MA3403 Algebraic Topology
Lecturer: Gereon Quick
Lecture 01

1. Introduction

Organization:

Lectures: Tuesdays and Thursdays, both days at 12.15-14.00 in S21 in Sentralbygg 2.

Exercises: We will have some suggested exercises, hopefully on a weekly basis. You should try to solve as many exercises as possible, not just the ones I suggest, but also all that you find in oter textbooks. We will not have an exercise class though. But you can discuss exercises with me at any time!

Course webpage: wiki.math.ntnu.no/ma3403/2018h/start
where all news about the class will be announced. You will also find lecture notes a few hours after class on the webpage.

Office hours: Upon request.

Just send me an email: gereon.quick@ntnu.no

Textbooks: We will not follow just one book... but there are many good texts out there. For example, you can look at

[H] A. Hatcher, Algebraic Topology. It's available online for free. It contains much more than we have time for during one semester.

[Mu] J.R. Munkres, Elements of Algebraic Topology.

[V] J.W. Vick, Homology Theory - An Introduction to Algebraic Topology.

Two books that you can use as an outlook to future topics:

[Ma] J.P. May, A Concise Course in Algebraic Topology. It's also online somewhere.

[MS] J.W. Milnor, J. Stasheff, Characteristic Classes.

There are many other good books and lecture notes out there. Ask me if you need more.

What is required?

I will assume that you are starting your third year at NTNU (or more). You should have taken the equivalent of **Calculus 1-3** or **MA1101-1103, MA 1201-1202**. So you should be familiar with Euclidean space $\mathbb{R}^n$, multivariable calculus and linear algebra. Ideally, you have taken TMA4190 Introduction to Topology and/or General Topology.

You should also know a bit about algebra, like what is a group, an abelian group, a field, ideally also what is a ring and module over a ring.

Finally, it would be good if you knew what a topoogical space is and you would know what the words **open, closed, compact, etc** mean. But, in fact, you could also just have some few examples of topologial spaces in mind, like n-spheres, torus etc. without knowing too many abstract stuff. For, the class is much more about the **ideas and methods** we develop than anything else. And these methods are useful almost everywhere.

Nevertheless, if you want to refresh your knowledge on Topology, you may want to have a look at the book

[J] K. Jänich, Topology.

What this class is about:

Note: If some of the following words do not yet make sense to you, no worries! For the moment we are just waving our hands and use fancy words. We will make sense of all this during the semester...

Very roughly speaking, **topology** studies spaces up to **continuous transformation** of one into the other.

The correct place to do this is the **category of topological spaces** whose objects are topological spaces and whose morphisms are continuous maps. The isomorphisms in this category are called **homeomorphisms**, i.e., a continuous map with a continuous (left- and right-) inverse, or a continuous bijective map with a continuous inverse.

We can then describe topology as the science which studies properties of spaces which do not change under homeomorphisms. You have seen many such properties already, e.g. **compactness** (to be recalled in a bit).

This gives rise to a typical question in topology:

Typical question in topology

Given two topological spaces X and Y. Are X and Y homeomorphic, i.e., is there a homeomorphism $\phi\colon X \xrightarrow{\approx} Y$?

Let us look at a familiar example and compare it to similar situations. Fix two natural numbers $n < m$.

- Is there a **linear isomorphism** (of vector spaces) $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$?
  Answer: **No**, since linear algebra tells us that isomorphic vector spaces have equal dimension.
- Is there a **diffeomorphism** (bijective differential map with differentiable inverse) $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$?
  Answer: **No**, since otherwise the derivative at 0 would be a linear isomorphism $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$ between tangent spaces.
- Is there a **bijective map** (of sets) $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$?
  Answer: **Yes**. Surprisingly enough one can construct such maps, and it is actually not that difficult.
- Is there a **homeomorphism** $\mathbb{R}^m \xrightarrow{\approx} \mathbb{R}^n$?

The answer to the last question is: **No**. But it is not so simple to show. In fact, one of the goals of algebraic topology is to develop tools that help us decide similar questions. For example, is there a homeomorphism between the 2-dimensional sphere S^2 and the torus? The answer is no. But how can we prove that? Both spaces are compact and (in some sense) two-dimensional and oriented...

