MA3403 Algebraic Topology
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Lecture 04

4. Singular homology, functoriality and H_0

Recall that we constructed, for any topological space X, the **singular chain complex** of X

$$\cdots \xrightarrow{\partial} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0.$$

The homomorphisms ∂_n satisfy the fundamental rule: $\partial \circ \partial = 0$.

The following definition of homology groups applies to any chain complex. However we formulate it only for the singular chain complex:

Definition: Singular homology

The **nth singular homology** group of X is defined to be the quotient group of n-cycles modulo n-boundaries:

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\mathrm{Ker}\,(\partial \colon S_n(X) \to S_{n-1}(X))}{\mathrm{Im}\,(\partial \colon S_{n+1}(X) \to S_n(X))}.$$

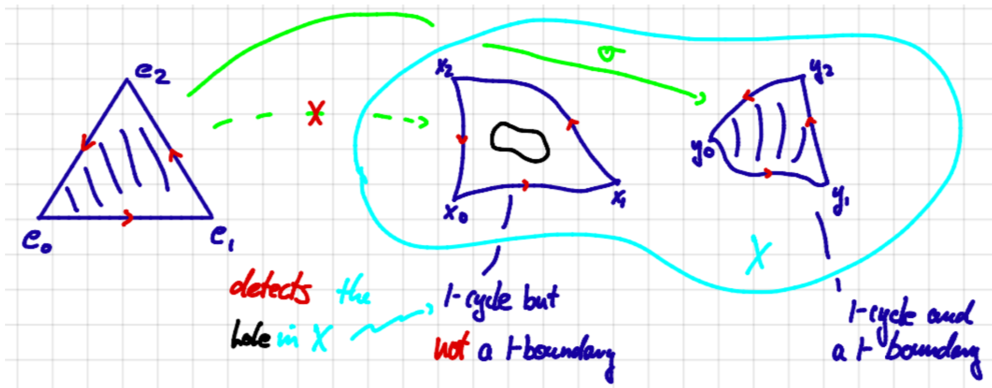
We are going to say that two cycles whose difference is a boundary are **homologous**.

Let us make a first attempt to understand what is going on here:

Singular what?

In algebra, homology is a way to **measure the difference between cycles and boundaries**. Singular homology is an application of homology in order to understand the structure of a space.
Given a space X, the group of n-cycles measures how often we can map an n-dimensional simplex into X without collapsing it to any of its $n-1$-dimensional faces.
Let $\sigma(\Delta^n)$ be the image in X of such a cycle. If we can even map an $(n+1)$-dimensional simplex $\sigma'(\Delta^{n+1})$ into X whose boudnary is $\sigma(\Delta^n)$, then we can continuously collapse all of $\sigma(\Delta^n)$ to a point. In this case, we would like to forget about this σ. For, from an n-dimensional point of view, this $\sigma(\Delta^n)$ is not interesting. That is what it means geometrically/topologically to take the quotient by $B_n(X)$.

But if we **cannot find** an $n+1$-dimensional simplex such that $\sigma(\Delta^n)$ is its boundary, then $\sigma(\Delta^n)$ potentially **carries interesting n-dimensional information** about X.
The slogan is: $H_n(X)$ measures n-dimensional wholes in X.



Before we see some examples of homology groups we go back to the idea of **"orientations of simplices"** and see why taking the quotient by boundaries is a good thing. We said that we think of the signs as orientations, but this is not completely correct. But **modulo boundaries** we are good:

