MA3403 Algebraic Topology
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Lecture 02

2. Cell complexes and homotopy

Our main goal today is to introduce cell complexes as an important type of topological spaces and the conecpt of homotopy which is a fundamental idea to simplify problems.

But we start with a super brief recollection of some basic notions in topology.

A crash course in topology

Roughly speaking, a topology on a set of points is a way to express that points are near to each other as a generalization of a space with a metric, i.e., a concrete distance function.

You know the fundamental example of a metric space. For, recall from Calculus 2 that the norm of a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is defined by

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \in \mathbb{R}.$$

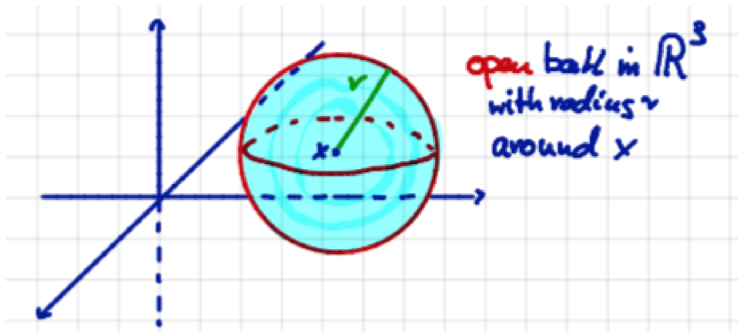
For any n, the space $\mathbb{R}^n$ with this norm is called **n-dimensional Euclidean space**. The norm induces a maetric, i.e., a distance function by

$$d(x,y) := |x - y| \text{ for } x,y \in \mathbb{R}^n.$$

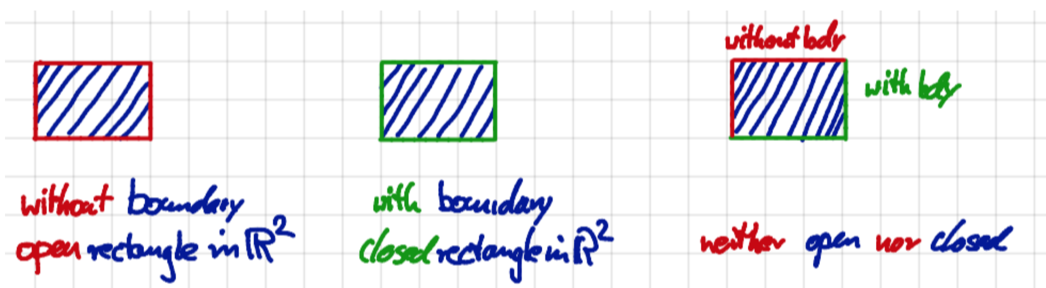
This turns $\mathbb{R}^n$ into a metric space and therefore an example of a **topological space** in the following way:

Open sets in $\mathbb{R}^n$

- Let x be a point in $\mathbb{R}^n$ and $r > 0$ a real number. The ball
$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$$
with radius ϵ around x is an **open** set in $\mathbb{R}^n$.
- The open balls $B_r(x)$ are the prototypes of open sets in $\mathbb{R}^n$.
- A subset $U \subseteq \mathbb{R}^n$ is called **open** if for every point $x \in U$ there exists a real number $\epsilon > 0$ such that $B_\epsilon(x)$ is contained in U.
- A subset $Z \subseteq \mathbb{R}^n$ is called **closed** if its complement $\mathbb{R}^n \setminus Z$ is open in $\mathbb{R}^n$.



- Familiar examples of open sets in $\mathbb{R}$ are open intervals, e.g. (0,1) etc.
- The cartesian product of n open intervals (an open rectangle) is open in $\mathbb{R}^n$.
- Similarly, closed intervals are examples of closed sets in $\mathbb{R}$.
- The cartesian product of n closed intervals (a closed rectangle) is closed in $\mathbb{R}^n$.
- The empty set $\emptyset$ and $\mathbb{R}^n$ itself are by both open and closed sets.
- Not every subset of $\mathbb{R}^n$ is open or closed. There are a lot of subsets which are neither open nor closed. For example, the interval (0,1] in $\mathbb{R}$; the product of an open and a closed interval in $\mathbb{R}^2$.



The set of open sets in $\mathbb{R}^n$

$$\mathcal{T}_{\mathbb{R}^n} = \{U \subseteq \mathbb{R}^n \text{ open}\}$$

is a subset of all subsets of R^n and has the following properties:

- $\emptyset, \mathbb{R}^n \in \mathcal{T}_{\mathbb{R}^n}$
- $U_j \in \mathcal{T}_{\mathbb{R}^n}$ for all $j \in J \Rightarrow \cup_{j \in J} U_j \in \mathcal{T}_{\mathbb{R}^n}$
- $U_1, U_2 \in \mathcal{T}_{\mathbb{R}^n} \Rightarrow U_1 \cap U_2 \in \mathcal{T}_{\mathbb{R}^n}$.

We take these three properties as the model for a topology:

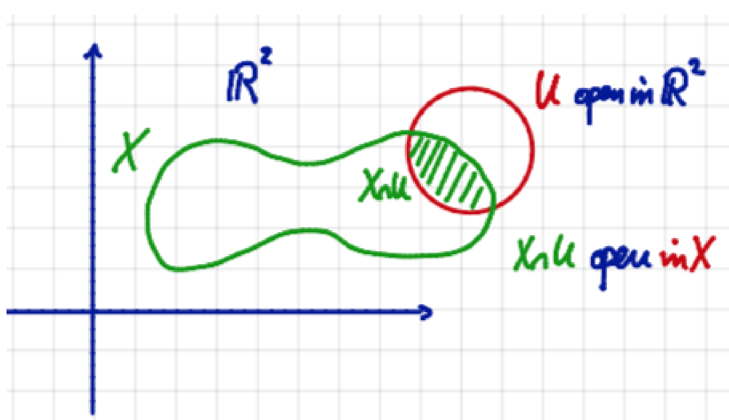
Definition: Topological spaces

Let X be a set together with a collection $\mathcal{T}_X$ of subsets which satisfy
(i) $\emptyset, X \in \mathcal{T}_X$
(ii) $U_j \in \mathcal{T}_X$ for all $j \in J \Rightarrow \cup_{j\in J} U_j \in \mathcal{T}_X$
(iii) $U_1, U_2 \in \mathcal{T}_X \Rightarrow U_1 \cap U_2 \in \mathcal{T}_X$.
(Note that in (ii), J can be an arbitrary indexing set.)
Then we say that the pair $(X,\mathcal{T}_X)$ is a topological space and the sets in $\mathcal{T}_X$ are called open. We also say that $\mathcal{T}_X$ defines a topology on X. We often drop mentioning $\mathcal{T}_X$ and just say X is a topological space (when the topology $\mathcal{T}_X$ is given otherwise). The complement of an open set is called a closed set.