Algebraic Topology in a nutshell

Translate problems in topology into problems in algebra which are (hopefully) easy to answer.
Key idea: develop **algebraic invariants** (numbers, groups, rings etc and homomorphisms between them) which decode the topological problem.
This should be done such that **homeomorphic** spaces should have the **same invariants** (that is where the name comes from).
In particular, this implies: if we find values of an invariant that differ for X and Y, then they cannot be homeomorphic.

Remark: We will later see that all the invariants we construct are preserved under **homotopy** equivalences, a **weaker** notion than homeomorphisms. This will finally lead to the idea of the stable homotopy category being the motive of topological spaces. We will not discuss this in class, but feel free to ask me about it. :)

For example, the first important tool that we are going to define soon is **singular homology**. It will allow us to use a simple algebraic argument to show that there cannot be a homeomorphism $\mathbb{R}^m \xrightarrow{\approx} \mathbb{R}^n$.

Just to make you taste a little more of what algebraic topology can do:

Multiplicative Structures on $\mathbb{R}^n$

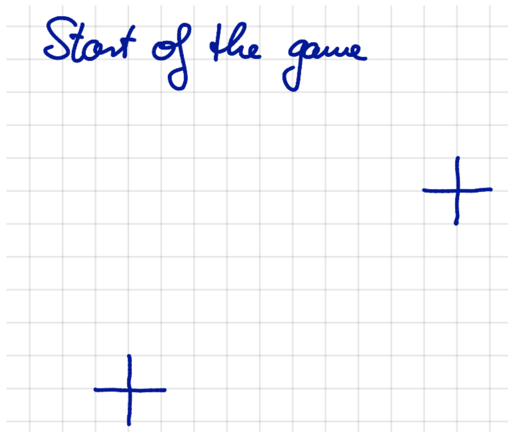
Let $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear map with two-sided identity element $e \neq 0$ and no zero-divisors. Then $n = 1, 2, 4,$ or 8.
What we are looking for is a "multiplication map". You know the cases $n = 1$ and $n = 2$ very well. It's just $\mathbb{R}$ and $\mathbb{C} \cong \mathbb{R}^2$. These are actually fields.
For $n = 4$, there are the Hamiltonians, or Quaternions, $\mathbb{H} \cong \mathbb{R}^4$ with a multiplication which as almost as good as the one in $\mathbb{C}$ and $\mathbb{R}$, but it is not commutative. (You add elements i, j, k to $\mathbb{R}$ with certain multiplication rules.)
For $n = 8$, there are the Octonions $\mathbb{O} \cong \mathbb{R}^8$. The multiplication is not associative and not commutative.
And that's it!

This is a really deep result!
The crucial and, at first glance maybe surprising, point to prove this fundamental result is that the statement has something to do with the behavior of tangent spaces on spheres. That's a topological problem. Frank Adams was the first to solve it.

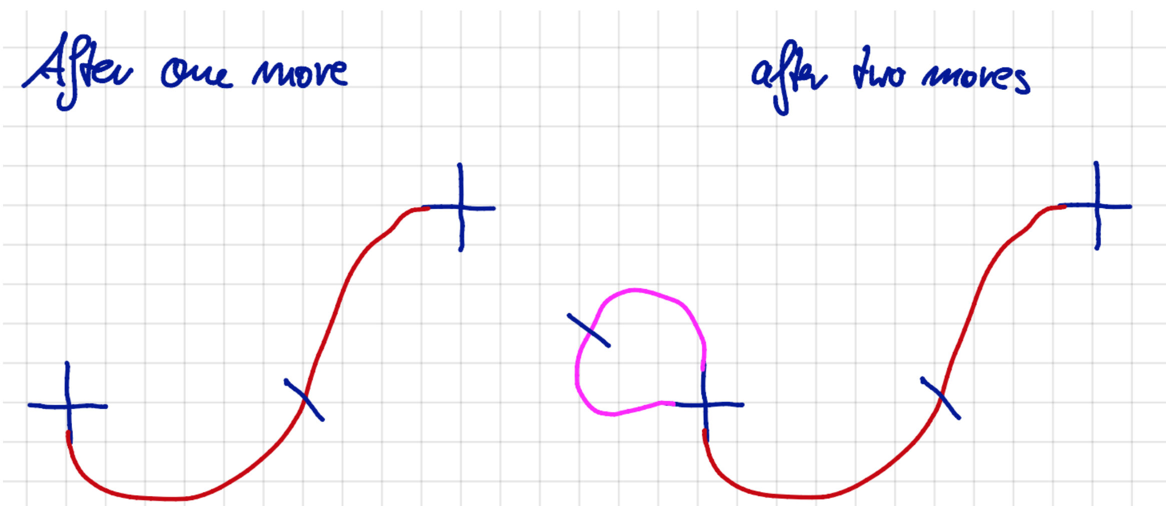
In this class we will start to walk on the path towards a proof of this problem. Unfortunately, we won't make it to the finish line within one semester. So, if you like, learn more about it in **Advanced Aglebraic Topology...**

