Norwegian University of Science and Technology
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MA3403 Algebraic Topology
Fall 2018

Solutions to exercise set 9

We start with proving some properties about the Tor-functor that we mentioned in the lecture.

$\boxed{1}$ Let M be an abelian group.

a) Let A be an abelian group and $0 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M. Consider the chain complex K_* given by

$$0 \to A \otimes F_1 \to A \otimes F_0 \to 0 \tag{1}$$

with $A \otimes F_1$ in dimension one and $A \otimes F_0$ in dimension zero.
Show that $H_1(K_*) = \mathrm{Tor}(A, M)$ and $H_0(K_*) = A \otimes M$.
Solution: By definition, $\mathrm{Tor}(A, M)$ is the kernel of $A \otimes F_1 \to A \otimes F_0$. Since the image of the differential into $A \otimes F_1$ is trivial, we get

$$H_1(K_*) = \mathrm{Ker}\,(A \otimes F_1 \to A \otimes F_0) = \mathrm{Tor}(A, M).$$

We also know H_0 of this complex: Since the augmented complex

$$A \otimes F_1 \to A \otimes F_0 \to A \otimes M \to 0$$

is exact, know

$$A \otimes M \cong \mathrm{Coker}(A \otimes F_1 \to A \otimes F_0) = (A \otimes F_0)/\mathrm{Im}\,(A \otimes F_1 \to A \otimes F_0).$$

Thus

$$\begin{aligned} H_0(K_*) &= \mathrm{Ker}\,(A \otimes F_0 \to 0)/\mathrm{Im}\,(A \otimes F_1 \to A \otimes F_0) \\ &(A \otimes F_0)/\mathrm{Im}\,(A \otimes F_1 \to A \otimes F_0) \\ &= A \otimes M. \end{aligned}$$

b) Show that for any short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

there is an associated long exact sequence

$$0 \to \mathrm{Tor}(A, M) \to \mathrm{Tor}(B, M) \to \mathrm{Tor}(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

Solution: Let $0 \to E_1 \to E_0 \to A \to 0$ be a free resolution of A, and $0 \to F_1 \to F_0 \to C \to 0$ be a free resolution of C. This data gives us a free resolution of B by forming direct sums:

$$0 \to E_1 \oplus F_1 \to E_0 \oplus F_0 \to B \to 0.$$

By the result of the previous lecture, we can lift the maps in the short exact sequence to maps of resolutions

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r, dotted] \arrow[d] & E_1 \oplus F_1 \arrow[r, dotted] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_0 \arrow[r, dotted] \arrow[d] & E_0 \oplus F_0 \arrow[r, dotted] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The horizontal sequences are short exact, since the middle term is a direct sum of the other terms. Hence we get a short exact sequence of chain complexes

$$0 \to E_* \to E_* \oplus F_* \to F_* \to 0.$$

Since all three complexes consist of free abelian groups, applying $- \otimes M$ yields a short exact sequence of chain complexes

$$0 \to E_* \otimes M \to E_* \otimes M \oplus F_* \otimes M \to F_* \otimes M \to 0$$

where we use for the middle term that tensor products distribute over direct sums.

By taking homology of these chain compelxes, we get an induced long exact sequence of the associated homology groups. This is the desired exact sequence together with the identifiction of H_1 with Tor and H_0 with the tensor product of the previous point.

c) For any abelian group A, show that $\mathrm{Tor}(A, M) = 0$ if A or M is a free abelian group.

Solution: If M is free, we can choose $F_0 = M$ and $F_1 = 0$ and get $0 \to M \xrightarrow{1} M \to 0$ as a free resolution. Hence $\mathrm{Tor}(A, M) = 0$ for all A.

If A is free and $0 \to F_1 \to F_0 \to M \to 0$ is a free resolution of M, then tensoring this resolution with A yields an exact sequence

$$0 \to A \otimes F_1 \to A \otimes F_0 \to A \otimes M \to 0,$$

since tensoring with free groups is an exact functor. Hence the kernel of the map $A \otimes F_1 \to A \otimes F_0$ is trivial, i.e., $\mathrm{Tor}(A, M) = 0$.

d) Show that, for any abelian group A, Tor is symmetric:

$$\mathrm{Tor}(A, M) \cong \mathrm{Tor}(M, A).$$

(Hint: Make sure that you understand what you need to show. Then apply Tor to a short exact sequence that arises as a free resolution.)

