**MA3403 Algebraic Topology**
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**Lecture 07**

## 7. Generators for $H_n(S^n)$ and first applications

### Generators for $H_n(S^n)$

Last time we calculated the homology groups of $S^n$ and the pair $(D^n, S^{n-1})$. To make this calculation a bit more concrete, let us try to figure out the generators of the infinite cyclic groups $H_n(D^n, S^{n-1})$ and $H_n(S^n)$:

• On the standard $n$-simplex, there is a special $n$-chain $S_n(\Delta^n)$, called the **fundamental $n$-simplex**, given by the **identity** map $\iota_n \colon \Delta^n \to \Delta^n$. We observed in a previous lecture that $\iota_n$ is **not a cycle**, since its boundary $\partial(\iota_n) \in S_{n-1}(\Delta^n)$ is the alternating sum of the faces of the $n$-simplex each of which is a generator in $S_{n-1}(\Delta^n)$.

$$\partial(\iota_n) = \sum_i (-1)^i \phi_i^n(\Delta^{n-1}) \neq 0.$$

However, each of these singular simplices lies in $\partial\Delta^n$, and hence

$$\partial(\iota_n) \in S_{n-1}(\partial\Delta^n).$$

Thus the image of $\iota_n$ in $S_n(\Delta^n, \partial\Delta^n)$ is a **relative cycle**. Let us denote its image also by $\iota_n$ and its class in $H^n(\Delta^n, \partial\Delta^n)$ by $[\iota_n]$.

If $H_n(\Delta^n, \partial\Delta^n)$ is nontrivial, then $[\iota_n]$ must be a **nontrivial generator**. For, if $\sigma \colon \Delta^n \to \Delta^n$ is any $n$-simplex of $\Delta^n$ which defines a nontrivial class $[\sigma]$ in $H_n(\Delta^n, \partial\Delta^n)$, then

$$[\sigma] = H_n(\sigma)([\iota_n]).$$

This is because $\iota_n$ is the identity map and $H_n(\sigma)([\iota_n])$ is defined by comoposing $\sigma$ and $\iota_n$. Hence if $[\iota_n]$ was trivial, then any class in $H_n(\Delta^n, \partial\Delta^n)$ would be trivial.

• Now we use this observation to find a generator of $H_n(D^n, S^{n-1})$. The standard $n$-simplex $\Delta^n$ and the unit $n$-disk $D^n$ are homeomorphic. In order to find a homeomorphism we just need to smoothen out the corners of $\Delta^n$. (Note that we cannot ask for a diffeomorphism, since $\Delta^n$ is not a smooth manifold.)

In fact, we can choose a **homeomorphism of pairs**

$$\varphi_n \colon (\Delta^n, \partial\Delta^n) \xrightarrow{\approx} (D^n, S^{n-1})$$

which maps $\partial\Delta^n$ homeomorphically to $S^{n-1}$. We will construct a concrete homeomorphism below. For the moment, let us accept that we have such a homeomorphism $\varphi_n$ for every $n$.

Then $\varphi_n$ induces an **isomorphism**

$$H_n(\varphi_n)\colon H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_n(D^n, S^{n-1}) \text{ with } [\iota_n] \mapsto [\varphi_n].$$

Since we now know $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$, we also have $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ and $[\iota_n]$ as a generator. **Thus $[\varphi_n]$ is a generator of $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$.**

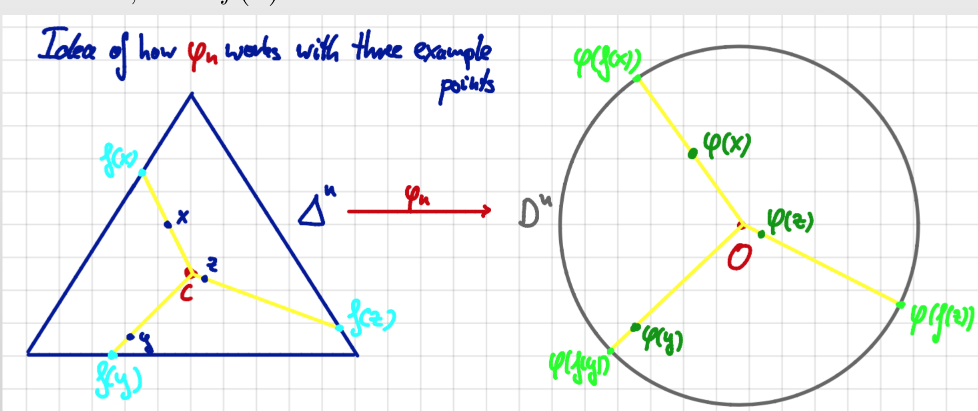
- Recall that we showed last time that the connecting homomorphism

$$\partial\colon H_n(D^n, S^{n-1}) \xrightarrow{\cong} H_{n-1}(S^{n-1})$$

an isomorphism. The image of $[\varphi_n]$ under $\partial$ is a generator. In other words, **$[\partial(\varphi_n)]$ is a generator of $H_{n-1}(S^{n-1})$** for all $n \geq 2$.

