Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3403 Algebraic Topology Fall 2018

Solutions to exercise set 10

1 Let M be an abelian group. Let X be a cell complex and let X_n denote the n-skeleton of X. We set

$$C^*(X;R) := H^n(X_n, X_{n-1}; M).$$

We would like to turn this into a cochain complex. We define the differential

$$d^n \colon C^n(X;M) \to C^{n+1}(X;M)$$

as the composite

$$\begin{array}{ccccc} C^n(X;M) = H^n(X_n, X_{n-1}; M) & & \xrightarrow{\quad d^n \quad} & & H^{n+1}(X_{n+1}, X_n; M) = C^{n+1}(X;M) \\ & \searrow^{j^n} & & \nearrow^{\partial^n} & \\ & & H^n(X_n; M) & & \end{array}$$

where ∂^n is the connecting homomorphism in the long exact sequence of cohomology groups of pairs and j^n is the homomorphism induced by the inclusion $(X_n, \emptyset) \hookrightarrow (X_n, X_{n-1})$. Define the **cellular cochain complex** of X with coefficients with M to be the cochain complex $(C^*(X;M), d^*)$.

Note that the cup product defines a product on the cellular cochain complex.

a) Show that $C^*(X;M)$ is in fact a complex, i.e., $d^n \circ d^{n-1} = 0$.

Solution: Since both j^n and ∂^n are part of an exact sequence, we have $\partial^n \circ j^{n-1} = 0$. Thus

$$d^n \circ d_{n-1} = (j^n \circ \partial^n) \circ (j^{n-1} \circ \partial^{n-1}) = j^n \circ (\partial^n \circ (j^{n-1}) \circ \partial^{n-1} = 0.$$

b) Show that $C^*(X;M)$ is isomorphic to the cochain complex $\mathrm{Hom}(C_*(X), M)$ where $C_*(X)$ is the cellular chain complex of X.
(Hint: Remember the Kronecker map κ.)

Solution: Consider the diagram

$$\begin{array}{ccccc} H^n(X_n, X_{n-1}; M) & \xrightarrow{j^n} & H^n(X_n; M) & \xrightarrow{\delta^n} & H^n(X_{n+1}, X_n; M) \\ \downarrow{\scriptstyle \kappa^n} & & \downarrow{\scriptstyle \kappa^n} & & \downarrow{\scriptstyle \kappa^{n+1}} \\ \mathrm{Hom}(H_n(X_n, X_{n-1}), M) & \xrightarrow[j_n^*]{} & \mathrm{Hom}(H_n(X_n), M) & \xrightarrow[\partial_n^*]{} & \mathrm{Hom}(H_{n+1}(X_{n+1}, X_n), M) \end{array}$$

where j_n and ∂_n are the corresponding maps for homology.
The composite of the top row is the differential d^n. The composite of the top row is the dual of the differential d_n of the cellular chain complex. We know that the outer vertical maps are isomorphisms by the Universal Coefficient Theorem, since the involved homology groups are free. Hence it remains to observe that both squares commute, since all the maps involved are functorial.

c) Use the UCT for cohomology and the isomorphism between $H_n(X)$ and $H_n(C_*(X))$ to show

$$H^n(X;M) \cong H^n(C^*(X;M)).$$

Note that the isomorphism we produce this way is not functorial.

Solution: The UCT for cohomology tells us that we have isomorphisms

$$H^n(X;M) \cong \operatorname{Ext}(H_{n-1}(X), M) \oplus \operatorname{Hom}(H_n(X), M)$$

and

$$H^n(C^*(X;M)) \cong \operatorname{Ext}(H_{n-1}(C_*(X), M) \oplus \operatorname{Hom}(H_n(C_*(X), M).$$

Together with the isomorphism $H_n(X) \cong H_n(C_*(X))$, we get the desired isomorphism of cohomology groups.

But, since the splitting of the UCT is not natural, this isomorphism is not functorial. The problem is that we do not construct it via a map of cochain complexes.

2 Let $X = M(\mathbb{Z}/m, n)$ be a Moore space constructed by starting with an n-sphere S^n and then forming X by attaching an $n+1$-dimensional cell to it via a map $f\colon S^n \to S^n$ of degree m

$$X = S^n \cup_f D^{n+1}.$$

Let

$$q\colon X \to X/S^n \approx S^{n+1}$$

be the quotient map.

Recall that we showed that q induces a trivial map on $\tilde{H}_i(-;\mathbb{Z})$ for all i.

a) Show $H^{n+1}(X;\mathbb{Z}/m) \cong \mathbb{Z}/m$ and that $H^{n+1}(q;\mathbb{Z}/m)$ is nontrivial.
(Hint: Use the UCT for cohomology.)