Here are some examples of topological spaces which also demonstrate that some topologies are more interesting than others:

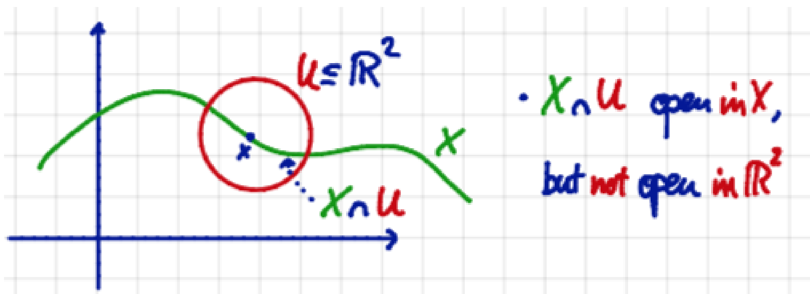
- $\mathbb{R}^n$ with $\mathcal{T}_{\mathbb{R}^n}$ as described above.
- An arbitrary set X with the **discrete topology** $\mathcal{T}_X = \mathcal{P}(X)$, where $Ph(X)$ is the power set of X, i.e., the set of all subsets of X. In the discrete topology, all subsets are open and hence all subsets are also closed.
- On an arbitrary set X, there is always the **coarse topology** $\mathcal{T}_X = \{\emptyset, X\}$.
- Let (X,d) be a **metric space**. Then we can imitate the construction of the standard topology on R^n and define the **induced topology** as the set of all $U \subseteq X$ such that for each $x \in U$ there exists an $r > 0$ so that $B(x,r) \subseteq U$. Here $B(x,r) = \{y \in X : d(x,y) < r\}$ is the metric ball of radius r centered at x.
- Let $(X,\mathcal{T}_X)$ be a topological space, let $Y \subset X$ be an arbitrary subset. The induced topology or **subspace topology of Y** is defined by

$$\mathcal{T}_Y := \{V \subset Y : \text{there is a } U \in \mathcal{T}_X \text{ such that } V = U \cap X\}.$$



Warning

It is important to note that that the property of being **an open subset** really depends on the bigger space we are looking at. Hence **open** always refers to being **open in** some given space.
For example, a set can be open in a space $X \subset \mathbb{R}^2$, but not be open in $\mathbb{R}^2$, see the picture.



Open sets are nice for a lot of reasons. First of all, they provide us with a way to talk about things that happen **close to** a point.

Definition: Open neighborhoods

We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of** x if there is an open subset $U \subseteq V$ with $x \in U$. If V itself is open, we call V an **open neighborhood**.

The type of maps that preserve open sets are the continuous maps:

Definition: Continuous maps

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is **continuous** if and only if, for every $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$, i.e., the **preimages of open sets are open**.
We denote the **set of continuous maps** $X \to Y$ by $C(X,Y)$.
Topological spaces form a **category** with morphisms given by continuous maps.

Examples of continuous maps include:

- Continuous maps $\mathbb{R}^n \to \mathbb{R}^m$ that you are familiar with from Calculus 2.

- If X carries the **discrete topology** then **every** map $f\colon X \to Y$ is continuous.
- If Y carries the **coarse topology** then **every** map $f\colon X \to Y$ is continuous.

Definition: Homeomorphisms

A continuous map $f\colon X \to Y$ is a **homeomorphism** if it is one-to-one and onto, and its inverse f^{-1} is continuous as well. Homeomorphisms preserve the topology in the sense that $U \subset X$ is open in X if and only if $f(U) \subset Y$ is open in Y.
Homeomorphisms are the isomorphisms in the category of topological spaces.

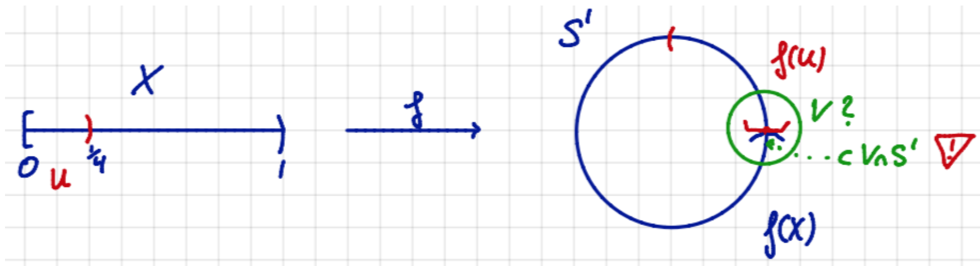
Some examples are:

- $\tan\colon (-\pi/2,\pi/2) \to \mathbb{R}$ is a homeomorphism.
- $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is a homeomorphism.

But not every continuous bijective map is a homeomorphism. Here is an example:

Example: A bijection which is not a homeomorphism

Let
$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$
be the unit circle considered as a subspace of $\mathbb{R}^2$. Define a map
$$f\colon [0,1) \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$
We know that f is bijective and continuous from Calculs and Trigonometry class. But the function f^{-1} **is not continuous**. For example, the image under f of the open subset $U = [0,\frac{1}{4})$ (open **in** $[0,1)$!) is not open in S^1. For the point $y = f(0)$ does not lie in any open subset V of $\mathbb{R}^2$ such that
$$V \cap S^1 = f(U).$$



Here is an extremely important property a subset in a topological space can have. We are going to use it quite often.