- Before we move on, let us play **a game** and see an invariant in action.

The rules: Take a piece of paper and draw two crosses, i.e. spots with four free ends.

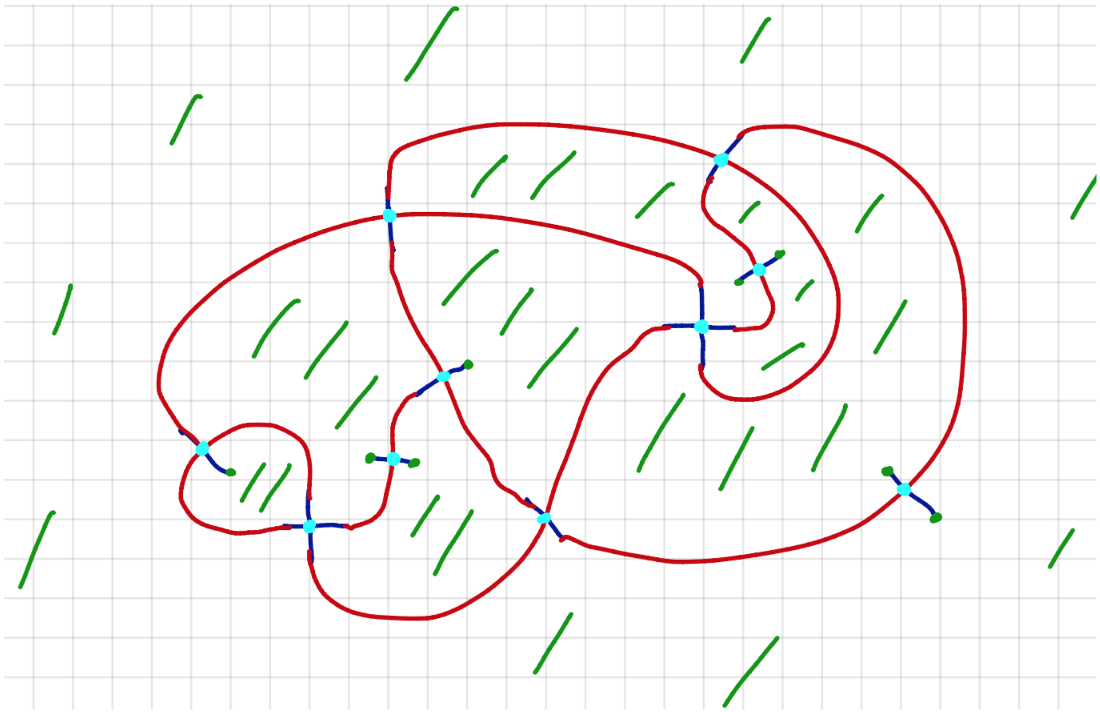


Each move involves joining two free ends with a curve which does not cross any existing line, and then putting a short stroke across the line to create two new free ends. The players play alternating moves.



If there are no legal moves left, **the player who made the last legal move wins**.

Let us assume that we know that the game ends after a finite number of moves, say m moves. At the end we will have created a connected, closed planar graph, in particular, a figure which has vertices, edges and faces.



We claim that no matter how you play, what strategy you use etc, there are always 8 moves before the game stops, it is always the second player who wins and there is a fixed number of vertices, edges and faces!