Solution: We know from the UCT for cohomology that the quotient map induces a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}(H_n(S^{n+1}), \mathbb{Z}/m) & \longrightarrow & H^{n+1}(S^{n+1};\mathbb{Z}) & \longrightarrow & \operatorname{Hom}(H_{n+1}(S^{n+1}), \mathbb{Z}/m) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle q^*} & & \big\downarrow & & \\
0 & \longrightarrow & \operatorname{Ext}(H_n(X), \mathbb{Z}/m) & \longrightarrow & H^{n+1}(X;\mathbb{Z}/m) & \longrightarrow & \operatorname{Hom}(H_{n+1}(X), \mathbb{Z}/m) & \longrightarrow & 0.
\end{array}$$

By construction of X, we know

$$H_q(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}/m & \text{for } q = n \\ 0 & \text{for } q = n+1 \end{cases}$$

whereas

$$H_q(S^{n+1};\mathbb{Z}) = \begin{cases} 0 & \text{for } q = n \\ \mathbb{Z} & \text{for } q = n+1. \end{cases}$$

Since $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/m) = \mathbb{Z}/m$ and $\mathrm{Hom}(\mathbb{Z}/m, \mathbb{Z}/m) = \mathbb{Z}/m$, the above diagram is isomorphic to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & H^{n+1}(S^{n+1}; \mathbb{Z}/m) & \longrightarrow & \mathbb{Z}/m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q^*} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/m & \longrightarrow & H^{n+1}(X; \mathbb{Z}/m) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Hence $H^{n+1}(X; \mathbb{Z}/m) = \mathbb{Z}/m$ and $H^{n+1}(S^{n+1}; \mathbb{Z}/m) = \mathbb{Z}/m$.
To show that q^* is nontrivial, we look at a piece of the the long exact sequence of the pair (X, S^n):

$$0 = H_{n+1}(S^n; \mathbb{Z}/m) \to H_{n+1}(S^{n+1}; \mathbb{Z}/m) \xrightarrow{q^*} H_{n+1}(X; \mathbb{Z}/m).$$

Since the left-hand group is 0, exactness implies that q^* is injective. Hence it is an isomorphism and nontrivial, since both $H^{n+1}(X; \mathbb{Z}/m)$ and $H^{n+1}(X/S^n; \mathbb{Z}/m)$ are isomorphic to $\mathbb{Z}/m$.

b) Use the previous example to show that the splitting in the UCT for cohomology cannot be functorial.
(Hint: You need to show that a certain square induced by the UCT does not commute.)
Solution: We look again at the commutative diagram induces by the UCT for cohomology and q. that the quotient map induces a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_n(S^{n+1}), \mathbb{Z}/m) & \longrightarrow & H^{n+1}(S^{n+1}; \mathbb{Z}/m) & \underset{s}{\overset{\cdots}{\rightleftarrows}} & \mathrm{Hom}(H_{n+1}(S^{n+1}), \mathbb{Z}/m) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (q_*)^E} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (q_*)^H} & & \\
0 & \longrightarrow & \mathrm{Ext}(H_n(X), \mathbb{Z}/m) & \longrightarrow & H^{n+1}(X; \mathbb{Z}/m) & \underset{t}{\rightleftarrows} & \mathrm{Hom}(H_{n+1}(X), \mathbb{Z}/m) & \longrightarrow & 0
\end{array}$$

where we use temporarily the superscripts E and H to distinguish between the two respective maps.
We know that the solid squares commute. Moreover, the UCT tells us that there are the dotted splittings of the short exact sequences.
However, as we observed above the diagram is isomorphic to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/m & \underset{s}{\rightleftarrows} & \mathbb{Z}/m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (q_*)^E} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (q_*)^H} & & \\
0 & \longrightarrow & \mathbb{Z}/m & \longrightarrow & \mathbb{Z}/m & \underset{t}{\rightleftarrows} & 0 & \longrightarrow & 0
\end{array}$$

We see that s is an isomorphism, since it could not split an isomorphism of $\mathbb{Z}/m$ otherwise. This implies that $q^* \circ s$ is an isomorphism, whereas

$$t \circ (q_*)^H = 0$$

is not an isomorphism. Thus, the right-hand square with horizontal maps s and t does not commute. This means that the splitting in the UCT is not functorial.

3 Show that if a map $g\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ induces a nontrivial homomorphism

$$g^*\colon H^1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}/2) \to H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2),$$

then $n \geq m$.