Orientations revisited

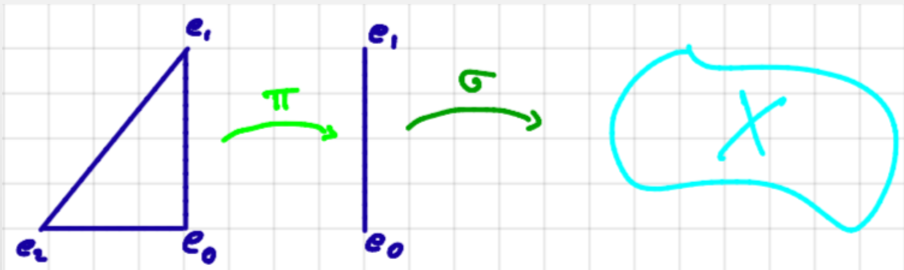
Let X be some space, and suppose we have a one-simplex $\sigma\colon \Delta^1 \to X$. Define

$$\phi\colon \Delta^1 \to \Delta^1,\ (t,1-t) \mapsto (1-t,t).$$

Precomposing with ϕ gives another singular simplex $\bar{\sigma} = \sigma \circ \phi$ which reverses the orientation of σ. It is **not true** that $\bar{\sigma} = -\sigma$ in $S_1(X)$.
However, we claim that

$$\bar{\sigma} \equiv -\sigma \mod B_1(X).$$

This means that there is a 2-chain in X whose boundary is $\bar{\sigma} + \sigma$. If $d_0(\sigma) = d_{(}(\sigma)$ such that $\sigma \in Z_1(X)$ is a 1-cycle, then $\bar{\sigma}$ and σ are homologous and $[\bar{\sigma}] = [\sigma]$ in $H_1(X)$.
To prove the claim we need to construct an appropriate 2-chain. Let $\pi\colon \Delta^2 \to \Delta^1$ be the affine map determined by sending e_0 and e_2 to e_0 and e_1 to e_1. For $x \in X$ and $n \geq 0$, we write $\kappa_x^n\colon \Delta^n \to X$ for the constant map with value x.



Now we calculate

$$\partial(\sigma \circ \pi) = \sigma \circ \pi \circ \phi_0^2 - \sigma \circ \pi \circ \phi_1^2 + \sigma \circ \pi \circ \phi_2^2 = \bar{\sigma} - \kappa^1_{\sigma(0)} + \sigma.$$

Hence up to the term $-\kappa^1_{\sigma(0)}$ we get what we want. So we would like to eliminate this term. To do that we define the constant 2-simplex $\kappa^2_{\sigma(0)} \colon \Delta^2 \to X$ at $\sigma(0)$. Its boundary is

$$\partial(\kappa^2_{\sigma(0)}) = \kappa^1_{\sigma(0)} - \kappa^1_{\sigma(0)} + \kappa^1_{\sigma(0)} = \kappa^1_{\sigma(0)}.$$

Thus

$$\bar{\sigma} + \sigma = \partial(\pi \circ \sigma + \kappa^2_{\sigma(0)})$$

which proves the claim.

Actually, we will have to get back to orientations almost regularly, in particular when we talk about simplicial complexes, and step by step improve our understanding and control.

Aside: The sequence of homology groups $\{H_n(X)\}_n$ also forms a graded abelian group. Note that even though $Z_n(X)$ and $B_n(X)$ are free abelian groups because they are subgroups of the free abelian group $S_n(X)$, the quotient $H_n(X)$ is **not necessarily free**. Moreover, while $Z_n(X)$ and $B_n(X)$ may be uncountably generated, $H_n(X)$ turns out to be finitely generated for the spaces we are interested in.

Let us look at two **simple examples**:

(1) **Let** $X = \emptyset$. Then $\mathrm{Sing}_*(\emptyset) = \emptyset$ and $S_*(\emptyset) = 0$ is just the trivial abelian group by convention. Hence $\cdots \to S_2 \to S_1 \to S_0$ is the zero chain complex and $Z_*(\emptyset) = B_*(\emptyset) = 0$. The homology in all dimensions is therefore 0.