Definition: Compactness

Let X be a topological space. A subset $Z \subset X$ is called **compact** if for any collection $\{U_i\}_{i\in I}$, $U_i \subset X$ open, with $Z \subset \bigcup_{i\in I} U_i$ there exist **finitely many** $i_1, \ldots, i_n \in I$ such that $Z \subset U_{i_1} \cup \cdots \cup U_{i_n}$.
In other words, a subset Z in a topological space is compact iff every open cover $\{U_i\}_i$ of Z has a **finite** subcover.

- By the Theorem of Heine-Borel, a subset $Z \subset \mathbb{R}^n$ is **compact** if and only if it is **closed and bounded**. Being bounded means, that there is some (possibly huge) $r >> 0$ such that $Z \subset B_r(0)$.
- In patricular, neither $\mathbb{R}$ nor any $\mathbb{R}^n$ is compact.
- The n-dimensional disk $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ are compact.
- Finite sets, i.e., a subset which contains only finitely many elements, are always compact.
- If X carries the discrete topology, then a subset $Z \subset X$ is compact if and only if it is finite.
- If X carries the coarse topology, then every $Z \subset X$ is compact.

Definition: Connectedness

A topological space X is called **connected** if it is not possible to split it into the union of two non-empty, disjoint subsets which are both open and closed at the same time.
In other words, a space is connected if and only if the empty set and the whole space are the only subsets which are both open and closed.
Note that the image $f(X)$ of a connected space X under a continuous map $f\colon X \to Y$ is again connected.

Simple examples of connected spaces are given by intervals in $\mathbb{R}$.

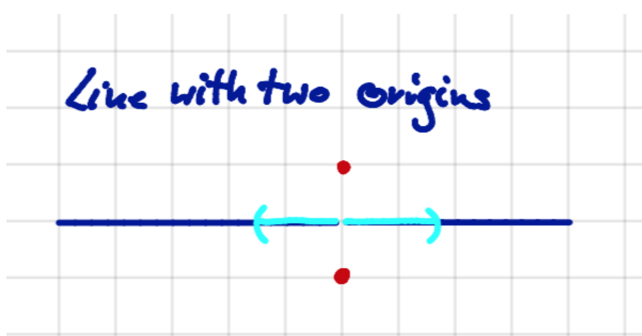
Definition: Hausdorff spaces

A topological space X is called **Hausdorff** if, for any two distinct points $x,y \in X$, there are two **disjoint** open subsets $U,V \subset X$ such that $x \in U$ and $y \in V$.
In other words, in a Hausdorff space we can separate points by open subsets.

Every subspace of $\mathbb{R}^N$ (with the relative topology) is a Hausdorff space. Moreover, basically all the spaces we look at will be Hausdorff. However, there are spaces which are not Hausdorff.

For a **typical counter-example**, consider two copies of the real line $Y_1 := \mathbb{R} \times \{1\}$ and $Y_2 := \mathbb{R} \times \{2\}$ as subspaces of $\mathbb{R}^2$. On $Y_1 \cup Y_2$, we define the equivalence realtion $(x,1) \sim (x,2)$ for all $x \neq 0$.

Let X be the set of equivalence classes. The topology on X is the quotient topology defined as follows (see also below): a subset $W \subset X$ is open in X if and only if both its preimages in $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{2\}$ are open.



Then X looks like the real line except that the origin is replaced with two different copies of the origin. Away from the double origin, X looks perfectly nice and we can separate points by open subsets. But every neighborhood of one of the origins contains the other. Hence we cannot separate the two origins by open subsets, and X is **not Hausdorff**.

Here are some useful facts about compact spaces:

Lemma: Closed in compact implies compact

1) Let X be a **compact** topological space. Let $Z \subset X$ be a **closed** subset. Then Z is **compact**.
2) Let Y be a **Hausdorff** space. Then any compact subset of Y is **closed**.

Let us prove the first assertion. The other one is left as a little exercise.

Proof: Let $\{U_i\}_{i \in I}$ be an open cover of Z. We set $U := X \setminus Z$. Then $\{U, U_i\}_{i \in I}$ is an open cover of X. Since X is compact, there exist $i_1, \ldots, i_n$ such that $X \subset U \cup U_{i_1} \cup \ldots \cup U_{i_n}$ and hence, by the definition of U, we have $Z \subset U_{i_1} \cup \ldots \cup U_{i_n}$. **QED**

Another useful fact:

Lemma: Continuous images of compact sets are compact

Let $f \colon X \to Y$ be continuous. Let $K \subset X$ be compact. Then $f(K) \subset Y$ is compact.
But, in general, if $Z \subset Y$ is compact, then $f^{-1}(Z) \subset X$ does not have to be compact.

As a consequence we can deduce a useful criterion for when continuous bijections are homeomorphisms:

Lemma: Continuous bijection from compact to Hausdorff is a homeomorphism

Let X be a compact space and Y be Hausdorff. If $f \colon X \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof: Since f is a bijection, there is a set-theoretic inverse map which we denote by $g := f^{-1} \colon Y \to X$. We need to show that g is continuous. So let $K \subset X$ be a closed subset. We are going to show that $g^{-1}(K) = f(K) \subset$ is closed in X. Since X is compact, K is also compact as a closed subset. Hence its image $f(K) \subset Y$ is compact. Since Y is Hausdorff, this implies that $f(K)$ is closed in Y. **QED**

Compactness, being Hausdorff, and being connected are important examples of topological properties:

Homeomorphisms preserve topological properties

Slogan: **Topology** is the study of properties which are preserved under homeomorphisms. From this point of view, a **topological property** is by definition a property that is preserved under homeomorphisms.
Hence, roughly speaking, from the point of view of a topologist, two spaces which are homeomorphic are basically the same.
For example, if $f: X \to Y$ is a **homeomorphism**, then **X is compact if and only if Y is compact**. For, both f and its inverse f^{-1} are continuous and surjective maps. Hence if X is compact, so is $f(X) = Y$; and if Y is compact, so is $f^{-1}(Y) = X$.