Solution: By the theorem of the lecture about the cohomology ring of real projective space, we know that the induced map g^* is a homomorphism of truncated polynomial rings

$$H^*(\mathbb{R}\mathrm{P}^m;\mathbb{Z}/2) = \mathbb{Z}/2[\alpha]/(\alpha^{m+1}) \xrightarrow{g^*} \mathbb{Z}/2[\alpha]/(\alpha^{n+1}) = H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2).$$

If g^* is nontrivial in degree 1

$$H^1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}/2) = \mathbb{Z}/2\{\alpha\} \xrightarrow{g^*} \mathbb{Z}/2\{\alpha\} = H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2),$$

then it must send α to α. But if $n > m$, then $\alpha^{m+1} \neq 0$ in $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2)$ whereas $\alpha^{m+1} = 0$ in $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2)$. This contradicts g^* being a ring homomorphism. Hence $n \geq m$, if g^* is nontrivial on H^1.

4 Show that there does not exist a homotopy equivalence between $\mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^2 \vee S^3$.

Solution: If there was a homotopy equivalence between these two spaces, it would induce an isomorphism of $\mathbb{F}_2$-algebras

$$\tilde{H}^*(\mathbb{R}\mathrm{P}^3;\mathbb{F}_2) \cong \tilde{H}^*(\mathbb{R}\mathrm{P}^2 \vee S^3;\mathbb{F}_2) \cong \tilde{H}^*(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \times \tilde{H}^*(S^3;\mathbb{F}_2).$$

But such an isomorphism would have to send the generator $x \in H^1(\mathbb{R}\mathrm{P}^3;\mathbb{F}_2)$ to the generator x in $H^*(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. But in $H^*(\mathbb{R}\mathrm{P}^3;\mathbb{F}_2)$ we have $x^3 \neq 0$, whereas $x^3 = 0$ in $H^*(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. Hence such a map does not exist.

The next exercise is a bit more challenging.

5 Let X be the cell complex obtained by attaching a 3-cell to $\mathbb{C}\mathrm{P}^2$ via a map

$$S^2 \to S^2 = \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$$

of degree p. Let $Y = M(Z/p,2) \vee S^4$ where $M(Z/p,2)$ is a Moore space. We observe that the cell complexes X and Y have the same 2-skeleton, but the 4-cell is attached via different maps.

a) Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$-coefficients.

Solution:

- The cohomology ring of $\mathbb{C}\mathrm{P}^2$ is isomorphic to $\mathbb{Z}[x,y]/(x^2 = y, x^3, xy, y^2)$ with $|x| = 2$ and $|y| = 4$.

Attaching the 2-cell induces p-torsion in degree 2. Hence in $H^2(X;\mathbb{Z})$ we have $px = 0$. Since $py \neq 0$ in $H^4(X;\mathbb{Z})$, we must have $x^2 = 0$. Thus the integral cohomology ring of X is

$$H^*(X;\mathbb{Z}) = \mathbb{Z}[x,y](x^2, px, xy, y^2) \text{ with } |x| = 2,\ |y| = 4.$$

• The UCT for cohomology tells us that that integral cohomology of $M(\mathbb{Z}/p,2)$ is given by

$$H^i(M(\mathbb{Z}/p,2),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i=0 \\ \mathbb{Z}/p & \text{for } i=2 \\ 0 & \text{else.} \end{cases}$$

Hence there is no nontrivial multiplication byond degree 0. Hence the cohomology ring of $M(Z/p,2)$ is $\mathbb{Z}[s]/(s^2,ps)$ with $|s|=2$.
Thus the integral cohomology ring of Y is

$$H^*(Y;\mathbb{Z}) = \mathbb{Z}[s,t](s^2,ps,st,t^2) \text{ with } |s|=2,\ |t|=4.$$

Hence the two integral cohomology rings look the same.

b) Show that the cohomology rings of X and Y with $\mathbb{Z}/p$-coefficients are not isomorphic.

Solution: With $\mathbb{Z}/p$-coefficients, multiplication with p is trivial and therefore the differentials in the cellular cochain complex vanish.
Thus we get

$$H^*(X;\mathbb{Z}/p) = \mathbb{Z}/p[x,z](x^3,z^2,xz) \text{ with } |x|=2,\ |z|=3.$$

but

$$H^*(Y;\mathbb{Z}/p) = \mathbb{Z}/p[s,r,t](s^2,r^2,t^2,rs,rt,st) \text{ with } |s|=2,\ |r|=3,\ |t|=4.$$