MA 3403Algebraic Topology
Lecturer: Gereon Quick
Lecture 03

3. Singular chains and homology

We would like to make the idea to study a topological space X by considering all continuous maps from test spaces into X precise. We start with defining an important class of **test spaces**:

Definiton: The standard n-simplex

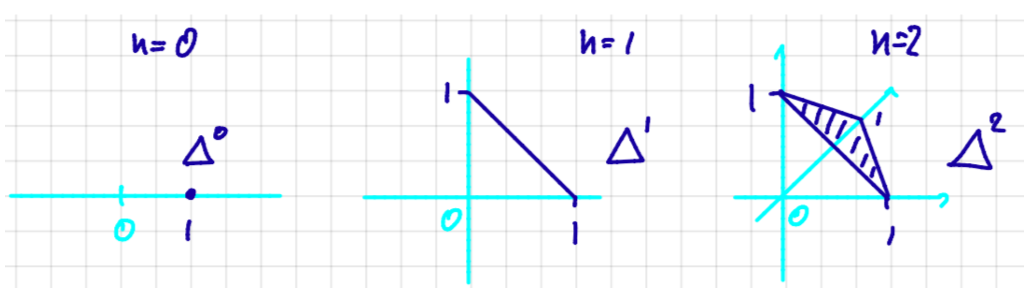
For $n \geq 0$, the **standard n-simplex Δ^n** is the set $\Delta^n \subset \mathbb{R}^{n+1}$ defined by

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for all } i\}.$$

Another way to describe Δ^n is to say that it is the **convex hull** of the standard basis $\{e_0, \ldots, e_n\}$ in $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{ \sum_i t_i e_i : \sum_i t_i = 1, t_i \geq 0 \right\}.$$

The t_i are called **barycentric coordinates**.
It will be convenient to keep both these descriptions in mind.



The standard simplices are related by **face maps** for $0 \leq i \leq n$ which can be described as

$$\phi_i^n(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

with the 0 inserted at the ith coordinate (t_0 is the 0th coordinate).

Using the **standard basis**, ϕ_i^n can be described as the affine linear map (a translation plus a linear map)

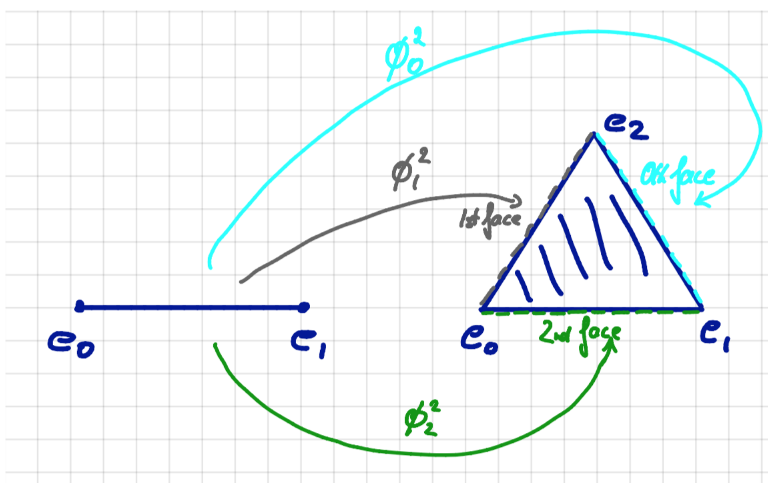
$$\phi_i^n\colon \Delta^{n-1} \hookrightarrow \Delta^n \text{determined by } \phi_i^n(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i. \end{cases}$$

A short way of expressing the above formula for ϕ_i^n is that it embedds Δ^{n-1} into Δ^n by omitting the ith vertex (that is what the hat in the following formula means):

$$\phi_i^n = [e_0, \ldots, e_{i-1}, \widehat{e_i}, e_{i+1}, \ldots, e_n]\colon \Delta^{n-1} \to \Delta^n.$$

Definiton: Faces

Note that ϕ_i^n maps Δ^{n-1} onto the subsimplex opposite to the ith corner, or in the standard basis, opposite to e_i. We call the image of ϕ_i^n the **ith face** of Δ^n (which is opposite to e_i).
Note that the union of the images of all the face inclusions is the boundary of Δ^n.



