MA3403 Algebraic Topology
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Exercise set 10

1 Let M be an abelian group. Let X be a cell complex and let X_n denote the n-skeleton of X. We set

$$C^*(X;R) := H^n(X_n, X_{n-1}; M).$$

We would like to turn this into a cochain complex. We define the differential

$$d^n \colon C^n(X;M) \to C^{n+1}(X;M)$$

as the composite

$$\begin{array}{ccc} C^n(X;M) = H^n(X_n, X_{n-1}; M) & \xrightarrow{d^n} & H^{n+1}(X_{n+1}, X_n; M) = C^{n+1}(X;M) \\ & {\scriptstyle j^n} \searrow \quad H^n(X_n; M) \quad \nearrow {\scriptstyle \partial^n} & \end{array}$$

where ∂^n is the connecting homomorphism in the long exact sequence of cohomology groups of pairs and j^n is the homomorphism induced by the inclusion $(X_n, \emptyset) \hookrightarrow (X_n, X_{n-1})$. Define the **cellular cochain complex** of X with coefficients with M to be the cochain complex $(C^*(X;M), d^*)$.

Note that the cup product defines a product on the cellular cochain complex.

a) Show that $C^*(X;M)$ is in fact a complex, i.e., $d^n \circ d^{n-1} = 0$.

b) Show that $C^*(X;M)$ is isomorphic to the cochain complex $\mathrm{Hom}(C_*(X), M)$ where $C_*(X)$ is the cellular chain complex of X.
(Hint: Remember the Kronecker map κ.)

c) Use the UCT for cohomology and the isomorphism between $H_n(X)$ and $H_n(C_*(X))$ to show

$$H^n(X;M) \cong H^n(C^*(X;M)).$$

Note that the isomorphism we produce this way is not functorial.

2 Let $X = M(\mathbb{Z}/m, n)$ be a Moore space constructed by starting with an n-sphere S^n and then forming X by attaching an $n+1$-dimensional cell to it via a map $f \colon S^n \to S^n$ of degree m

$$X = S^n \cup_f D^{n+1}.$$

Let

$$q\colon X \to X/S^n \approx S^{n+1}$$

be the quotient map.

Recall that we showed that q induces a trivial map on $\tilde{H}_i(-;\mathbb{Z})$ for all i.

a) Show $H^{n+1}(X;\mathbb{Z}/m) \cong \mathbb{Z}/m$ and that $H^{n+1}(q;\mathbb{Z}/m)$ is nontrivial.
(Hint: Use the UCT for cohomology.)

b) Use the previous example to show that the splitting in the UCT for cohomology cannot be functorial.
(Hint: You need to show that a certain square induced by the UCT does not commute.)

3 Show that if a map $g\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ induces a nontrivial homomorphism

$$g^*\colon H^1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}/2) \to H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2),$$

then $n \geq m$.

4 Show that there does not exist a homotopy equivalence between $\mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^2 \vee S^3$.

The next exercise is a bit more challenging.

5 Let X be the cell complex obtained by attaching a 3-cell to $\mathbb{C}\mathrm{P}^2$ via a map

$$S^2 \to S^2 = \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$$

of degree p. Let $Y = M(Z/p,2) \vee S^4$ where $M(Z/p,2)$ is a Moore space. We observe that the cell complexes X and Y have the same 2-skeleton, but the 4-cell is attached via different maps.

a) Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$-coefficients.

b) Show that the cohomology rings of X and Y with $\mathbb{Z}/p$-coefficients are not isomorphic.