We will remind ourselves of many other important topological properties along the way.

Constructing new spaces out of old

There are sveral ways to construct topological spaces. Here are two important constructions that we are going to use:

Definition: Product topology

Let X and Y be two topological spaces. The **product topology** on $X \times Y$ is the coarsest topology, i.e., the topology with fewest open sets, such that the projection maps $X \times Y \to X$ and $X \times Y \to Y$ are both continuous. More concretely, a subset $W \subset X \times Y$ is open in the product topology if for every point $w = (x,y) \in W$ there are open subsets $x \in U \subset X$ and $y \in V \subset Y$ with $U \times V \subset W$.

Definition: Disjoint unions or sums of spaces

Let X and Y be two topological spaces. We denote by $X \sqcup Y$ the **disjoint union** (or sum) of X and Y. Recall that as a set we can define $X \sqcup Y$ as

$$X \sqcup Y = X \times \{0\} \cup Y \times \{1\}.$$

(In other words, we take one copy of X and one copy of Y and by the indexing we make sure that we keep them apart.)

The disjoint union inherits a topology by defining

$$\mathcal{T}_{X\sqcup Y} = \{U \sqcup V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}.$$

Another important construction for producing new topological spaces is to take quotients.

- **Quotient Spaces**

Let X be a topological space. Let $\sim$ be an equivalence relation on X. For any $x \in X$ let $[x]$ be the equivalence class of x. We denote as usually the set of equivalence classes by

$$X/\sim := \{\text{set of equivalence classes under } \sim\} = \{[x] : x \in X\}.$$

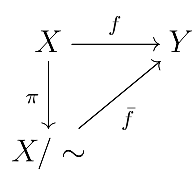
Let $\pi\colon X \to X/\sim,\ x \mapsto [x]$ be the natural projection. The **quotient topology** is defined by

$$U \subset X/\sim \text{ open} \iff \pi^{-1}(U) \subset X \text{ open}.$$

Note that the map $\pi\colon X \to X/\sim$ is continuous by definition.

The quotient topology is the coarsest topology, in the sense that it has fewest open sets, such that the quotient map π is continuous.

The quotient topology has the following **universal property**: For any topological space Y and for any maps $f\colon X \to Y$ which descends to a map $\bar{f}\colon X/\sim \to Y$, i.e., f is constant on equivalence classes, such that the diagram



commutes, the map f is continuous iff $\bar{f}$ is continuous.

Many important examples of spaces that we will study arise as follows:

- Take a subset $X \subset \mathbb{R}^n$ and consider it with the induced topology as a subset.
- Consider an interesting equivalence relation $\sim$ on X and take the quotient topological space $X/\sim$.

Let us look at some **examples** of this procedure:

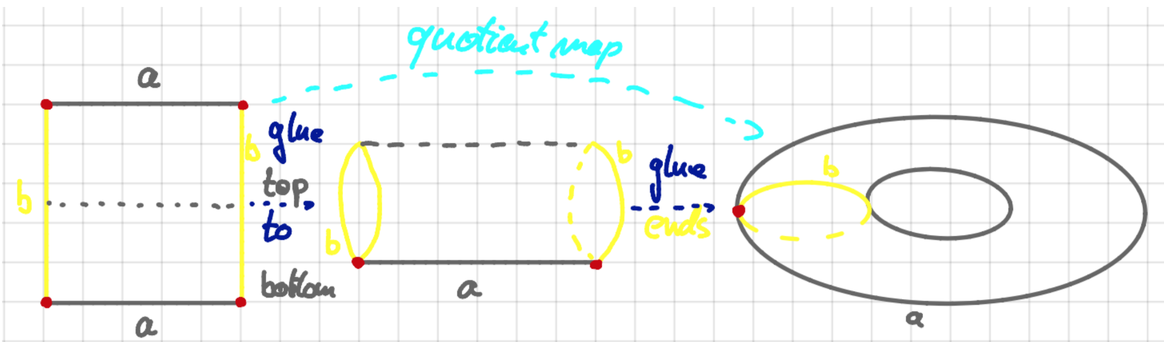
Torus

We start with the square

$$S := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1\} \subset \mathbb{R}^2$$

with the subspace topology induced from the topology of $\mathbb{R}^2$. Now we would like to glue opposite sides to each. This corresponds to taking the **quotient**

$$T := S/((x,0) \sim (x,1) \text{ and } (0,y) \sim (1,y)).$$



Real projective space

Real projective space $\mathbb{RP}^n$ is the space of lines in $\mathbb{R}^{n+1}$ through the origin. As a topological space it can be constructed as follows:
We define the equivalence relation $\sim$ on the n-sphere S^n by identifying antipodal points, i.e., $x \sim y \iff y = -x$. Then we have

$$\mathbb{RP}^n = S^n / \sim$$

and equip it with the quotient topology. Since S^n is compact and $\mathbb{RP}^n$ is the continuous image of S^n (under the quotient map), we see that **$\mathbb{RP}^n$ is compact**.

There is also a complex version:

Complex projective space

Again, complex projective space $\mathbb{CP}^n$ is the space of one-dimensional $\mathbb{C}$-vector subspaces in $\mathbb{C}^{n+1}$. It can be topologized as follows:

We define the equivalence relation $\sim$ on the sphere S^{2n+1} by $x \sim y$ if and only if there is a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $y = \lambda x$ where we think S^{2n+1} as the subspace of points x in $\mathbb{C}^n$ with $|x| = 1$. Then we have

$$\mathbb{C}\mathrm{P}^n = S^{2n+1}/\sim$$

and equip it with the quotient topology. Since S^{2n+1} is compact, $\mathbb{C}\mathrm{P}^n$ **is compact**.
Aside: Note that the "topological dimension" of $\mathbb{C}\mathrm{P}^n$ is $2n$ (we have not said what that means though). The n rather refers to the dimension as a complex manifold.

