MA3403 Algebraic Topology
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Lecture 06

6. The Eilenberg-Steenrod Axioms

Singular homology can in fact be unquely characterized by a quite short list of properties some of which we have already checked. This list of properties is called the **Eilenberg-Steenrod Axioms**. We are now going to formulate them in general and will then discuss the relation to singular homology as we defined it.

First some preparations:
We denote by $\mathbf{Top_2}$ the category of pairs of topological spaces. Two continuous maps $f_0, f_1 \colon (X,A) \to (Y,B)$ between pairs are called **homotopic**, denoted $f_0 \simeq f_1$, if there is a continuous map

$$h \colon X \times [0,1] \to Y$$

such that, for all $x \in X$,

$$h(x,0) = f_0(x),\ h(x,1) = f_1(x), \text{ and } h(A \times [0,1]) \subset B.$$

For $A \subset X$, we call

$$A^\circ = \bigcup_{U \subset A} U \text{ with } U \text{ open in } X$$

the **interior** of A and

$$\bar{A} = \bigcap_{A \subset Z} Z \text{ with } Z \text{ closed in } X$$

the **closure** of A.

Eilenberg-Steenrod Axioms

A **homology theory** (for topological spaces) h **consists of**:

- a sequence of functors $h_n \colon \mathbf{Top_2} \to \mathbf{Ab}$ for all $n \in \mathbb{Z}$ and
- a sequence of functorial connecting homomorphisms

$$\partial \colon h_n(X,A) \to h_{n-1}(A,\emptyset) =: h_{n-1}(A)$$

which satisfy the following **properties**:

- **Dimension Axiom:** $h_q(\mathrm{pt})$ is nonzero only if $q = 0$.

- **Long exact sequences:** For any pair (X,A), the sequence
$$\cdots \xrightarrow{\partial} h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(j)} h_n(X,A) \xrightarrow{\partial} h_{n-1}(A) \xrightarrow{\partial} \cdots$$
is exact, where we write $h_n(X) := h_n(X,\emptyset)$.
- **Homotopy Axiom:** If $f_0,f_1\colon (X,A) \to (Y,B)$ are homotopic, then the induced maps on homology
$$h_n(f_0) = h_n(f_1)\colon h_n(X,A) \to h_n(Y,B)$$
for all $n \in \mathbb{Z}$.
- **Excision Axiom:** For every pair of spaces (X,A) and every $U \subset A$ with $\bar{U} \subset A^\circ$ the homomorphism
$$h_n(k)\colon h_n(X \setminus U, A \setminus U) \to h_n(X,A)$$
induced by the inclusion map $k\colon (X \setminus U, A \setminus U) \hookrightarrow (X,A)$ is an **isomorphism**.
- **Additivity Axiom:** If $X = \sqcup_\alpha X_\alpha$ is a disjoint union, then the inclusion maps $i_\alpha\colon X_\alpha \hookrightarrow X$ induce an isomorphism for every n
$$\oplus_\alpha h_n(i_\alpha)\colon \bigoplus_\alpha h_n(X_\alpha) \xrightarrow{\cong} h_n(\sqcup_\alpha X_\alpha).$$

We have already shown that singular homology satisfies the dimension axiom and the connectiong homomorphism fits into long exact sequences. It remains to check homotopy invariance and excision. But before we do that we will assume these properties for a moment and use them to compute some homology groups.

First an important consequence of the **homotopy axiom**:

Proposition: Homotopy invariance of homology

Let $f\colon (X,A) \to (Y,B)$ be a map of pairs which is a **homotopy equivalence**, i.e., there is a map $g\colon (Y,B) \to (X,A)$ such that $g \circ f \simeq \mathrm{id}_{(X,A)}$ and $f \circ g \simeq \mathrm{id}_{(Y,B)}$. Then f induces an isomorphism
$$H_n(f)\colon H_n(X,A) \xrightarrow{\cong} H_n(Y,B)$$
in homology for all n.
In other words, **homology is invariant under homotopy** equivalences, not just homeomorphisms.