Solution: Let $0 \to E_1 \to E_0 \to A \to 0$ be a free resolution of A. Since E_1 and E_0 are free, we know $\mathrm{Tor}(E_1, M) = 0$ and $\mathrm{Tor}(E_0, M) = 0$. By the previous point on short exact sequences, applying $\mathrm{Tor}(-, M)$ to this sequence yields an exact sequence of the form

$$0 \to \mathrm{Tor}(A, M) \to E_1 \otimes M \to E_0 \otimes M \to A \otimes M \to 0.$$

On the other hand, by definition of $\mathrm{Tor}(-, A)$ we have an exact sequence

$$0 \to \mathrm{Tor}(M, A) \to M \otimes E_1 \to M \otimes E_0 \to M \otimes A \to 0.$$

Since the tensor product is symmetric, these two sequences are related by solid maps

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(A, M) \arrow[r] \arrow[d] & E_1 \otimes M \arrow[r] \arrow[d, "\cong"] & E_0 \otimes M \arrow[r] \arrow[d, "\cong"] & A \otimes M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Tor}(M, A) \arrow[r] & M \otimes E_1 \arrow[r] & M \otimes E_0 \arrow[r] & M \otimes A \arrow[r] & 0.
\end{tikzcd}$$

Since the solid squares commute, there is an induced dotted map on the Tor-groups. The Five-Lemma then implies that this map is an isomorphism.

e) For any abelian group A, show that $\mathrm{Tor}(A, M) = 0$ if A or M is torsion-free, i.e., the subgroup of torsion elements vanishes.

(Hint: You may want to use the fact that a finitely generated torsion-free abelian group is free.)

Solution: Let A be torsion-free and let $0 \to F_1 \xrightarrow{i} F_0 \to M \to 0$ be a free resolution of M. We would like to show that $A \otimes F_1 \xrightarrow{1 \otimes i} A \otimes F_0$ is injective. So suppose that $y := \sum_{k=1}^{n} a_k \otimes x_k$ is in the kernel of $1 \otimes i$. Let A_0 be the subgroup generated by the elements $a_1, \ldots, a_n$. Hence y lies in the kernel of the map $A_0 \otimes F_1 \to A_0 \otimes F_0$. But this kernel is by definition $\mathrm{Tor}(A_0, M)$. Now we use the hint that the finitely generated torsion-free abelian group A_0 is free. By a previous point, this implies $\mathrm{Tor}(A_0, M) = 0$ and hence $y = 0$. Thus $1 \otimes i$ is injective.

f) Let A be an abelian group and let $T(A)$ denote the subgroup of torsion elements in A. Show $\mathrm{Tor}(A, M) = \mathrm{Tor}(T(A), M)$.

(Hint: $A/T(A)$ is torsion-free.)

Solution: We have a short exact sequence

$$0 \to T(A) \to A \to A/T(A) \to 0.$$

By one of the previous points, this induces an exact sequence (of which we write down only a part)

$$0 \to \mathrm{Tor}(T(A), M) \to \mathrm{Tor}(A, M) \to \mathrm{Tor}(A/T(A), M) \to T(A) \otimes M.$$

Since $A/T(A)$ is torsion-free, we know by the previous point that

$$\mathrm{Tor}(A/T(A), M) = 0.$$

Since the sequence is exact, this implies that

$$\mathrm{Tor}(T(A), M) \xrightarrow{\cong} \mathrm{Tor}(A, M)$$

is an isomorphism.

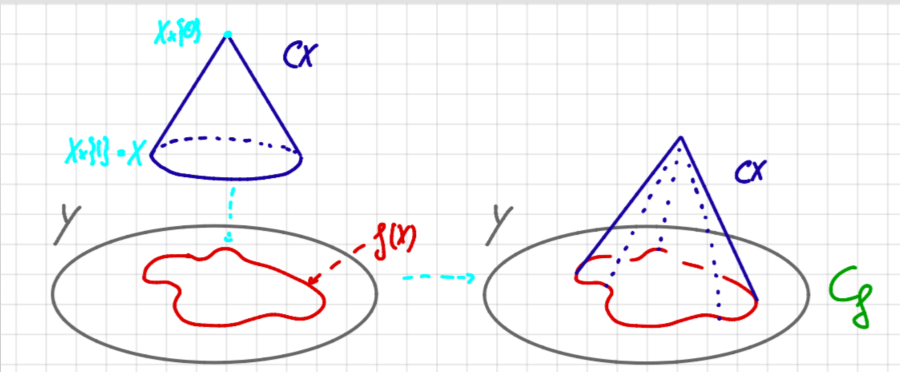
Definition: Mapping cone

For a space X, the **cone** over X is defined as the quotient space

$$CX := (X \times [0,1])/(X \times \{0\}).$$

Let $f\colon X \to Y$ be a continuous map. The **mapping cone of f** is defined to be te quotient space

$$C_f := (CX \sqcup Y)/((x,1) \sim f(x)) = Y \cup_f CX.$$



$\boxed{2}$ Let $f\colon X \to Y$ be a continuous map, and let M be an abelian group.