Projective spaces play an important role in geometry and topology. We will meet them quite frequently during this course (and future courses).

It happens also that it might be necessary to present a well-known space in a different form. For example, we can write spheres as quotients. We will see that this is just one example of a whole class of interesting spaces.

Sphere as a quotient

For every $n \geq 1$, there is a homeomorphism

$$\bar{\rho}\colon D^n/\partial D^n \xrightarrow{\approx} S^n.$$

There are in fact many different ways to construct such a homeomorphism. Let us write down one in concrete terms for the special case $n = 2$. The general case follows by throwing in more coordinates.
We define a continuous map $\rho\colon D^2 \to S^2$ such that

$$\begin{cases} \rho(0,0) = (0,0,-1) & \text{and} \\ \rho(x,y) = (0,0,+1) & \text{for all } (x,y) \in \partial D^2 = S^1. \end{cases}$$

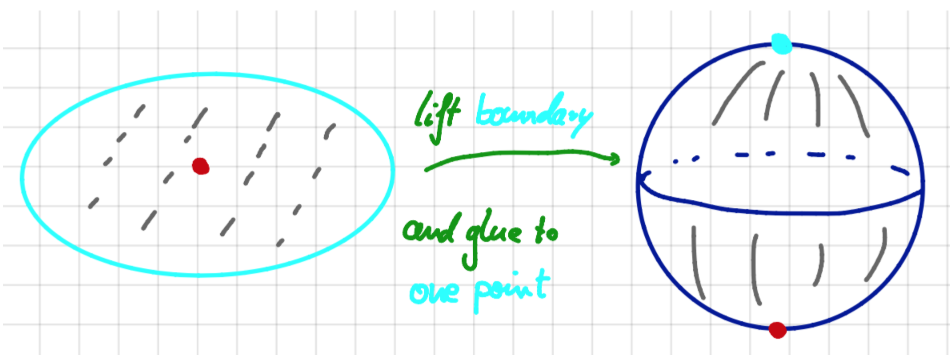
Since ρ will be constant on ∂D^n, it will induce a map $\bar{\rho}$ on the quotient $D^2/\partial D^2$.
We define ρ as a rotation invariant map which sends the inner part of D^2 of points with radius less than $1/2$ mapping onto the lower hemisphere of S^2 and the outer part of D^2 of points with radius greater than $1/2$ mapping onto the upper hemisphere $\rho\colon D^2 \to S^2$ by

$$\rho(x,y) = \begin{cases} \left(2x,2y,-\sqrt{1-4(x^2+y^2)}\right) & \text{if } x^2+y^2 \leq 1/4 \\ \left(f(x,y)x,f(x,y)y,\sqrt{1-f(x,y)^2(x^2+y^2)}\right) & \text{if } x^2+y^2 \geq 1/4 \end{cases}$$

where we denote $f(x,y) = 4 - 4\sqrt{x^2+y^2}$ (to make the formula fit in a frame). This map is well-defined also for points with $x^2 + y^2 = 1/4$. Moreover, ρ is contiunuous, as a composite of continuous functions, and constant on ∂D^2.
An inverse map can be defined by

$$S^2 \to D^2/\partial D^2,\ (x,y,z) \mapsto \begin{cases} (\frac{1}{2}x, \frac{1}{2}y) & \text{if } -1 \leq z \leq 0 \\ (g(x,y)x, g(x,y)y) & \text{if } 0 \leq z < 1 \\ \text{class of } \partial D^2 & \text{for } (0,0,1) \end{cases}$$

where we denote $g(x,y) = \frac{1-\sqrt{1-\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}$. Note that this map is well-defined also for $z = 0$, since then $x^2 + y^2 = 1$ and $g(x,y) = 1/2$.



- **Compactifications**

The concrete maps we wrote down in the previous example are kind of ugly. But there is another way to show that there is such a homeomorphism $D^n/\partial D^n \approx S^n$.

For we can also consider S^n as the **one-point compactification** of $\mathbb{R}^n$. Let us first say what that means:

Definition: One-point compactification

1) If Y is a compact Hausdorff space and $X \subset Y$ is a proper subspace whose closure equals Y, then Y is called a **compactification** of X.
If $Y \setminus X$ consists of a single point, then Y is called the **one-point compactification** of X.

2) Let X be a topological space with topology $\mathcal{T}_X$. Let ∞ denote an abstract point which is not in X and let $\hat{X} := X \cup \{\infty\}$. We define a topology $\mathcal{T}_{\hat{X}}$ on $\hat{X}$ as follows:

- each open set in X is an open set in $\hat{X}$, i.e., $\mathcal{T}_X \subset \mathcal{T}_{\hat{X}}$ and
- for each compact subset $K \subseteq X$, define an open subset $U_K \in \mathcal{T}_{\hat{X}}$ by $U_K := (X \setminus K) \cup \{\infty\}$.

Then $\hat{X}$ is a **one-point compactification** of X.
To see that $\hat{X}$ actually is compact, take any open cover of $\hat{X}$. Then at least one of the open sets contains ∞. Hence that set covers $(X \setminus K) \cup \{\infty\}$ for some compact set K. Since K is compact, finitely many of the remaining open sets suffice to cover K and therefore all of $\hat{X}$.

Examples of one-point compactifications are **spheres**. For S^n is the one-point compactification of $\mathbb{R}^n$. For $n = 1$, one can think of S^1 as taking the real number line and connect the two ends at infinity in one point ∞ to close the circle. More generally, one can construct a homeomorphism via **stereographic projection**.