The face maps satisfy a useful identity, sometimes called simplicial identity:

Lemma: A useful identity

For all $0 \leq j < i \leq n+1$ we have

$$\phi_i^n \circ \phi_j^{n-1} = \phi_j^n \circ \phi_{i-1}^{n-1}. \tag{1}$$

The first composition, $\phi_i^n \circ \phi_j^{n-1}$, results in a 0 at the jth and $i+1$st place. The second composition, $\phi_j^n \circ \phi_{i-1}^{n-1}$, has the effect to insert a 0 at the $(i-1)$st place and then one at the jth place. But since $j < i$, this means that, in both cases, we have an extra 0 at the jth and at the $(i+1)$st spot.
Thus both compositions yield

$$\begin{aligned} &\phi_i^n \circ \phi_j^{n-1}(t_0, \dots, t_{n-2}) \\ =&(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{i-2}, 0, t_{i-1}, \dots, t_{n-2}) \\ =&\phi_j^n \circ \phi_{i-1}^{n-1}(t_0, \dots, t_{n-2}). \end{aligned}$$

We are going to study a topological space X by looking at all the continuous maps from simplices into X. We give those sets of maps a name:

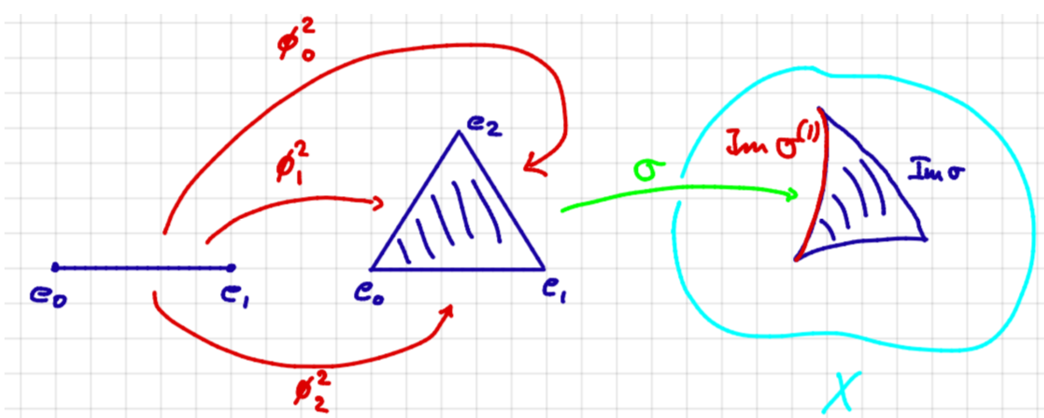
Definition: Singular n-simplices

Let X be any topological space. A **singular n-simplex in X** is a continuous map $\sigma \colon \Delta^n \to X$. We denote by $\mathrm{Sing}_n(X)$ the set of all n-simplices in X. For example, $\mathrm{Sing}_0(X)$ is just the set of points of X. But, in general, $\mathrm{Sing}_n(X)$ carries more interesting information for $n \geq 1$.

For $0 \leq i \leq n$, we can use the face maps ϕ_i^n to define maps

$$d_i^n \colon \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X),\ \sigma \mapsto \sigma \circ \phi_i^n$$

by sending an n-simplex σ to the $n-1$-simplex defined by precomposition with the ith face inclusion. The image $d_i^n(\sigma) = \sigma \circ \phi_i^n$ is called the **ith face of σ**. We will sometimes use the notation $\sigma^{(i)} := \sigma \circ \phi_i^n$ for the ith face.