Recall that, for $n \geq 1$, we write D^n for the n-dimensional unit disk

$$D^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x|^2 = \sum_i x_i^2 \leq 1\}.$$

Recall that D^n is homotopy equivalent to a point. For, the constant map $D^n \to \{0\}$ is a strong deformation retraction with homotopy

$$h \colon D^n \times [0,1] \to D^n,\ (x,t) \mapsto (1-t)x$$

between the identity map of D^n and the constant map.

As a consequence of the **homotopy axiom** and our computation of $H_n(\mathrm{pt})$ we get:

Corollary for contractible spaces

Recall that a space which is homotopy equivalent to a one-point space is called contractible. For every **contractible** space X, we have

$$H_q(X) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0. \end{cases}$$

Before we look at an application of the axioms, a remark on homology theories with a brief outlook to the future (of your studies in algebraic topology):

A remark on homology theories

In fact, if we require $h_0(\mathrm{pt}) = \mathbb{Z}$ the above propierties or axioms characterize singular homology uniquely. In other words, if h satisfies the Eilenberg-Steerod axioms, then h must be singular homology.
We will see later that we can define variations of singular homology with coefficients different from $\mathbb{Z}$. If R is a commutative ring with unit and M an R-module, then there are singular homology groups $H_n(X; M)$ which fit into a homology theory which satisfies the dimension axiom with $h_0(\mathrm{pt}) = M$.
We can even go a step further (in a different class) where we drop the dimension assumption allow $h_q(\mathrm{pt}) \neq 0$ for infinitely many n. This leads to **generalized homology theories**, for example **K-theory** or **cobordism**, which are extremely useful for the solution of many fundamental problems, not just in topology. For example, **complex K-theory** can be used to show the theorem me mentioned in the notes of the first lecture: only in dimensions 1, 2, 4, and 8 there is a nice multiplication on $\mathbb{R}^n$.

- **Homology of the sphere**

As a fundamental example we are going to compute the homology of the k-dimensional sphere S^n. Actually, we already know one case. For, S^0 is just the disjoint union of two points. Hence $H_q(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $q = 0$ is 0 for all other n.

Theorem: Homology of the sphere

For $n \geq 1$, we have

$$H_q(S^n) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \text{ or } q = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & q = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof of the Theorem: For $n \geq 1$, the n-sphere S^n is **path-connected**. By our previous result, that implies $H_0(S^n) \cong \mathbb{Z}$.

The proof will proceed by **induction** using the long exact sequence in homology for pairs of spaces. This explains why we compute $H_q(S^n)$ and $H_q(D^n, S^{n-1})$ at the same time.

For $n = 1$ and $q = 0$, the pair (D^1, S^0), with $i \colon S^0 \hookrightarrow D^1$, is equipped with the **exact sequence**

$$\begin{tikzcd} H_0(S^0) \arrow[r, "H_0(i)"] \arrow[d, equal] & H_0(D^1) \arrow[r] \arrow[d, equal] & H_0(D^1, S^0) \arrow[r] \arrow[d, equal] & 0 \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & ? \arrow[r] & 0. \end{tikzcd}$$

The map $H_0(i)$ is induced by the map $S_0(S^0) \to S_0(D^1)$ which sends a 0-simplex $\Delta^0 \to S^0$ to the composite $\Delta^0 \to S^0 \hookrightarrow D^1$.

The image of S^0 in D^1 consists of the two endpoints of D^1 and both points are homologous as 0-simplices of D^1. Hence they both represent the class of the generator of $H_0(D^1)$. Hence the map $H_0(i)$ **sends each generator of $H_0(S^0)$ to the generator of $H_0(D^1)$**. Writing (1,0) and 0,1) for the generators of $\mathbb{Z} \oplus \mathbb{Z}$, any $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ is of the form $a \cdot (1,0) + b \cdot (0,1)$. Hence (a,b) is sent to $a + b$ under $H_n(i)$.