As an application, we give a new proof $D^n/\partial D^n \approx S^n$:

Sphere as a quotient revisited

For every $n \geq 1$, there is a homeomorphism

$$\bar{\rho}\colon D^n/\partial D^n \xrightarrow{\approx} S^n.$$

Since $S^n \approx \mathbb{R}^n \cup \{\infty\}$, it suffices to construct homeomorphism

$$\rho\colon D^n \xrightarrow{\approx} \mathbb{R}^n \cup \{\infty\},\ x \mapsto \begin{cases} \frac{x}{1-|x|} & \text{if } |x| < 1 \\ \infty & \text{if } |x| = 1. \end{cases}$$

We claim that the map ρ is **continuous**. To show this, we use the sequential criterion of continuity. Let (a_n) be a sequence in D^n with $\lim_{n\to\infty} a_n = c$. If $c \in D^n \setminus \partial D^n$ is an interior point, then $\rho(c) \in \mathbb{R}^n$ and we know $\lim_{n\to\infty} \rho(a_n) = \rho(c)$, since the restriction of ρ to $D^n \setminus \partial D^n$ is a composite of continuous maps and the a_n will all be in $D^n \setminus \partial D^n$ for n sufficiently large. If $c \in \partial D^n$ is a boundary point, then $\rho(c) = \infty$. Since $a_n \to c$, the sequence $(\rho(a_n))$ is **unbounded**, since the denominator of $\rho(a_n)$ tends to 0 while the norm of the nominator tends to 1.
Hence for any compact subset K in $\mathbb{R}^n$, i.e., for any closed and bounded $K \subset \mathbb{R}^n$, there is a natural number $N(K)$ such that $\rho(a_n) \notin K$ for all

$n \geq N(K)$. That means that the sequence $(\rho(a_n))$ **converges in the topology** of $\mathbb{R}^n \cup \{\infty\}$ to $\rho(c) = \infty$. This shows that ρ is continuous.
We also know that $\bar{\rho}$ is **bijective**, since the restriction $\rho\colon D^n \setminus \partial D^n \to \mathbb{R}^n$ is bijective and ρ sends ∂D^n to ∞. Hence $\bar{\rho}$ is a continuous bijection from a compact space to a Hausdorff space. As we have seen above, this implies that $\bar{\rho}$ is a homeomorphism.

- **Cell complexes**

Another way to think of the above procedure is the following. The sphere consists of two parts that we **glue together**:

- an open n-disk, i.e., the open interior $D^n \setminus \partial D^n$,
- and a single point, which corresponds to the class of the boundary ∂D^n; on S^2 we can picture this point as the northpole (the light blue dot in the above picture).

Topologists think of such **building blocks** as the **cells** of a space. However, not all spaces can be built this way. So let us make precise what is needed:

Definition: Cell complexes

A **cell complex** or ***CW*-complex** is a space X which results from the following inductive procedure:

(1) Start with a discrete set X^0. The points of X^0 will be the **0-cells** of X.

(2) If X^{n-1} is defined, we construct the ***n*-skeleton** X^n by attaching ***n*-cells** e^n_α to X^{n-1} via continuous maps $\varphi_\alpha\colon S^{n-1} \to X^{n-1}$. This means that X^n is the **quotient space** of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$ and $\varphi_\alpha\colon \partial D^{n-1}_\alpha = S^{n-1}_\alpha \to X^{n-1}$. Thus, as a set, X^n consists of X^{n-1} together with a union of n-cells e^n_α each of which is an open n-disk $D^n_\alpha \setminus \partial D^n_\alpha$.

(3) If this process stops after finitely many steps, say N, then $X = X^N$. But it is also allowed to continue with the inductive process indefinitely. In this case, one defines $X = \bigcup_n X^n$ and equips X with the **weak topology**, i.e., a set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

We have already seen some **examples** of cell complexes:

- The **sphere** S^n is a cell complex with just two cells: **one 0-cell** e^0 (that is a point) and **one n-cell** e^n which is attached to e^0 via the constant map $S^{n-1} \to e^0$. Geometrically, this corresponds to expressing S^n as $D^n/\partial D^n$: we take the open n-disk $e^n = D^n \setminus \partial D^n$ and collapse the boudnary ∂D^n to a single point which is e^0.
- **Real projective space** $\mathbb{RP}^n$ is a cell complex with **one cell in each dimension** up to n. To show this we proceed inductively. We know that $\mathbb{RP}^0$ consists of a single point, since it is S^0 whose two antipodal points are identified. Now we would like to understand how $\mathbb{RP}^n$ can be constructed from $\mathbb{RP}^{n-1}$: We embedd D^n as the **upper hemisphere** into S^n, i.e., we consider D^n as $\{(x_0, \ldots, x_n) \in S^n : x_0 \geq 0\}$. Then

$$\mathbb{RP}^n = S^n/x \sim -x = D^n/(x \sim -x \text{ for } \textbf{boundary} \text{ points } x \in \partial D^n).$$

But ∂D^n is just S^{n-1}. Thus the quotient map

$$S^{n-1} \to S^{n-1}/\sim\, = \mathbb{RP}^{n-1}$$

attaches an n-cell e^n, the open interior of D^n, at $\mathbb{RP}^{n-1}$. Thus we obtain $\mathbb{RP}^n$ from $\mathbb{RP}^{n-1}$ by attaching one n-cell via the quotient map $S^{n-1} \to \mathbb{RP}^{n-1}$. Summarizing, we have shown that $\mathbb{RP}^n$ is a cell complex with one cell in each dimension from 0 to n:

$$\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n.$$

- We can continue this process and build the **infinite projective space** $\mathbb{RP}^\infty := \bigcup_n \mathbb{RP}^n$. It is a cell complex with one cell in each dimension. We can think of $\mathbb{RP}^\infty$ as the space of lines in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

The **torus** is a cell complex with **one 0-cell**, **two 1-cells** and **one 2-cells**. This should be apparent from the construction of the torus as a quotient of a square that we have seen above. Starting with X^0 being a point p, the red dot in the picture above. Then we attach two open 1-cells $e^1_a, e^1_b \subset D^1$ via the two constant maps