Since the collection of all face inclusions ϕ_i^n forms the boundary of Δ^n, we can use the maps d_i^n to talk about the **boundary** of an n-simplex. The boundaries of simplices will actually play a crucial role in the story.

We need to make this precise. First let us look at a simple example. Let X be some space and $\sigma\colon \Delta^1 \to X$ be a 1-simplex in X. Assume $\sigma(e_0) = x_0 \neq x_1 = \sigma(e_1)$. Then we would like to say that the boundary of σ is given by x_0 and x_1.

Now let us assume that $\sigma\colon \Delta^1 \to X$ is another 1-simplex in X which forms a **closed loop**, i.e., $\sigma(e_0) = \sigma(e_1) = x \in X$. Now we would like to say that σ has no boundary (since it is a loop). Our face maps express $\sigma(e_0) = \sigma(e_1)$ as

$$d_0^1(\sigma) = d_1^1(\sigma).$$

It would be nice if we had a short way to formulate that the boundary of σ vanishes. For example, it would be nice if we were allowed to rewrite this equation as

$$\partial(\sigma) = d_0^1(\sigma) - d_1^1(\sigma) = 0.$$

But, so far, $\mathrm{Sing}_0(X)$ is just a set and we are not allowed to add or subtract elements. We are now going to remedy this defect, since algebraic operations make life much easier. Therefore, we **formally extend** $\mathrm{Sing}_n(X)$ into an abelian group.

The general way to turn a set B into an abelian group, is to form the associated free abelian group. **The idea is to add the minimal amount of structure and relations** to turn B into an abelian group. Since this is an important construction, we recall how this works:

Good to know about free abelian groups

- Any abelian group A can be seen as a $\mathbb{Z}$-module with $n \cdot a := a + \cdots + a$ (n summands), for $n \in \mathbb{N}$ and $a \in A$, and $(-n) \cdot a := -n \cdot a$. Thus, abelian groups are in bijection with $\mathbb{Z}$-modules. An abelian group A is called **free** over a subset $B \subset A$ if B is a **$\mathbb{Z}$-basis**, i.e., if any element $a \in \mathbb{Z}$ can be **written uniquely as a $\mathbb{Z}$-linear combination** of elements in B. The cardinality of a basis is the same for any choice of basis and is called the **rank of A**.
- The group $\mathbb{Z}^r$ is free abelian with **basis** $\{e_1, \ldots, e_r\}$ with $e_i = (0, \ldots, i, 0, \ldots, 0)$ (the 1 in the ith position).
- Note that, for example, the group $\mathbb{Z}/2\mathbb{Z}$ is **not free**, since it does not admit a basis: the vector $1 \in \mathbb{Z}/2\mathbb{Z}$ cannot be in a basis since $2 \cdot 1 = 0$.

- Given a set B, there is an **associated free abelian group** $\mathbb{Z}B$ with basis B which is characterized by the following **universal property**: any map $f\colon B \to A$ of sets into an arbitrary abelian group A can be **extended uniquely** to a group homomorphism $\phi\colon \mathbb{Z}B \to A$ with $\phi(b) = f(b)$ for all $b \in B$.

  In terms of category theory, this means that the functor **AbGroups** $\to$ **Sets** which forgets the group structure, is right adjoint to the functor

  $$\textbf{Sets} \to \textbf{AbGroups},\ B \mapsto \mathbb{Z}B.$$

  In other words,

  $$\mathrm{Hom}_{\textbf{Sets}}(B,A) = \mathrm{Hom}_{\textbf{AbGroups}}(\mathbb{Z}B,A).$$

- Any **subgroup** of a free abelian group F is a free abelian group.