Why? Well, the number of moves and everything about the figure we create is determined by **Euler's formula** $v - e + f = 2$. The number 2 is an **example of an algebraic invariant**.

To understand how this works, we need to determine how the number v of vertices, the number e of edges, and the number f of faces depends on the number m of moves.

For the vertices, when we start the game we have two vertices. In each move, we create one new vertex. Thus we get

$$v = 2 + m.$$

For the edges, when we start the game we have no edges. In each move, we create one line, but we split it into two edges by adding a vertex in the middle. Hence in each move, we create 2 edges. Thus we get

$$e = 2m.$$

For the faces, we have to think backwards. At the end, there is exactly one free (or loose) end pointing into each face that we created. For, if there was a face with two free ends pointing into one face, then we could connect these two ends within that face and the game would not have stopped. Note that there is also exactly one lose end pointing out of the figure. (Again, of there were two we could connect them by going around the figure.)

Now we need to check how many free ends we produce. We start with 4 free ends per cross, that is 8 free ends. In each move, we connect two free ends, but we also create two new ones. Thus the number of free ends does not change during the whole game. Hence we get

$$f = 8.$$

In total we get

$$\begin{aligned} 2 &= v - e + f \\ 2 &= 2 + m - 2m + 8 \\ 0 &= -m + 8 \\ m &= 8. \end{aligned}$$

Hence no matter how we play, the game ends after 8 moves. Since this number is even, the second player always wins. Moreover, we always get $v = 10$, $e = 16$, and $f = 8$.

Alternative: Changing the starting setup changes the outcome of the game. For if we start with n crosses (or nodes), then we get with the same reasoning as above

$$\begin{aligned} v &= n + m \\ e &= 2m \\ f &= 4n. \end{aligned}$$

Euler's formula then yields

$$\begin{aligned} 2 &= v - e + f \\ 2 &= n + m - 2m + 4n \\ m &= 5n - 2. \end{aligned}$$

Thus the game ends after $m = 5n - 2$ moves. For example, if $n = 3$, this is an odd number and the first player always wins.

Here is the idea why Euler's formula holds: We can draw any connected planar graph (a graph we can draw in the plane such that its edges only intersect in the vertices and we can walk along the edges between any two vertices) as folllows:

1) We start with a graph consisting of **just one vertex** and no edges, so $v = 1$ and $e = 0$. And we have one face, the outer face or the plane around the vertex, so $f = 1$. So in total the formula holds $v - e + f = 1 - 0 + 1 = 2$.

2) Now we can extend the graph by either
   - **a)** adding one vertex and connect it via an edge to the first one; that is we change $v \to v + 1$ and $e \to e + 1$ or
   - **b)** draw an edge from the existing vertex to itself; this way we create a new face as well, hence we change $e \to e + 1$ and $f \to f + 1$.

   Thus after both operations the formula $v - e + f = 2$ still holds. Now we continue this process until we have created the planar graph we had in mind.

Here is another example of the use of an algebraic invariant:

- **Football pattern:**

Question: How many pentagons and hexagons are there on a classical football?

We set $P :=$ **# of pentagons** and $H :=$ **# of hexagons**. The collection of all vertices, edges and faces of all the pentagons and hexagons on the football forms a **graph on the surface of the football**. This graph and therefore the pattern on the football is governed by **Euler's formula** $v - e + f = 2$. Hence we need to calculate the number of vertices v, the number of edges e and the number of faces f.

The number of faces is obviously given by

$$f = P + H.$$

To calculate the number of edges e we observe that every pentagon has 5 edges and every hexagon has 6 edges. That yields $5P + 6H$ edges. **But** we have counted too many edges. For at each edge, there are **two faces which meet**. Thus we need to divide our number by 2 and get

$$e = \frac{5P + 6H}{2}.$$

To calculate the number of vertices v we observe again that every pentagon has 5 vertices and every hexagon has 6 vertices. That yields $5P+6H$ edges. **But again** we have counted too many vertices. For at each vertex, there are **three faces which meet**. Thus we need to divide our number by 3 and get

