Norwegian University of Science
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MA3403 Algebraic Topology
Fall 2018

Exercise set 9

We start with proving some properties about the Tor-functor that we mentioned in the lecture.

1 Let M be an abelian group.

a) Let A be an abelian group and $0 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M. Consider the chain complex K_* given by

$$0 \to A \otimes F_1 \to A \otimes F_0 \to 0 \tag{1}$$

with $A \otimes F_1$ in dimension one and $A \otimes F_0$ in dimension zero.
Show that $H_1(K_*) = \mathrm{Tor}(A, M)$ and $H_0(K_*) = A \otimes M$.

b) Show that for any short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

there is an associated long exact sequence

$$0 \to \mathrm{Tor}(A, M) \to \mathrm{Tor}(B, M) \to \mathrm{Tor}(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

c) For any abelian group A, show that $\mathrm{Tor}(A, M) = 0$ if A or M is a free abelian group.

d) Show that, for any abelian group A, Tor is symmetric:

$$\mathrm{Tor}(A, M) \cong \mathrm{Tor}(M, A).$$

(Note: It takes a wile to show this without any additional tools from homological algebra. It is important though that you think about what you have to do to prove the statement and that you try. The arguments used in the lecture are useful here, too.)

e) For any abelian group A, show that $\mathrm{Tor}(A, M) = 0$ if A or M is torsion-free, i.e., the subgroup of torsion elements vanishes.
(Hint: You may want to use the fact that a finitely generated torsion-free abelian group is free.)

f) Let A be an abelian group and let $T(A)$ denote the subgroup of torsion elements in A. Show $\mathrm{Tor}(A, M) = \mathrm{Tor}(T(A), M)$.
(Hint: $A/T(A)$ is torsion-free.)

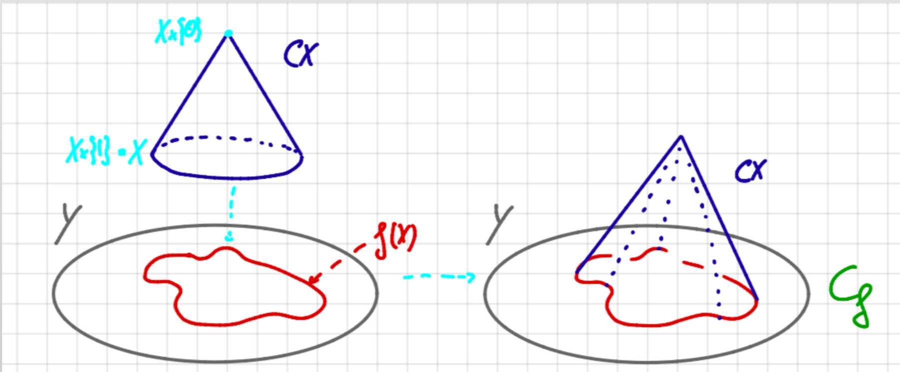
Definition: Mapping cone

For a space X, the **cone** over X is defined as the quotient space

$$CX := (X \times [0,1])/(X \times \{0\}).$$

Let $f\colon X \to Y$ be a continuous map. The **mapping cone of f** is defined to be te quotient space

$$C_f := (CX \sqcup Y)/((x,1) \sim f(x)) = Y \cup_f CX.$$



2 Let $f\colon X \to Y$ be a continuous map, and let M be an abelian group.

a) Show that the homology of the mapping cone of f fits into a long exact sequence

$$\cdots \to \tilde{H}_{n+1}(C_f;M) \to \tilde{H}_n(X;M) \xrightarrow{f_*} \tilde{H}_n(Y;M) \xrightarrow{i_*} \tilde{H}_n(C_f;M) \to \tilde{H}_{n-1}(X;M) \to \cdots$$

(Hint: Relate C_f to the mapping cylinder M_f from a previous exercise set.)

b) Show that f induces an isomorphism in homology with coefficients in M if and only if $\tilde{H}_*(C_f;M) = 0$.

The first part of the next exercise requires some familiarity with Tor beyond the discussion of the lecture. But you should think about it anyway and definitely note the statement.

3 Let X and Y be topological spaces.

a) Show that $\tilde{H}_*(X;\mathbb{Z}) = 0$ if and only if $H_*(X;\mathbb{Q}) = 0$ and $H_*(X;\mathbb{F}_p) = 0$ for all primes p.
(Hint: Use the UCT with $\mathbb{F}_p$-coefficients to control the torsion part and with $\mathbb{Q}$-coefficients to control the torsion-free part. Then apply suitable points of the first exercise.)

b) Show that a map $f\colon X \to Y$ induces an isomorphism in integral homology if and only if it induces an isomorphism in homology with rational coefficients and in homology with $\mathbb{F}_p$-coefficients for all primes p.

4 Let X be a finite cell complex, and let $\mathbb{F}_p$ be a field with p elements. Show that the Euler characteristic $\chi(X)$ can be computed by the formula

$$\chi(X) = \sum_i \dim_{\mathbb{F}_p} (-1)^i H_i(X; \mathbb{F}_p).$$

In other words, $\chi(X)$ is the alternating sum of the dimensions of the $\mathbb{F}_p$-vector spaces $H_i(X; \mathbb{F}_p)$.

(Hint: Use the UCT.)

5 Use the Künneth Theorem of the lecture to show:

a) The homology of the product $X \times S^k$ is satisfies

$$H_n(X \times S^k) \cong H_n(X) \oplus H_{n-k}(X).$$

b) The homology of the n-torus T^n defined as the n-fold product $T^n = S^1 \times \ldots \times S^1$ is given by

$$H_i(T^n) \cong \mathbb{Z}^{\binom{n}{i}}.$$