a) Show that the homology of the mapping cone of f fits into a long exact sequence

$$\cdots \to \tilde{H}_{n+1}(C_f; M) \to \tilde{H}_n(X; M) \xrightarrow{f_*} \tilde{H}_n(Y; M) \xrightarrow{i_*} \tilde{H}_n(C_f; M) \to \tilde{H}_{n-1}(X; M) \to \cdots$$

(Hint: Relate C_f to the mapping cylinder M_f from a previous exercise set.)

Solution: The sequence looks very much like the long exact sequence of a pair. Except that X is not a subspace of Y. The idea of the mapping cylinder M_f and the mapping cone C_f is to fix this issue.

The point is that $X \times \{0\}$ is in fact a subspace of M_f. To make this work, we recall from a previous exercise set that observe that $Y \hookrightarrow C_f$ is a deformation retract. Thus, by homotopy invariance, the inclusion induces an isomorphism in homology with coefficients in M.

Moreover, $X \times [0, 1/2)$ is a subspace of C_f such that

$$X \times \{0\} \hookrightarrow X \times [0, 1/2)$$

is a deformation retract. A homotopy is given by

$$h\colon X \times [0,1/2) \times I \to X \times [0,1/2),\ ([x,s],t) \mapsto [x, s(1-t)].$$

Now we can apply a result of the lecture (the proof of which also works with M-coefficients) to deduce

$$\tilde{H}_*(C_f;M) = \tilde{H}_*(M_f/(X \times \{0\});M) \cong \tilde{H}_*(M_f,(X \times \{0\});M)$$

Finally, we note $\tilde{H}_*(X \times \{0\};M) = \tilde{H}_*(X);M$.
Now we apply the long exact sequence associated to the pair $(X \times \{0\}, M_f)$ together with the above identifications to get the desired long exact sequence.

b) Show that f induces an isomorphism in homology with coefficients in M if and only if $\tilde{H}_*(C_f;M) = 0$.
Solution: This follows from the previous point and exactness.

The first part of the next exercise requires some familiarity with Tor beyond the discussion of the lecture. But you should think about it anyway and definitely note the statement.

$\boxed{3}$ Let X and Y be topological spaces.

a) Show that $\tilde{H}_*(X;\mathbb{Z}) = 0$ if and only if $H_*(X;\mathbb{Q}) = 0$ and $H_*(X;\mathbb{F}_p) = 0$ for all primes p.
(Hint: Use the UCT with $\mathbb{F}_p$-coefficients to control the torsion part and with $\mathbb{Q}$-coefficients to control the torsion-free part. Then apply suitable points of the first exercise.)
Solution: The Universal Coefficient Theorem implies that if $\tilde{H}_*(X;\mathbb{Z}) = 0$, then $H_*(X;\mathbb{Q}) = 0$ and $H_*(X;\mathbb{F}_p) = 0$ for all primes p. For, the left- and right-hand terms in the short exact sequence vanish, and hence the middle term has to vanish as well.
Now assume we know $H_*(X;\mathbb{Q}) = 0$ and $H_*(X;\mathbb{F}_p) = 0$ for all primes p. Since the outer terms in a short exact sequence have to vanish if the middle term does, the Universal Coefficient Theorem reduces the task to showing the following: If an abelian group A satisfies $A \otimes \mathbb{Q} = 0$ and $\mathrm{Tor}(A,\mathbb{F}_p) = 0$ for all p, then $A = 0$.
So assume A satisfies $A \otimes \mathbb{Q} = 0$ and $\mathrm{Tor}(A,\mathbb{F}_p) = 0$ for all p. By a previous exercise, the short exact sequences

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

induces an exact sequences

$$0 \to \mathrm{Tor}(A,\mathbb{Z}/p) \to A \xrightarrow{p} A \to A/pA \to 0.$$

Since $\mathrm{Tor}(A,\mathbb{Z}/p) = 0$ by assumption, multiplication with p is injective in A for all primes p. Thus A is torsion-free.
On the other hand, the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

induces an exact sequence

$$0 \to \mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) \to A \otimes \mathbb{Z} = A \to A \otimes \mathbb{Q} \to A \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

Since A is torsion-free, the previous exercise implies that $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) = 0$. Hence the map $A \to A \otimes \mathbb{Q}$ is injective. Since $A \otimes \mathbb{Q} = 0$ by assumption, this implies $A = 0$.

b) Show that a map $f\colon X \to Y$ induces an isomorphism in integral homology if and only if it induces an isomorphism in homology with rational coefficients and in homology with $\mathbb{F}_p$-coefficients for all primes p.