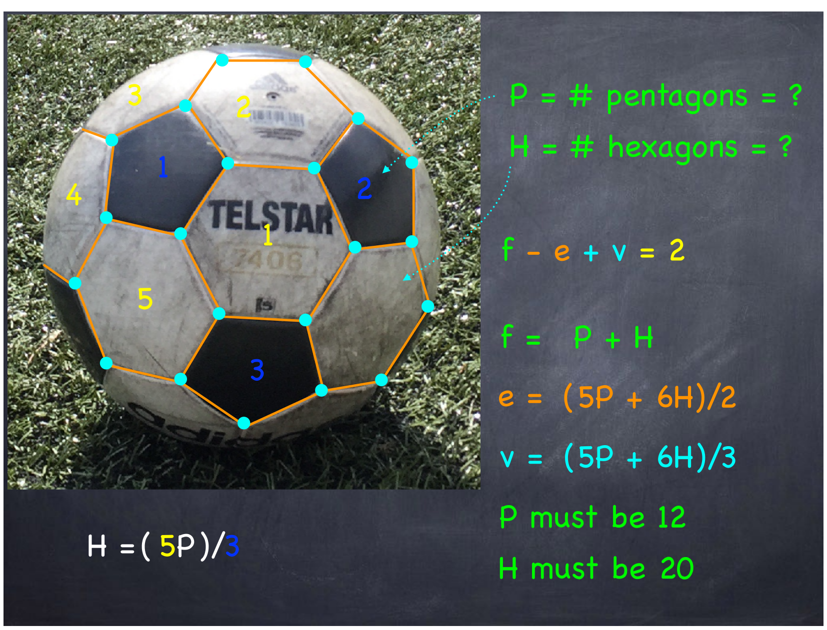
$$v = \frac{5P+6H}{3}.$$

Now we apply Euler's formula:

$$\begin{aligned} v - e + f &= 2 \\ \frac{5P+6H}{3} - \frac{5P+6H}{2} + P + H &= 2 && \text{(multiply by 6)} \\ 10P + 12H - 15P - 18H + 6P + 6H &= 2 && \text{(simplify)} \\ P &= 12. \end{aligned}$$

To get H, we count how many hexagons there are per pentagon: Each pentagon is surrounded by 5 hexagons which would yield $H = 5P$. But each hexagon is attached to 3 pentagons at the same time. Hence we have counted three times as many hexagons as there really are. This yields

$$H = \frac{5P}{3} = \frac{5 \cdot 12}{3} = \frac{60}{3} = 20.$$



The Euler characteristic:

The Euler characteristic is a topological invariant of a space, that means it does not change if we transform the space continuously. Fopr a space X, it is denoted by $\chi(X)$. It is always an integer number.

Here are some examples:

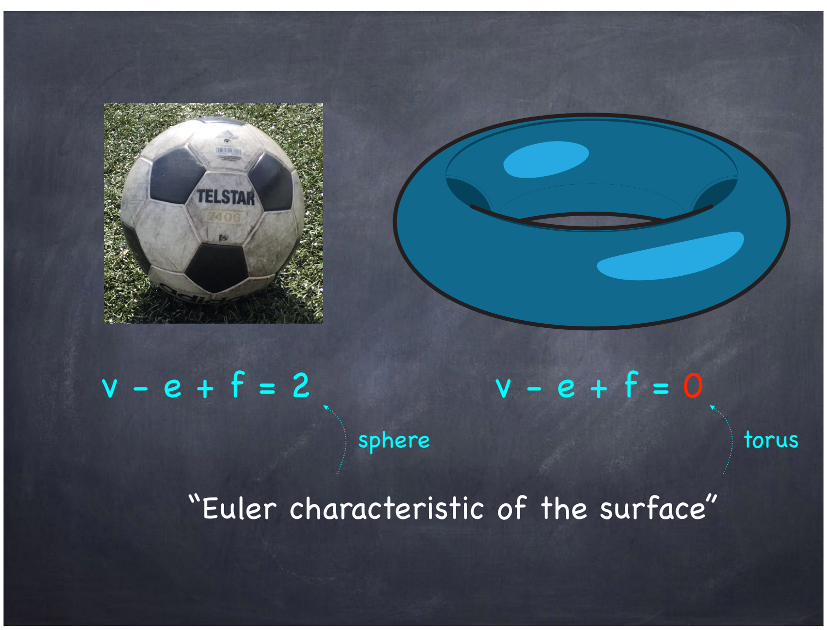
- For a sphere $X = S^2$, it 2: $\chi(S^2) = 2$.
- For a torus $X = T^2$, it 0: $\chi(T^2) = 0$.
- For a surface with two holes, it is -2.
- In general, for a surface S with g holes, it is $\chi(S) = 2 - 2g$.