## Constructing $\varphi_n$

For each $\Delta^n$ the point $c = (t_0, \ldots, t_n)$ with $t_i = \frac{1}{n+1}$ for all $i$ is the **barycenter** of $\Delta^n$.
For every point $x \in \Delta^n$ which is not $c$, there is a unique ray from $c$ to $x$. We denote the unique point where this ray hits $\partial\Delta^n$ by $f(x)$. In particular, if $x \in \partial\Delta^n$, then $f(x) = x$.




Now we define the map

$$\varphi_n\colon \Delta^n \to D^n,\ x \mapsto \begin{cases} \frac{x-c}{|f(x)-c|} & \text{if } x \neq 0 \\ 0 & \text{if } x = c. \end{cases}$$

It is clear that $\varphi_n$ is continuous except possibly at $x = c$. But since there is a strictly positive lower bound for $|f(x) - c| > 0$, we know $|\varphi_n(x)| \leq M|x - c|$ for some real number $M$. This implies that $\varphi_n$ is also continuous at $x = c$. Moreover, $\varphi_n$ is a bijection, since it is one restricted to each ray. Since $\Delta^n$ is compact and $\varphi_n$ is a continuous bijection, it is a homeomorphism.

Finally, we write down a generator for the unit circle.

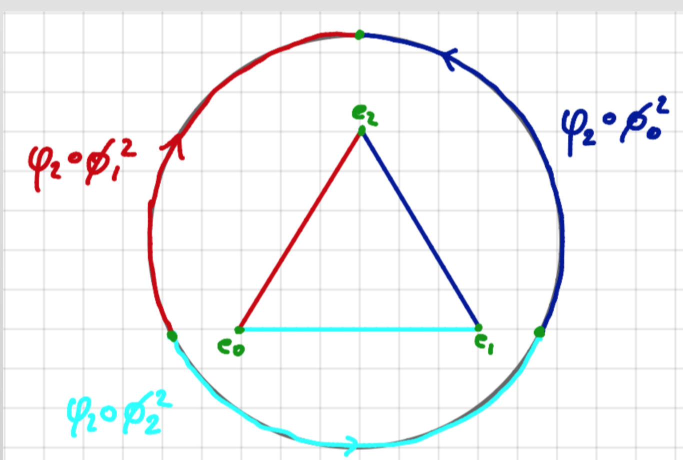
## A concrete generator of $H_1(S^1)$

We just learned that the class $[\partial(\varphi_2)]$ is a generator of $H_1(S^1)$. We can describe this class as follows:
By definition, $\partial(\varphi_2)$ is the 1-cycle

$$\begin{aligned}\partial(\varphi_2) &= d_0\varphi_2 - d_1\varphi_2 + d_2\varphi_2 \\ &= \varphi_2 \circ \phi_0^2 - \varphi_2 \circ \phi_1^2 + \varphi_2 \circ \phi_2^2.\end{aligned}$$

Recall that $\varphi_2$ maps $\partial\Delta^2$ homeomorphically to $S^1$. With this in mind, the summands look like

$$\begin{aligned}\varphi_2 \circ \phi_0^2(1-t,t) &= e^{i\pi(-\frac{1}{6}+t\frac{2}{3})} \\ \varphi_2 \circ \phi_1^2(1-t,t) &= e^{i\pi(\frac{7}{6}-t\frac{2}{3})} \\ \varphi_2 \circ \phi_2^2(1-t,t) &= e^{i\pi(\frac{7}{6}+t\frac{2}{3})}.\end{aligned}$$




We proved in Lecture 03 that the 1-simplex

$$\Delta^1 \to \Delta^1,\ t \mapsto \varphi_2 \circ \phi_1^2(1-t,t)$$

is **homologous** to the 1-simplex

$$\Delta^1 \to \Delta^1,\ t \mapsto \varphi_2 \circ \phi_1^2(t,1-t)$$

which **reverses the direction** of the walk from one vertex to the other.
In an exercise, we will show that after splitting a path into different steps, the 1-chain associated to the initial path is homologous to the sum of the 1-chains associated to the parts. This result implies that the 1-cycles $\partial\varphi_2$ **is homologous** to the 1-cycle corresponding to the familiar path

$$\gamma\colon \Delta^1 \to S^1,\ (1-t,t) \mapsto e^{2\pi i t}$$

which **walks once around the circle**.
In summary, we showed that $[\gamma] = [\partial\varphi_2]$ is the **desired generator of** $H_1(S^1)$.