We apply this construction to the set $B = \mathrm{Sing}_n(X)$:

Definition: Singular n-chains

The group $S_n(X)$ of **singular n-chains in X** is the **free abelian group** generated by n-simplices

$$S_n(X) := \mathbb{Z}\mathrm{Sing}_n(X).$$

Thus an **n-chain** is a finite $\mathbb{Z}$-linear combination of simplices

$$\sum_{i=1}^{k} a_i\sigma_i,\ a_i \in \mathbb{Z},\ \sigma_i \in \mathrm{Sing}_n(X).$$

Note: If $n < 0$, $\mathrm{Sing}_n(X)$ is defined to be empty and $S_n(X)$ is the trivial abelian group $\{0\}$. So whenever we talk about n-chains, n will be assumed to be nonnegative.

Definition: Boundary operators

We define the **boundary operator** by

$$\partial_n\colon \mathrm{Sing}_n(X) \to S_{n-1}(X),\ \partial(\sigma) = \sum_{i=0}^{n}(-1)^i d_i^n\sigma = \sum_{i=0}^{n}(-1)^i\sigma^{(i)}.$$

We can then extend this to a homomorphism, which we also call boundary operator, by additivity, i.e.,

$$\partial_n \colon S_n(X) \to S_{n-1}(X),\ \partial\left(\sum_{j=1}^{m} a_j\sigma_j\right) := \sum_{j=1}^{m} a_j\partial(\sigma_j).$$

Note that we will often just write ∂ instead of ∂_n.

In particular, for the loop σ we considered above we are allowed to write in $S_0(X)$

$$\partial_1(\sigma) = d_0^1(\sigma) + (-1)d_1^1(\sigma) = d_0^1(\sigma) - d_1^1(\sigma) = 0.$$

A loop is an example of a particularly **important class of chains**. For, the equation $\partial(\sigma) = 0$ **expresses algebraically** that σ has **no boundary**. We give such chains a special name:

Definition: Cycles

An **n-cycle in X** is an n-chain $c \in S_n(X)$ with $\partial_n c = 0$. We denote the **group of n-cycles** by

$$\begin{aligned} Z_n(X) &:= \operatorname{Ker}\left(\partial_n \colon S_n(X) \to S_{n-1}(X)\right) \\ &= \{c \in S_n(X) : \partial_n(c) = 0\} \subseteq S_n(X). \end{aligned}$$

Note that the group of **0-cycles** is all of $S_0(X)$, since every 0-chain is mapped to 0:

$$Z_0(X) = S_0(X).$$

To find another **example** of a 1-**cycle** we could consider a 1-chain $c = \alpha+\beta+\gamma$ where $\alpha, \beta, \gamma \colon \Delta^1 \to X$ are singular 1-simplices such that

$$\alpha(e_1) = \beta(e_0), \beta(e_1) = \gamma(e_0), \gamma(e_1) = \alpha(e_0).$$

For then we get

$$\begin{aligned} \partial(c) &= d_0(\alpha) - d_1(\alpha) + d_0(\beta) - d_1(\beta) + d_0(\gamma) - d_1(\gamma) \\ &= \alpha(e_1) - \alpha(e_0) + \beta(e_1) - \beta(e_0) + \gamma(e_1) - \gamma(e_0) \\ &= 0. \end{aligned}$$

As the notation suggests, we are going to think of a chain of the form $\partial(c)$ as the boundary of c:

Definition: Boundaries

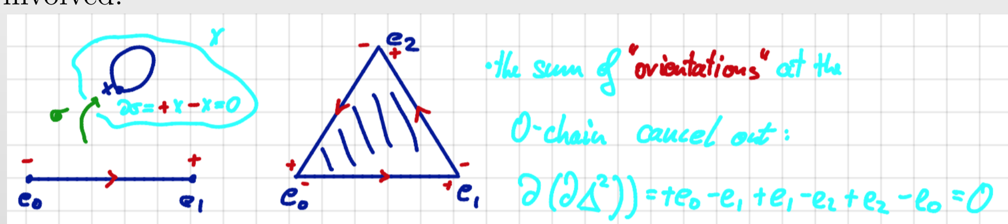
An **n-dimensional boundary in X** is an n-chain $c \in S_n(X)$ such that there exists an $(n+1)$-chain b with $\partial_{n+1} b = c$. We denote the group of n-boudnaries by

$$\begin{aligned} B_n(X) &:= \mathrm{Im}\,(\partial_{n+1} \colon S_n(X) \to S_{n-1}(X)) \\ &= \{c \in S_n(X) : \text{there is a } b \in S_{n+1}(X) \text{ with } \partial_{n+1}(b) = c\}. \end{aligned}$$

As an aside, here is another way of thinking of the algebraic process.