The Euler characteristic can be defined (in a more abstract way) for any topological space. It is an important example of an **algebraic invariant** of a space which only depends on its topology.

Assume we have two spaces X and Y, defined in some complicated way which makes it difficult to understand how they look. But let us assume we can calculate their Euler characteristics by some method. Then, if $\chi(X) \neq \chi(Y)$, we know that we cannot transform X continuously into Y.

And there are also more positive examples. It often happens that an invariant defined one way turns out to encode a lot of other information as well.

You will learn more about these things soon...



The rough initiating idea for our algebraic invariants

Now back to the general situation. Let us try to get a first idea of how algebraic topologists think and come up with their fancy invariants. Let us say we have a space X and we want to characterize it, or at least be able to distinguish it from other spaces.

The initiating idea is to study X by taking **test spaces** we understand well and looking at the **space of all maps** from these test spaces into X. This may sound like an awkward detour, but it turns out to be pretty smart.

So what are those test spaces? The most simple space is a point. So let $T = \bullet$ the one-point space and let $C(\bullet,X)$ be the set/space of all (continuous) maps from $\bullet$ into X. Since any map in $C(\bullet,X)$ is determined by the one-point image, we just get that $C(\bullet,X)$ is the set of points of X.

So what happens if we take a **one-dimensional test space** like the unit interval [0,1]? A continuous map $\gamma\colon [0,1] \to X$ is a path in X from $\gamma(0)$ to $\gamma(1)$. Each $\gamma(0)$ and $\gamma(1)$ also gives us also an element in $C(\bullet,X)$. Hence if we look at $C(\bullet,X)$ modulo those which can be connected by a path in $C([0,1],X)$, then we can read off how many "pieces" X has. Making this more precise gives us the set $\pi_0(X)$ of connected components of X. The set $\pi_0(X)$ is the first example of an **algebraic invariant**.

Let us keep going with this. A **two-dimenionsal test space** might be the square $[0,1] \times [0,1]$. Given a continuous map $\alpha\colon [0,1] \times [0,1] \to X$, the two restrictions $\alpha(0,t)$ and $\alpha(1,t)$ define two paths in X. If we assume that they have the same start and end points, then we get a relation on the set $C([0,1],X)$ of paths in X.

This leads, by looking only at paths which are loops, to the **fundamental group** $\pi_1(X)$ of X, the next algebraic invariant.

Continuing this way and to look at maps from an **n-dimensional test space modulo relations** that come from maps from a corresponding $n+1$-dimensional test space we can produce a sequence of algebraic invariants $\pi_2(X), \pi_3(X), \dots$ which are called the **higher homotopy groups** of X. (Actually, for the nth homotopy group one uses the n-dimensional sphere, the n-dimensional space with the maximal symmetry.)

The collection of all homotopy groups encodes a lot of information abot X. In fact, in many cases it contains all the information about X **up to homotopy**, i.e., continuous deformation of X.

However, **homotopy groups** are notoriously **difficult to compute**. That is why one also uses a **different type of test spaces**.

Starting again with the unit interval [0,1] in dimension one, we could also proceed as follows. In dimension two we take an equilateral **triangle**, called a two-simplex. In dimension three we take a regular **tetrahedron**, called a three-simplex. In dimension four, we continue with a regular four-dimensional simplex and so forth.

This leads to the **singular homology groups** $H_n(X)$ of X. These groups will be the main object of our studies for a while. in general, they carry less information than homotopy groups. However, their big advantage is that they are **computable**! During the next couple of weeks we will develop the machinery to compute homology groups. Along the way we will witness many fundamental ideas that turned out to be extremely useful in many areas of mathematics...