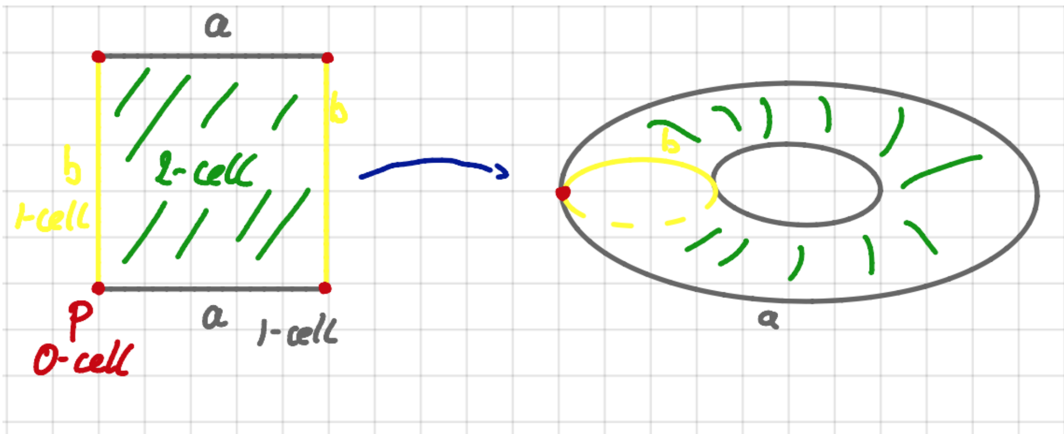
$$\varphi^1_a, \varphi^1_b \colon S^0 \to X^0$$

where we think of $e^1 = (0,1) \subset [0,1] = D^1$ as the open unit interval. (In the picture they look like two straight lines, but we should think of the end points being attached to p.)

Finally, we attach an open 2-cell $e^2 \subset D^2$ via the attaching map

$$\varphi^2 \colon S^1 \to X^1,\ \varphi^2(x,y) = \begin{cases} x \in e^1_{\mathbf{a}} & \text{if } y > 0 \\ x \in e^1_{\mathbf{b}} & \text{if } y < 0 \\ p & \text{if } y = 0. \end{cases}$$

Note that this is a well-defined map, since we have identfied the endpoints in X^1 with p and hence $(1,0)$ and $(-1,0)$ are sent to the same point p.



- Actually, every **compact smooth manifold** can be turned into a finite cell complex. This illustrates the vast scope and importance of cell complexes in algebraic topology.

What makes topology unique

Note that the ability to **build spaces by gluing** together cells (or other specific spaces) makes life as a topologist particularly comfortable. For example, we will see that this procedure will often allow us to create spaces with given algebraic invariant. This flexibility together with the concept of **homotopy**, which we will explore next, puts **algebraic topologists** in a unique position and led to the **solution of a lot of problems**, not just in topology. **Geometry**, in its various forms, is usually much **more rigid** and does not allow us to perform such maneuvers.

Note that there is a direct way to define the **Euler characteristic of cell complexes**. We will later see the reason why this is the correct definition using homology. Right now we can already check at the example of a tetrahedron that this definition agrees with Euler's formula we saw in the first lecture.

Definition: Euler characteristic for cell complexes

The Euler number of a cell complex X (with cells in dimension at most n) is defined to be the integer

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \#\{k - \text{dimensional cells that are attached to } X^{k-1}\}.$$

For **example**, the Euler characteristic of S^n is

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

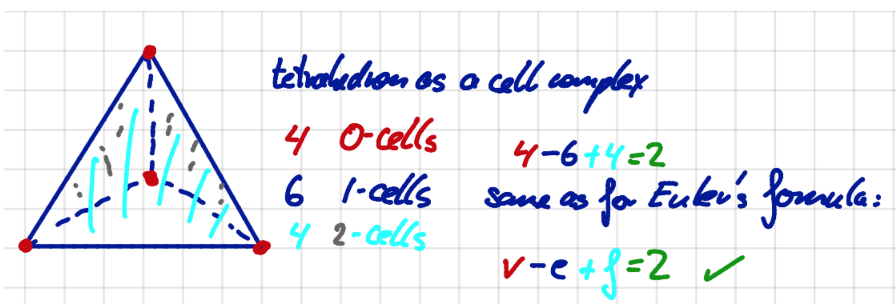
For real projective n-space we get

$$\chi(\mathbb{R}\mathrm{P}^n) = 1 - 1 + 1 - \cdots + (-1)^n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

For the torus, we get

$$\chi(T) = 1 - 2 + 1 = 0.$$

To compare this definition with Euler's formula we used in the first lecture, let us look at the tetrahedron which is also a cell complex:



- **Homotopy**

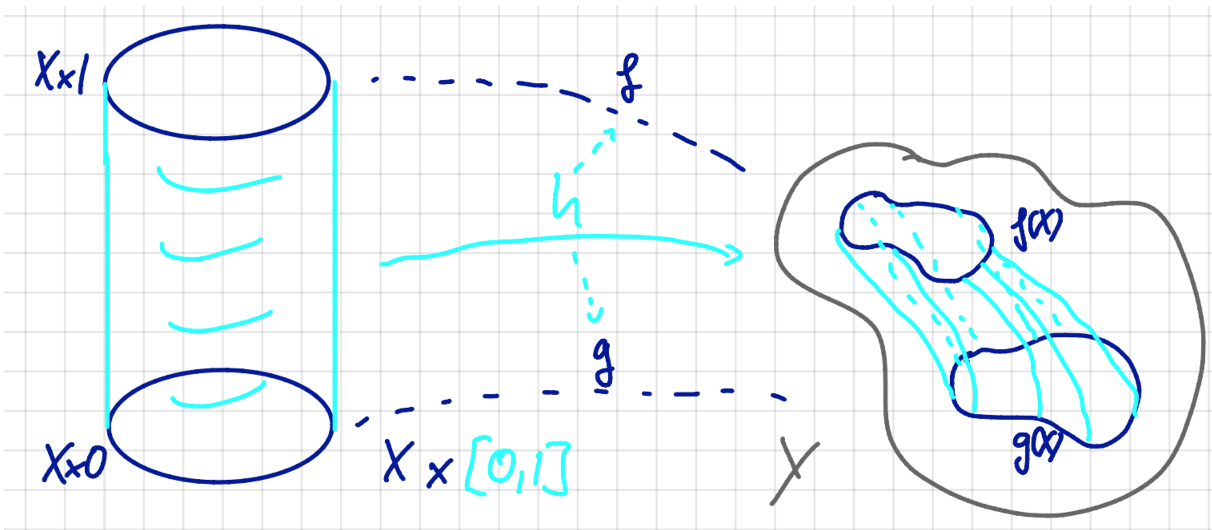
Homotopy is a fundamental notion in topology. Let us start with a definition and then try to make sense of this.