(2) **Let** $X = \mathrm{pt}$ be a one-point space. Then, for each n, there is only one singular n-simplex, namely the constant map $\sigma_n \colon \Delta^n \to \mathrm{pt}$. In other words, $S_n(X) = \mathbb{Z} \cdot \sigma_n$ is generated by a single element. Hence $\sigma_n^{(i)} =$

$\sigma_n \circ \phi_i^n = \sigma_{n-1}$ and

$$\partial\sigma_n = \sum_{i=0}^{n}(-1)^i\sigma_n^{(i)} = \sum_{i=0}^{n}(-1)\sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \\ 0 & n = 0. \end{cases}$$

For cycles and boundaries this means

$$Z_n(X) = \begin{cases} \mathbb{Z}\cdot\sigma_n & n \text{ odd or } n = 0 \\ 0 & n \text{ even and } n \neq 0, \end{cases}$$

and

$$B_n(X) = \begin{cases} \mathbb{Z}\cdot\sigma_n & n \text{ odd or} \\ 0 & n \text{ even.} \end{cases}$$

For the homology groups we get

$$H_n(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

To complete the picture, the singular chain complex looks like

$$\cdots \xrightarrow{\partial=\mathrm{id}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\mathrm{id}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z} \to 0.$$

- **Functoriality**

Now that we have defined homology we can ask how it behaves under continuous maps. So let X and Y be topological spaces and $f\colon X \to Y$ be a continuous map. Since singular simplices are just maps, we can define an **induced map**

$$f_*\colon \mathrm{Sing}_n(X) \to \mathrm{Sing}_n(Y),\ \sigma \mapsto f\circ\sigma$$

just by composition with f.

The same construction yields an induced map on chains:

$$f_* = S_n(f)\colon S_n(X) \to S_n(Y),\ \sum_{j=1}^{m} a_j\sigma_j \mapsto \sum_{j=1}^{m} a_j(f\circ\sigma_j).$$

The induced map is compatible with the boundary operator in the following way:

Lemma: The singular chain complex is natural

For every $n \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{S_n(f)} & S_n(Y) \\ \Big\downarrow{\scriptstyle \partial_X} & & \Big\downarrow{\scriptstyle \partial_Y} \\ S_{n-1}(X) & \xrightarrow[S_{n-1}(f)]{} & S_{n-1}(Y). \end{array}$$

Proof: We just calculate and check that both ways have the same outcome for any singular n-simplex σ on X:

$$\begin{aligned} \partial_Y(S_n(f))(\sigma) &= \sum_{i=0}^{n}(-1)^i(f \circ \sigma) \circ \phi_i^n \\ &= \sum_{i=0}^{n}(-1)^i f \circ (\sigma \circ \phi_i^n) \\ &= S_{n-1}(f)\left(\sum_{i=0}^{n}(-1)^i \sigma \circ \phi_i^n\right) \\ &= S_{n-1}(f)(\partial_X \sigma). \end{aligned}$$

QED

The lemma has the important consequence that

$$S_n(f)(Z_n(X)) \subset Z_n(Y) \text{ and } S_n(f)(B_n(X)) \subset B_n(Y).$$

For, if $c \in Z_n(X)$, then

$$\partial_Y(S_n(f)(c)) = S_{n-1}(f)(\partial_X(c)) = S_{n-1}(f)(0) = 0;$$

and, if $c \in B_n(X)$, then there is a $b \in B_{n+1}(X)$ with $\partial_X(b) = c$ and

$$\partial_Y(S_{n+1}(f)(b)) = S_n(f)(\partial_X(b)) = S_n(f)(c),$$

i.e., there is an element, $b' = S_{n+1}(f)(b)$, with $\partial_Y(b') = S_n(f)(c)$.

Proposition: Homology is functorial

Thus we get a well defined induced homomorphism on homology groups

$$H_n(f) \colon H_n(X) \to H_n(Y),\ [c] \mapsto [S_n(f)(c)].$$

The homomorphisms $S_n(f)$ and $H_n(f)$ have the following **properties**:

- $S_n(\mathrm{id}_X) = \mathrm{id}_{S_n(X)}$ and $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$
- $S_n(f \circ g) = S_n(f) \circ S_n(g)$ and $H_n(f \circ g) = H_n(f) \circ H_n(g)$.