## First applications

The calculation of the homology of spheres has many interesting consequences. We will discuss some of them today and will see many more soon.

We start with a result we advertised in the first lecture:

**Theorem: Invariance of dimension**

For $n \neq m$, the space $\mathbb{R}^n$ is **not** homeomorphic to $\mathbb{R}^m$.

**Proof: Assume there was a homeomorphism** $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Then the restricted map

$$f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$$

is **also a homeomorphism**, since these are open subsets and $f_{|\mathbb{R}^n\setminus\{0\}}$ and $(f^{-1})_{|\mathbb{R}^m\setminus\{f(0)\}}$ are still continuous mutual inverses.

We showed as an exercise that, for any $k \geq 1$, $S^{k-1}$ is a strong deformation retract of $\mathbb{R}^k\setminus\{0\}$. In particular, we showed $S^{k-1} \simeq \mathbb{R}^n\setminus\{0\}$. Since the translation $\mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto y + x$ is a homeomorphism for any $x \in \mathbb{R}^n$, this implies that

$$S^{k-1} \simeq \mathbb{R}^k \setminus \{x\} \text{ for every } x \in \mathbb{R}^k.$$

Hence, if the homeomorphism $f$ existed, we would get an **induced isomorphism**

$$H_q(S^{n-1}) \cong H_q(\mathbb{R}^n \setminus \{0\}) \xrightarrow{\cong} H_q(\mathbb{R}^m \setminus \{f(0)\}) \cong H_q(S^{m-1}).$$

But by our calculation of the $H_q(S^{n-1})$, such an isomorphism can only exist if $n-1 = q = m-1$. This **contradicts the assumption** $n \neq m$. **QED**

We can also give a short proof of Brouwer's famous Fixed-Point Theorem:

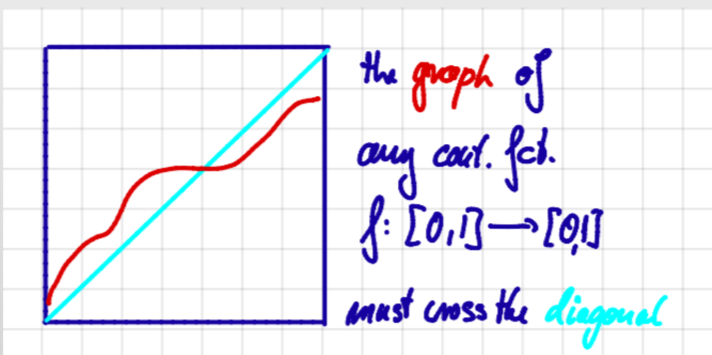
## Brouwer Fixed-Point Theorem

Let $f\colon D^n \to D^n$ be a **continuous** map of the closed unit disk into itself. Then $f$ must have a **fixed point**, i.e. there is an $x \in D^n$ with $f(x) = x$.

Before we prove the theorem, let us have a look at dimension one, where the result is very familiar:

## Brouwer FPT is familiar in dimension one

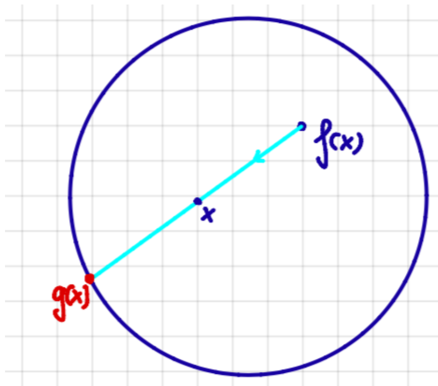
Note that you have seen this theorem for $n = 1$ in Calculus 1. Let $f\colon [0,1] \to [0,1]$ be a continuous map. Then it must have a fixed point. For, if not, then $g(x) = f(x) - x$ is a continuous map defined on $[0,1]$. We have $g(0) \geq 0$ and $g(1) \leq 0$, since $f(0) \geq 0$ and $f(1) \leq 1$.




If $g(0) = 0$ or $g(1) = 1$, we are done. But if $g(0) > 0$ and $g(1) < 1$, then the **Intermediate Value Theorem** implies that there is an $x_0 \in (0,1)$ with $g(x_0) = 0$, i.e. $f(x_0) = x_0$.