This implies that $H_0(i)$ is **surjective**. Since the above sequence is **exact**, this implies

$$H_0(D^1, S^0) = 0.$$

For $n \geq 2$, the exact sequence becomes

$$\begin{tikzcd}
H_0(S^{n-1}) \arrow[r, "H_0(i)"] \arrow[d, equal] & H_0(D^n) \arrow[r] \arrow[d, equal] & H_0(D^n, S^{n-1}) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & ? \arrow[r] & 0.
\end{tikzcd}$$

Since both S^{n-1} and D^n are **path-connected**, their 0th homology is isomorphic to $\mathbb{Z}$ and the generator of $H_0(S^{n-1})$, the class of any constant map $\kappa_x^0 \colon \Delta^0 \to S^{n-1}$, is sent to the generator of $H_0(D^n)$, the class of $\kappa_x^0 \colon \Delta^0 \to D^n$ corresponding to the image point $x \in S^{n-1} \subset D^n$. Hence $H_0(i)$ is surjective and we have again

$$H_0(D^n, S^{n-1}) = 0.$$

This **finishes the argument for H_0**.

For $q = 1$, we start with the exact sequence

(1)
$$\begin{tikzcd}
H_1(D^1) \arrow[r] \arrow[d, equal] & H_1(D^1, S^0) \arrow[r] \arrow[d, equal] & H_0(S^0) \arrow[r, "H_0(i)"] \arrow[d, equal] & H_0(D^1) \arrow[d, equal] \\
0 \arrow[r] & ? \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z}.
\end{tikzcd}$$

Since the sequence is exact, this shows that $H_1(D^1, S^0)$ is isomorphic to the **kernel** of

$$H_n(i) \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z},\ (a,b) \mapsto a + b.$$

Thus

(2)
$$H_1(D^1, S^0) \cong \mathbb{Z}.$$

For $n \geq 2$, we get the sequence

$$\begin{tikzcd}
H_1(D^n) \arrow[r] \arrow[d, equal] & H_1(D^n, S^{n-1}) \arrow[r] \arrow[d, equal] & H_0(S^{n-1}) \arrow[r, "\cong"] \arrow[d, equal] & H_0(D^n) \arrow[d, equal] \\
0 \arrow[r] & ? \arrow[r] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z}.
\end{tikzcd}$$

Since the **right most map is an isomorphism**, we get

(3)
$$H_1(D^n, S^{n-1}) = 0 \text{ for all } n \geq 2.$$

In order to study further groups, we consider the subspaces

$$D^n_+ := \{(x_0, \ldots, x_n) \in S^n : x_0 \geq 0\} \text{ and } D^n_- := \{(x_0, \ldots, x_n) \in S^n : x_0 \leq 0\}$$

which correspond to the **upper and lower hemisphere** (including the equator), respectively, of S^n.

For $n \geq 1$, we have the exact sequence

$$\begin{tikzcd} H_1(D^n_-) \arrow[r] \arrow[d, equal] & H_1(S^n) \arrow[r, "\cong"] & H_1(S^n, D^n_-) \arrow[r, "\partial = 0"] & H_0(D^n_-) \arrow[r, "\cong"] \arrow[d, equal] & H_0(S^n) \arrow[d, equal] \\ 0 & & & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z}. \end{tikzcd}$$

Since D^n_- is **contractible**, we know $H_1(D^n_-) = 0$. Hence the map $H_1(S^n) \to H_1(S^n, D^n_-)$ is **injective**. Since the map

$$\mathbb{Z} \cong H_0(D^n_-) \to H_0(S^n) \cong \mathbb{Z}$$

is an **isomorphism**, the **connecting hmomorphism** ∂ **is** 0. Since the sequence is **exact**, this implies that the map $H_1(S^n) \to H_1(S^n, D^n_-)$ is also **surjective**.