Solution: This follows from the previous point applied to the mapping cone and the previous exercise.

$\boxed{4}$ Let X be a finite cell complex, and let $\mathbb{F}_p$ be a field with p elements. Show that the Euler characteristic $\chi(X)$ can be computed by the formula

$$\chi(X) = \sum_i \dim_{\mathbb{F}_p} (-1)^i H_i(X; \mathbb{F}_p).$$

In other words, $\chi(X)$ is the alternating sum of the dimensions of the $\mathbb{F}_p$-vector spaces $H_i(X; \mathbb{F}_p)$.

(Hint: Use the UCT.)

Solution: Since $H_i(X)$ is a finitely generated abelian group, is has a decomposition

$$H_i(X) \cong \mathbb{Z}^{r_i} \oplus \mathrm{Torsion}(H_i(X)).$$

We showed in the lectures that $\chi(X)$ equals $\sum_i (-1)^i r_i$.

To show the assertion, we need to relate the rank r_i to the dimensions of the corresponding $\mathbb{F}_p$-vector spaces. The torsion of $H_i(X)$ can be further split into finitely summands $\mathbb{Z}/p^n$ for varying n and into torsion which is prime to p. Taking the tensor product with $\mathbb{F}_p = \mathbb{Z}/p$ kills the prime to p-torsion.

Let s_i be the number of summands of the form $\mathbb{Z}/p^n$ in $H_i(X)$. Then we get

$$\mathrm{Tor}(\mathbb{Z}/p, H_{i-1}(X)) \cong \mathbb{F}_p^{s_{i-1}} \text{ and } H_i(X) \otimes \mathbb{F}_p \cong \mathbb{F}_p^{r_i + s_i}.$$

Now the UCT gives us a direct sum decomposition

$$H_i(X; \mathbb{F}_p) \cong (H_i(X) \otimes \mathbb{F}_p) \oplus \mathrm{Tor}(\mathbb{F}_p, H_{i-1}(X)).$$

This implies

$$\dim_{\mathbb{F}_p} H_i(X; \mathbb{F}_p) = r_i + s_i + s_{i-1}.$$

Now we compute

$$\begin{aligned} \sum_i \dim_{\mathbb{F}_p} (-1)^i H_i(X; \mathbb{F}_p) &= \sum_i (-1)^i (r_i + s_i + s_{i-1}) \\ &= \sum_i (-1)^i r_i, \end{aligned}$$

since the contributions of s_i and s_{i-1} with alternating signs cancel out. This is formula for $\chi(X)$ we proved in the lecture.

5 Use the Künneth Theorem of the lecture to show:

a) The homology of the product $X \times S^k$ is satisfies

$$H_n(X \times S^k) \cong H_n(X) \oplus H_{n-k}(X).$$

Solution: Again, the homology of S^k is torsion-free. Hence the Künneth Theorem gives us a direct sum decomposition

$$\begin{aligned} H_i(X \times S^k) &\cong \bigoplus_{p+q=n} (H_p(X) \otimes H_q(S^k)) \\ &\cong H_n(X) \otimes H_0(S^k) \oplus H_{n-k}(X) \otimes H_k(S^k) \\ &\cong H_n(X) \oplus H_{n-k}(X) \end{aligned}$$

since $H_0(S^k) \cong \mathbb{Z}$ and $H^k(S^k) \cong \mathbb{Z}$ are the only non-zero homology groups of S^k.

b) The homology of the n-torus T^n defined as the n-fold product $T^n = S^1 \times \ldots \times S^1$ is given by

$$H_i(T^n) \cong \mathbb{Z}^{\binom{n}{i}}.$$

Solution: Since the homology of S^1 is torsion-free, the Tor-term in the Künneth Theorem vanishes. Moreover, the only non-zero homology groups of S^1 are $H_0(S^1) \cong \mathbb{Z}$ and $H_1(S^1) \cong \mathbb{Z}$. Hence the theorem implies a direct sum decomposition

$$H_i(T^n) \cong \bigoplus_{i_1+\cdots+i_n=i} (H_{i_1}(S^1) \otimes \cdots \otimes H_{i_n}(S^1))$$

where each i_j is either 0 or 1. Hence there are exactly $\binom{n}{i}$ many summands. Since all the tensor products are isomorphic to $\mathbb{Z}$, this shows the assertion.