To **summarize** our observations: $S_n(-)$ and $H_n(-)$ are **functors** from the category of topological spaces to the category of abelian groups. For the sequence of all $S_n(-)$ even more is true: $S_*(-)$ is a functor from the category of topological spaces to the category of chain complexes of abelian groups (with chain maps as morphisms).

Invariance

As a consequence, if $f\colon X \to Y$ is a **homeomorphism**, then $H_n(f)$ is an **isomorphism** of abelian groups. In other words, homology groups only depend on the topology of a space.

In fact, we will soon see that homology is a coarser invariant in the sense that homotopic maps induce the same map in homology.

- **The homology group H_0**

Let us try to understand the simplest of the homology groups.

Lemma: Augmentation

For any topological space X, there is a homomorphism

$$\epsilon\colon H_0(X) \to \mathbb{Z}$$

which is nontrivial whenever $X \neq \emptyset$.

Proof: If $X = \emptyset$, then $H_*(\emptyset) = 0$ by definition. In this case, we define ϵ to be the zero homomorphism.
Now let $X \neq \emptyset$. Then there is a unique map $X \to \mathrm{pt}$ from X to the one-point space. By functoriality, it induces a homomorphism

$$\epsilon\colon H_0(X) \to H_0(\mathrm{pt}) = \mathbb{Z}.$$

QED

Let us try to understand this ϵ a bit better. The map $X \to \mathrm{pt}$ induces a homomorphism of chain complexes $S_*(X) \to S_*(\mathrm{pt})$ which sends any 0-simplex

$\sigma\colon \Delta^0 \to X$ to the **constant map**

$$\kappa^0\colon \Delta^0 \xrightarrow{\sigma} X \to \mathrm{pt}$$

which is the generator of $S_0(\mathrm{pt}) = \mathbb{Z}$. Hence we get a map $\sigma \mapsto 1$ which extends to a homomorphism $\tilde{\epsilon}\colon S_0(X) \to \mathbb{Z}$ by additivity, i.e.

$$\tilde{\epsilon}(\sum_j a_j\sigma_j) = \sum_j a_j \in \mathbb{Z}.$$

To double check that this map **descends to a homomomorphism** ϵ on $H_0(X)$ we need to show that it maps boundaries to 0. (We know this already, but let us do it anyway.)

So let b be a 0-chain which is the boundary of a 1-chain c, i.e., $b = \partial c$, and let $c = \sum_j a_j\gamma_j$ with **finitely many** 1-simplices $\gamma_j\colon \Delta^1 \to X$. Then each $\gamma_j \circ \phi_0^1$ and $\gamma_j \circ \phi_1^1$ are 0-simplices and are sent to 1 by $\tilde{\epsilon}$. Thus we get

$$\epsilon(b) = \epsilon(\partial c) = \epsilon(\sum_j a_j(\gamma_j \circ \phi_0^1 - \gamma_j \circ \phi_1^1)) = \sum_j a_j - \sum_j a_j = 0.$$

We learn from this discussion that, since a 0-simplex $\Delta^0 \to X$ can be identified with its image point, **ϵ counts the points on X, with multiplicities**. And if two points can be connected by a 1-simplex, i.e., by a path in X, then they add up to 0. This leads us to:

Theorem: H_0 for path-connected spaces

If X is path-connected and non-empty, then ϵ is an isomorphism

$$\epsilon\colon H_0(X) \xrightarrow{\cong} \mathbb{Z}.$$

Proof: Since X is non-empty, there is a point $x \in X$. The 0-simplex $\sigma = \kappa_x^0$ with value x is an element in $S_0(X)$ which is sent to $1 \in \mathbb{Z}$. Additivity implies that ϵ is **surjective**. To show that ϵ is also injective, we need to show that the classes of the 0-simplices given by constant maps at any two points are homologous.