Thus, in total, we have an **isomorphism**

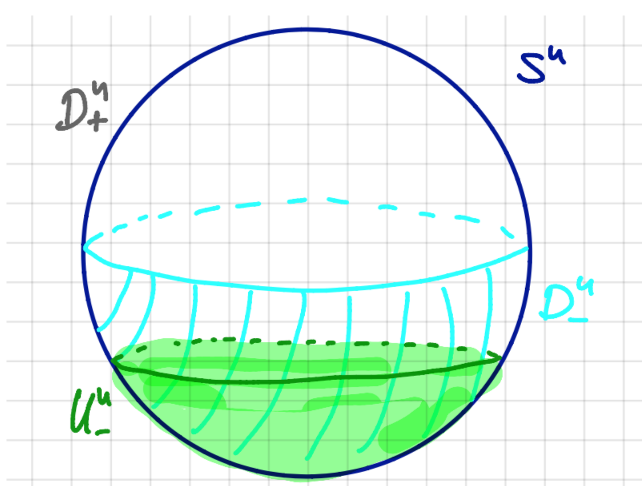
$$H_1(S^n) \xrightarrow{\cong} H_1(S^n, D^n_-).$$

To finish the analysis for $q = 1$, we consider the open subspace

$$U^n_- := \{(x_0, \ldots, x_n) \in S^n : x_0 < -1/2\} \subset D^n_-.$$

Its closure is still contained in the open interior of D^n_-, i.e.,

$$\bar{U}^n_- \subset (D^n_-)^\circ$$



Hence we can apply the **excision axiom** to the inclusion of pairs

$$k\colon (S^n \setminus U^n_-, D^n_- \setminus U^n_-) \hookrightarrow (S^n, D^n_-)$$

and obtain an isomorphism

$$H_q(k)\colon H_q(S^n \setminus U^n_-, D^n_- \setminus U^n_-) \xrightarrow{\cong} H_n(S^n, D^n_-).$$

But we also know

$$(S^n \setminus U^n_-, D^n_- \setminus U^n_-) \simeq (D^n_+, S^{n-1}) \xrightarrow{\approx} (D^n, S^{n-1})$$

where the last **homoeomorphism is given by vertical projection**, and the **homotopy equivalence is the natural retraction**.

In particular, we get an **isomorphism**

$$H_q(S^n, D^n_-) \cong H_q(D^n, S^{n-1}). \tag{4}$$

For H_1, this implies

$$H_1(S^n) \cong H_1(S^n, D^n_-) \cong H_1(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \text{ by (2).} \\ 0 & \text{else by (3).} \end{cases}$$

This **finishes the case** $q = 1$. In particular, we now know $H_1(S^1) \cong \mathbb{Z}$.

Finally, for $q \geq 2$, we proceed by induction. The pair (S^n,D^n_-) yields the exact sequence

$$\begin{tikzcd} H_q(D^n_-) \arrow[r] \arrow[d, equal] & H_q(S^n) \arrow[r, "\cong"] & H_q(S^n, D^n_-) \arrow[r] & H_{q-1}(D^n) \arrow[d, equal] \\ 0 & & & 0. \end{tikzcd} \tag{5}$$

Whereas the pair (D^n,S^{n-1}) yields the exact sequence

$$\begin{tikzcd} H_q(D^n) \arrow[r] \arrow[d, equal] & H_q(D^n, S^{n-1}) \arrow[r, "\cong"] & H_{q-1}(S^{n-1}) \arrow[r] & H_{q-1}(D^n) \arrow[d, equal] \\ 0 & & & 0. \end{tikzcd}$$

Together with isomorphism (4), we conclude

$$H_q(S^n) \cong H_q(S^n, D^n_-) \cong H_q(D^n, S^{n-1}) \cong H_{q-1}(S^{n-1}).$$

Hence knowing $H_1(S^1) = \mathbb{Z}$ implies $H_2(S^2) = \mathbb{Z}$ and $H_2(D^2,S^1) = \mathbb{Z}$. Continuing by **induction on** q yields the assertion of the theorem. **QED**