**Proof of Brouwer's FPT:** Since we know the theorem for $n = 1$, we assume $n \geq 2$. Suppose that there exists an $f$ without fixed points, i.e., $f(x) \neq x$ for all $x \in D^n$. Then, for every $x \in D^n$, **the two distinct points $x$ and $f(x)$ determine a line**. Let $g(x)$ be the point where the line segment starting at $f(x)$ and passing through $x$ hits the boundary $\partial D^n$. This defines a continuous map

$$g\colon D^n \to \partial D^n.$$




Let $i\colon S^{n-1} = \partial D^n \hookrightarrow D^n$ denote the inclusion map. Note that **if** $x \in \partial D^n$**, then** $g(x) = x$. In other words,

$$g \circ i = \mathrm{id}_{S^{n-1}}.$$

Applying the homology functor yields a **commutative diagram**

$$\begin{array}{ccc} \mathbb{Z} \cong H_{n-1}(S^{n-1}) & \xrightarrow{\mathrm{id}_{H_{n-1}(S^{n-1})}} & H_{n-1}(S^{n-1}) \cong \mathbb{Z} \\ & {\scriptstyle H_n(i)} \searrow \qquad \nearrow {\scriptstyle H_n(g)} & \\ & H_{n-1}(D^n) = 0. & \end{array}$$

But the **identity homomorphism on** $\mathbb{Z}$ **cannot factor through** $0$. This contradicts the assumption that $f$ has no fixed point. **QED**

**Typical application of homology theory**

The previous arguments are in fact a typical examples of proves in Algebraic Topology:

- The topological assumption that a homeomorphism $\mathbb{R}^n \xrightarrow{\approx} \mathbb{R}^m$ exists, is **translated** by applying homology to a **statement about an isomorphism of groups**. For groups, the existence of such an isomorphism is easily checked to be false.
- The geometric assumption that there is no fixed point be expressed in terms of maps and their compositions. Applying the **homology functor translates** this statement into an analogous **statement about groups and homomorphisms and their compositions**. Since the resulting statement about groups is obviously false, the original statement about spaces must be false as well.

- **The degree of a map $S^n \to S^n$**

The calculation of the homology of the sphere leads to another important algebraic invariant.

## Definition: The degree

For $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map. Then the induced homomorphism

$$\mathbb{Z} \cong H_n(S^n) \xrightarrow{H_n(S^n)} H_n(S^n) \cong \mathbb{Z}$$

is given by multiplication with an integer, the image of 1.
We denote this integer by $\deg(f)$ and call it the **degree of $f$**.

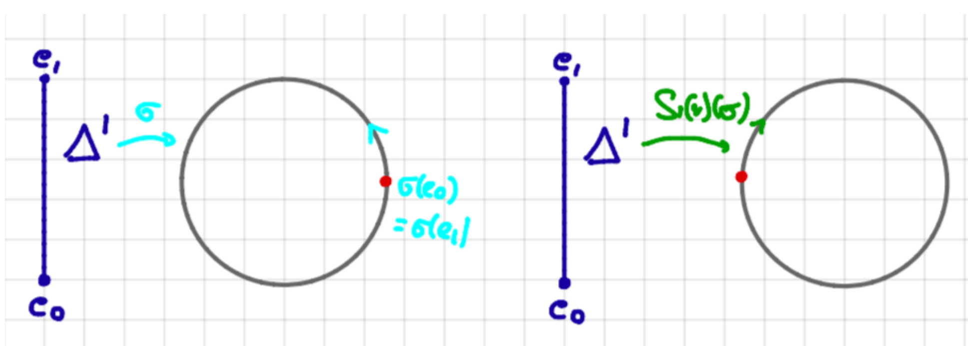
Let us calculate a first example:

## Theorem: The degree of a reflection

Let $r\colon S^n \to S^n$ be the **reflection map** defined by reversal of the first coordinate

$$r\colon (x_0,x_1,\ldots,x_n) \mapsto (-x_0,x_1,\ldots,x_n).$$

Then $\deg(r) = -1$.




Before we start the proof, let us have a look at what happens for the reflection map

$$r\colon D^1 = [-1,1] \to [-1,1] = D^1,\ t \mapsto -t$$

and **its restriction to $S^0$**. Recall that $S^0$ constists of just two points, $x = 1$ and $y = -1$ (on the real line $\mathbb{R}$). The effect of $r$ on $S^0$ is to **interchage $x$ and $y$**.