Signs are like orientations... just not exactly

We want to express the fact that a loop has **no boundary** by saying that the signs of the **boundary points cancel out**. The following picture illustrates that the something similar happens when several vertices are involved:



In general, we can **think of the signs** as giving the faces of the simplices an **orientation**. And if an n-simplex is a **face** of an $(n+1)$-simplex, then it inherits an **induced orientation** which is determined by how it fits into the bigger simplex. **Going down two steps** of inherited signs means things cancel out.
However, thinking of signs as orientations is formally not correct as we will notice in an example below. But, as we will see soon, we can **algebraically remedy this defect**.

As the above picture suggests, **every boundary is a cycle**:

Theorem: Boundaries of boundaries vanish

For every topological space X, the boundary operator satisfies $\partial \circ \partial = 0$, or more precisely

$$\partial_n \circ \partial_{n+1} = 0 \colon S_{n+1}(X) \to S_{n-1}(X).$$

Proof: It suffices to check this for an $(n+1)$-simplex σ. The general case follows, since each ∂ is a homomorphism. For σ, we just calculate:

$$\begin{aligned}
\partial_n \circ \partial_{n+1}(\sigma) &= \partial_n \left(\sum_{i=0}^{n+1} (-1)^i d_i^{n+1} \sigma \right) = \sum_{i=0}^{n+1} \partial_n (\sigma \circ \phi_i^{n+1}) \\
&= \sum_{i=0}^{n+1} (-1)^i \sum_{j=0}^{n} (-1)^j \sigma \circ \phi_i^{n+1} \circ \phi_j^n \\
&= \sum_{0 \le j < i \le n+1} (-1)^{i+j} \sigma \circ \phi_i^{n+1} \circ \phi_j^n + \sum_{0 \le i \le j \le n} (-1)^{i+j} \sigma \circ \phi_i^{n+1} \circ \phi_j^n \\
&\overset{(*)}{=} \sum_{0 \le j < i \le n+1} (-1)^{i+j} \sigma \circ \phi_j^{n+1} \circ \phi_{i-1}^n + \sum_{0 \le j' < i' \le n+1} (-1)^{j'+i'-1} \sigma \circ \phi_{j'}^{n+1} \circ \phi_{i'-1}^n \\
&= 0.
\end{aligned}$$

Note that at $(*)$ we applied identity (1) to the left hand sum and just changed the labels of the indices as $i \to j'$ and $j \to i' - 1$. Since both sums run over the same indices (it does not matter how we label them) and the right hand sum is the left hand sum multiplied by (-1), both sums cancel out. **QED**

As an immediate consequence we get:

Corollary: Every boundary is a cycle

For every $n \geq 0$, we have

$$B_n(X) \subseteq Z_n(X).$$

This basic resut shows that the sequence $\{S_n(X), \partial_n\}_n$ has an important property:

Definition: Chain complexes

A graded abelian group is a sequence of abelian groups, indexed by the integers. A **chain complex** is a graded abelian group $\{A_n\}_n$ together with homomorphisms $\partial_n \colon A_n \to A_{n-1}$ with the property that $\partial_{n-1} \circ \partial_n = 0$.

Hence we have shown that we obtain for any topological space X a complex of (free) abelian groups

$$\cdots \xrightarrow{\partial} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0.$$

It is called the **singular chain complex** of X. We will see next lecture what such chain complexes are good for.