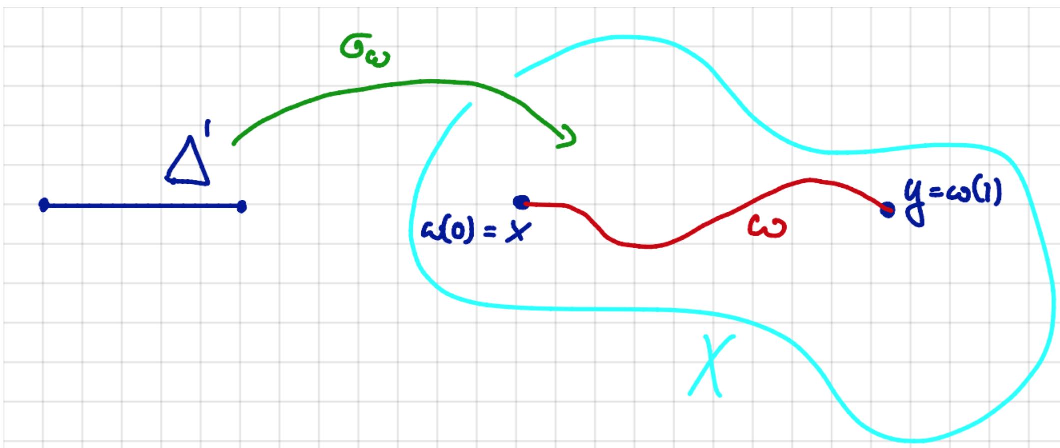
So let $y \in X$ be another point. Since X is path-connected, there is a path $\omega\colon [0,1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. We define a 1-simplex σ_ω by

$$\sigma_\omega(t_0,t_1) := \omega(1 - t_0) = \omega(t_1) \text{ for } t_0 + t_1 = 1, 0 \le t_0,t_1 \le 1.$$

The boundary of σ_ω is

$$\begin{aligned}\partial(\sigma_\omega) &= d_0(\sigma_\omega) - d_1(\sigma_\omega) = \sigma_\omega(e_1) - \sigma_\omega(e_0) \\ &= \sigma_\omega(0,1) - \sigma_\omega(1,0)(= \omega(0) - \omega(1)) \\ &= \kappa_x^0 - \kappa_y^0\end{aligned}$$

(where we identify 0-simplices and their image points). Hence the 0-simplices κ_x^0 and κ_y^0 are **homologous**. Since 0-simplices generate $H_0(X)$ and ϵ is a homomorphism, this implies that ϵ is **injective**. **QED**



Corollary: H_0 is generated by path components

If X is a disjoint union $X = \bigsqcup_{i \in I} X_i$ where each X_i is path-connected and non-empty, then, for all $n \geq 0$,

$$H_n(X) \cong \bigoplus_{i \in I} H_n(X_i).$$

In particular, for $n = 0$ we get

$$H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}.$$

In other words, $H_0(X)$ is the free abelian group generated by the set of path-components of X.

Proof: If $\sigma \colon \Delta^n \to X$ is an n-simplex, then its image lies in exactly one connected component X_i. Otherwise, we could write Δ^n as the disjoint union of two open and closed subsets contradicting the fact that Δ^n is connected. Hence σ factors into $\Delta^n \to X_i \hookrightarrow X$.

Since singular n-chains are freely generated by n-simplices, this shows that the singular chain complex of X splits into a direct sum

$$S_*(X) = \bigoplus_{i \in I} S_*(X_i).$$

For the same reason the boundary operators

$$\partial\colon S_n(X) \cong \bigoplus_{i \in I} S_n(X_i) \to \bigoplus_{i \in I} S_{n-1}(X_i) \cong S_{n-1}(X)$$

split into components $\partial_{X_i}\colon S_n(X_i) \to S_{n-1}(X_i)$. Hence we get an isono The statement for $n = 0$ then follows from the previous result on path-connected spaces. **QED**