The inclusion maps $i_x$ and $i_y$ induce an isomorphism

$$H_0(\{x\}) \oplus H_0(\{y\}) \xrightarrow{\cong} H_0(S^0).$$

Thus $H_0(r)$ can be viewed as

$$H_0(r)\colon H_0(S^0) \to H_0(S^0),\ (a,b) \mapsto (b,a).$$

During the calculation of $H_n(S^n)$, we remarked that the map $\epsilon := H_0(i)\colon H_0(S^0) \to H_0(D^1)$ induced by the inclusion $i\colon S^0 \hookrightarrow D^1$ can be identified with the homomorphism

$$\epsilon\colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z},\ (a,b) \mapsto a + b.$$

Let $\mathrm{Ker}\,(\epsilon) = \{(a, -a) \in H_0(S^0) : a \in \mathbb{Z}\}$ be the kernel of $\epsilon$. Then we get that the **effect of $H_0(r)$ on $\mathrm{Ker}\,(\epsilon)$ is given by multiplication by $-1$**:

$$H_0(r)\colon \mathrm{Ker}\,(\epsilon) \to \mathrm{Ker}\,(\epsilon),\ (a, -a) \mapsto (-a,a) = -(a, -a).$$

Now we can address the actual proof.

**Proof of the Theorem:** For $n \geq 1$, let

$$D^n_+ := \{(x_0, \ldots, x_n) \in S^n : x_n \geq 0\} \text{ and } D^n_- := \{(x_0, \ldots, x_n) \in S^n : x_n \leq 0\}$$

be the **upper and lower hemispheres** on $S^n$, respectively. We also denote by $r$ the reflection map on $D^n_+$ and $D^n_-$. (Note that we defined $D^n_+$ and $D^n_-$ using a **different coordinate** than for defining $r$ so that $r(D^n_+) \subset D^n_+$ and $r(D^n_-) \subset D^n_-$.)

Then we have a **commutative diagram**

$$\begin{array}{ccccccc}
H_1(S^1) & \xrightarrow{\cong} & H_1(S^1, D^1_+) & \xleftarrow{\cong} & H_1(D^1_-, S^0) & \xrightarrow{\cong} & \mathrm{Ker}\,(\epsilon) \\
\downarrow{\scriptstyle H_1(r)} & & \downarrow{\scriptstyle H_1(r)} & & \downarrow{\scriptstyle H_1(r)} & & \downarrow{\scriptstyle H_0(r)} \\
H_1(S^1) & \xrightarrow{\cong} & H_1(S^1, D^1_+) & \xleftarrow{\cong} & H_1(D^1_-, S^0) & \xrightarrow{\cong} & \mathrm{Ker}\,(\epsilon).
\end{array}$$

The right hand square commutes, since the isomorphism

$$H_1(D^1_-, S^0) \xrightarrow{\cong} \mathrm{Ker}\,(\epsilon)$$

is part of the exact sequence induced by the pair $(D^1_-, S^0)$:

$$\begin{array}{ccccccc}
H_1(D^1_-, S^0) & \longrightarrow & H_0(S^0) & \xrightarrow{\epsilon} & H_0(D^1_-) & \longrightarrow & H_0(D^1_-,S^0) \\
\| & & \| & & \| & & \| \\
\mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

We know that the left hand and central squares commute, since the inclusion and the reflection commute. We know that the horizontal maps are isomorphisms from the calculation of these groups.

Thus, knowing $H_0(r) = -1$ on $\mathrm{Ker}\,(\epsilon)$, we see that $H_1(r)$ **is also multiplication by** $-1$ **on** $H_1(S^1)$.

Now we can proceed by **induction**: For $n \geq 2$, we have again a **commutative diagram** from the calculation of $H_n(S^n)$:

$$\begin{array}{ccccccc} H_n(S^n) & \xrightarrow{\cong} & H_n(S^n, D^n_+) & \xleftarrow{\cong} & H_n(D^n_-, S^{n-1}) & \xrightarrow{\cong} & H_{n-1}(S^{n-1}) \\ \downarrow{\scriptstyle H_n(r)} & & \downarrow{\scriptstyle H_n(r)} & & \downarrow{\scriptstyle H_n(r)} & & \downarrow{\scriptstyle H_{n-1}(r)} \\ H_n(S^n) & \xrightarrow{\cong} & H_n(S^n, D^n_+) & \xleftarrow{\cong} & H_n(D^n_-, S^{n-1}) & \xrightarrow{\cong} & H_{n-1}(S^{n-1}). \end{array}$$

The right most square commutes by an exercise from last week. The left hand and central squares commute, since the inclusion and the reflection commute.

Assuming the assertion for $n-1$, i.e., $H_{n-1}(r)$ is multiplication by $-1$, we see that $H_n(r)$ is also multiplication by $-1$. **QED**