Definition: Homotopies

Let $f_0, f_1 \colon X \to Y$ be two continuous maps. Then f_0 and f_1 are called **homotopic**, denoted $f_0 \simeq f_1$, if there is a continuous map $h \colon X \times [0,1] \to Y$ such that, for all $x \in X$,

$$h(x,0) = f_0(x), \text{ and } h(x,1) = f_1(x).$$

Homotopy defines an equivalence relation (exercise!) on the set of continuous maps from X to Y. The set of equivalence classes of continuous maps from X to Y modulo homotopy is denoted by $[X,Y]$.



Definition: Homotopy equivalences and contractible spaces

- A continuous map $f\colon X \to Y$ is called a **homotopy equivalence** if there is a continuous map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
- Two spaces X and Y are called **homotopy equivalent** if there exists a homotopy equivalence f between X and Y. This is often denoted by $X \simeq Y$.
- A space which is homotopy equivalent to a one-point space is called **contractible**.

For **example**, $\mathbb{R}^n$ is contractible, since

$$h\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n,\ (t,x) \mapsto (1-t)x$$

defines a homotopy between the identity map on $\mathbb{R}^n$ and the constant map $\mathbb{R}^n \to \{0\} \subset \mathbb{R}^n$ to the one-point space consisting of the origin. For the same reason, the n-disk D^n is contractible.

However, it is not always obvious which spaces are homotopy equivalent to each other. So it will be useful to develop some intuition for homotopy equivalences. There is a particular type that is easier to spot:

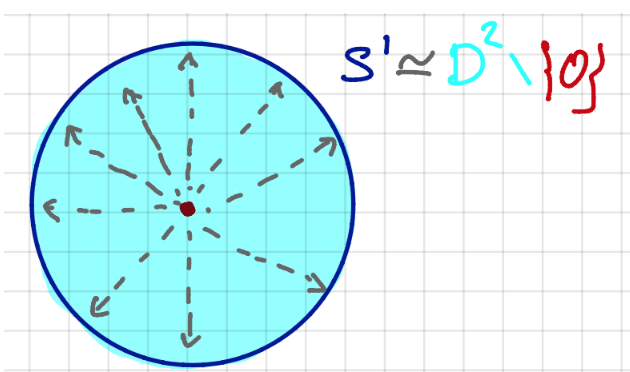
Definition: Deformation retracts

Let X be a topological space and $A \subset X$ a subspace.

- Then A is called a **retract** of X if there is a **retraction** $\rho\colon X \to A$, i.e., there is a continuous map $\rho\colon X \to A$ with $\rho_{|A} = \mathrm{id}_A$.
- Note that we can **consider** ρ also as a **map** $X \to X$ via the inclusion $X \xrightarrow{\rho} A \subset X$. If ρ is then in addition homotopic to the identity of X, then A is called a **deformation retract** of X. In this case, ρ is called a **deformation retraction**. Note that in this case, ρ and the inclusion $A \subset X$ are **mutual homotopy inverses**.
- If this homotopy between ρ and id_X can be chosen such that all points of A **remain fixed**, i.e., the homotopy $h(t,a) = a$ for all $a \in A$ and all $t \in [0,1]$, then ρ is called a **strong deformation retraction** and A is called a **strong deformation retract** of X.

For a deformation retraction, one can think of the homotopy h as a map which during the time from 0 to 1 pulls back all the points of X into the subspace A, and leaves the whole time the points in A fixed. Here are some examples:

- The origin $\{0\}$ is a strong deformation retract fo $\mathbb{R}^n$ and of the n-disk D^n.
- For any topological space Y, the product $Y \times \{0\}$ is a strong deformation retract of $Y \times \mathbb{R}^n$ and $Y \times D^n$. For example, the circle $S^1 \times \{0\}$ is a strong deformation retract of the solid torus $S^1 \times D^2$.
- The n-sphere S^n is a strong deformation retract of the punctured disk $D^{n+1} \setminus \{0\}$ and also of $\mathbb{R}^{n+1} \setminus \{0\}$.



Why homotopy?

The simplest reason why we consider the homotopy relation is that **it works**. It is **fine enough** such that all the tools that we are going to define are invariant under homotopy, i.e., they are constant on equivalence classes. But it is also **coarse enough** that it identifies enough things such that many problems become simpler and in fact solvable.
With respect to first point, one can consider the **homotopy category hoTop** of spaces, i.e., the category whose objects are topological spaces and whose sets of morphisms from X to Y are the sets of homotopy classes of maps $[X,Y]$, ssatifies a universal property for invariants.
With respect to the second point, we just indicate that life in **hoTop** is much easier because there are much **fewer morphisms**. For example, there are many and complicated continuous maps $S^1 \to \mathbb{C} \setminus \{0\}$. But there are very few homotopy classes of such maps, since $[S^1,\mathbb{C} \setminus \{0\}] = \mathbb{Z}$, **up to homotopy** a map $S^1 \to \mathbb{C} \setminus \{0\}$ is determined by the winding number, i.e., the number of times it goes around the origin.
To convince ourselves that homotopy actually works, we remark that homotopy is even fine enough to detect diffeomorphism classes between smooth manifolds and helped for example to classify manifolds up to bordism. But this is a story we save for a future lecture/class.
If you are still not convinced, then let us remark that to study things up-to-homotopy is **so useful** that mathematicians work hard to find analogs of the homotopy relation and the **homotopy category** in many different areas. If you want to learn more about this, have a look at **Quillen**'s highly influential book on **Homotopical Algebra**. You will also see an example in **homological algebra** where one talks about homotopies between chain complexes.