. \end{cases}$$

c) For each subcomplex Y of X, compute the homology of Y and of the quotient space X/Y.

Solution: First of all, we observe that the homology groups of all these spaces and quotients vanish in dimensions ≥ 3, since the complexes are at most two-dimensional.

We know the homologies of the subcomplexes $*$ and S^1. The cellular chain complex of A is

$$\mathbb{Z}[e_1^2] \xrightarrow{d_2^A=2} \mathbb{Z}[e^1] \xrightarrow{d_1=0} \mathbb{Z}[e^0] \to 0.$$

Hence

$$H_n(A) \cong \begin{cases} 0 & n=2 \\ \mathbb{Z}/2 & n=1 \\ \mathbb{Z} & n=0. \end{cases}$$

Similarly, the cellular chain complex of B is

$$\mathbb{Z}[e_2^2] \xrightarrow{d_2^B=3} \mathbb{Z}[e^1] \xrightarrow{d_1=0} \mathbb{Z}[e^0] \to 0.$$

$$H_n(B) \cong \begin{cases} 0 & n=2 \\ \mathbb{Z}/3 & n=1 \\ \mathbb{Z} & n=0. \end{cases}$$

Now we look at the quotients. Since all spaces are path-connected, we know H_0 is always isomorphic to $\mathbb{Z}$. For the subcomplex $*$, we have $X/* = X$. Since we consider subcomplexes, we just learned that the corresponding quotients are again cell complexes. Hence we can use their celllar chain complexes.
For X/S^1, we get

$$0 \to \mathbb{Z}[e_1^2] \oplus \mathbb{Z}[e_2^2] \to 0 \to \mathbb{Z}[e^0] \to 0.$$

Hence

$$H_n(X/S^1) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=2 \\ 0 & n=1 \\ \mathbb{Z} & n=0. \end{cases}$$

For X/A, we get

$$0 \to \mathbb{Z}[e_2^2] \to 0 \to \mathbb{Z}[e^0] \to 0.$$

Hence

$$H_n(X/A) \cong \begin{cases} \mathbb{Z} & n=2 \\ 0 & n=1 \\ \mathbb{Z} & n=0. \end{cases}$$

And similarly for X/B:

$$0 \to \mathbb{Z}[e_1^2] \to 0 \to \mathbb{Z} \to 0.$$

Hence

$$H_n(X/B) \cong \begin{cases} \mathbb{Z} & n=2 \\ 0 & n=1 \\ \mathbb{Z} & n=0. \end{cases}$$

Actually, these computations could also have been deduced by the observation that there are homeomorphisms

$$X/S^1 \approx S^2 \vee S^2, \ X/A \approx e^0 \cup e_2^2 = S^2, \text{ and } X/B \approx e^0 \cup e_2^1 = S^2.$$

d) As a more challenging task show that the only subcomplex Y of X for which $X \xrightarrow{q} X/Y$ is a homotopy equivalence is the trivial subcomplex consisting only of the 0-cell.

(Hint: Study the effect of $H_2(q)$.)

Note that one can nevertheless show that X is homotopy equivalent to S^2. But we are lacking some results in homotopy theory to prove this.

Solution: The map $X \to X/* = X$ is obviously a homotopy equivalence. Since $H_2(X/S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ but $H_2(X) \cong \mathbb{Z}$, $X \to X/S^1$ cannot be a homotopy equivalence.

For X/B we cannot use a similar argument, since both X and X/B have the same homology. However, the quotient map $q\colon X \to X/B$ is not a homotopy equivalence. To show this, we compare the two cellular chain complexes. The map q induces a commutative diagram

$$\begin{array}{ccccccc} \mathbb{Z}[e_1^2, e_2^2] & \xrightarrow{d_2=(2,3)} & \mathbb{Z}[e^1] & \longrightarrow & \mathbb{Z}[e^0] & \longrightarrow & 0 \\ \downarrow{\scriptstyle C_2(q)} & & \downarrow & & \downarrow & & \\ \mathbb{Z}[e_1^2] & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[e^0] & \longrightarrow & 0. \end{array}$$

The left-hand vertical map $C_2(q)$ sends the generator corresponding to e_1^2 to e_1^2 and e_2^2 to 0. But the elements in the kernel of $d_2 = (2,3)$ are elements of the form $m(3e_1^2, -2e_2^2)$ for $m \in \mathbb{Z}$. Hence an element $m(3e_1^2, -2e_2^2)$ in the kernel of d_2 is sent to $3me_1^2$ by $C_2(q)$. That implies that the image of

$$H_2(q)\colon H_2(X) \to H_2(X/B)$$

is isomorphic to $3\mathbb{Z}$ in $\mathbb{Z}$. Hence $H_2(q)$ is not surjective, and therefore q is not a homotopy equivalence.

A similar argument with the roles of e_1^2 and e_2^2 reversed yields the result for $X \to X/A$.

For the next exercise, note that if X and Y are cell complexes, then $X \times Y$ is a cell complex with cells the products $e^n_{\alpha,X} \times e^m_{\beta,Y}$ where $e^n_{\alpha,X}$ ranges over the cells of X and $e^m_{\beta,Y}$ ranges over the cells of Y.

5 Show that the Euler characteristic has the following properties:

a) If X and Y are finite cell complexes, then

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

Solution: Let b_n be the number of n-cells in X and c_m be the number of m-cells in Y. Then $X \times Y$ has $b_n c_m$ many $n+m$-cells. Thus its Euler characteristic is

$$\chi(X \times Y) = \sum_{n,m} (-1)^{n+m} b_n c_m = (\sum_n (-1)^n b_n)(\sum_n (-1)^m c_m) = \chi(X)\chi(Y).$$

b) Assume the finite cell complex X is the union of the two union of two subcomplexes A and B. Then

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Solution: Since X is a finite cell complex, so are A, B and $A \cap B$, since they are subcomplexes of X. Hence their Euler characteristics are defined.

We could prove the statement by merely counting the numbers of cells in each complex. But here we present the argument that works for the Euler characteristic in general.

So we apply the Mayer-Vietoris sequence and the rank formula of the lecture. Since the terms $H_n(X)$ and $H_n(A \cap B)$ are always to steps away from each other in the MVS, there ranks have the same sign, whereas the rank of $H_n(A) \oplus H_n(B)$ has the opposite sign. Hence we get

$$\begin{aligned} 0 &= \sum_n (-1)^n \left(\operatorname{rank}(H_n(X)) + \operatorname{rank}(H_n(A \cap B)) - \operatorname{rank}(H_n(A) \oplus H_n(B))\right) \\ &= \sum_n (-1)^n \left(\operatorname{rank}(H_n(X)) + \operatorname{rank}(H_n(A \cap B)) - \operatorname{rank}(H_n(A)) + \operatorname{rank}(H_n(B))\right) \\ &= \chi(X) + \chi(A \cap B) - \chi(A) - \chi(B). \end{aligned}$$