MA3403 Algebraic Topology
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Lecture 08

8. Calculating degrees

In last week's exercises we showed many useful properties of the degree and calculated the degree of some interesting maps. Today, we are going to continue our study of the degree.

But before we move on, another reason why the degree is so important:

Brouwer degree

Let p be an arbitrary point in S^n. We consider p as the **base point** of S^n. Let $C(S^n,S^n)_*$ be the set of **pointed continuous maps**, i.e., maps $f\colon S^n \to S^n$ with $f(p) = p$. Pointed homotopy defines an **equivalence relation** on this set. Hence we can define the quotient set

$$[S^n, S^n]_* := C(S^n, S^n)_*/\simeq$$

where we identify f and g if they are homotopic to each other $f \simeq g$. Now the degree defines a function from $C(S^n,S^n)_*$ to the integers $\mathbb{Z}$. Since the degree is invariant under homotopy, i.e., $f_0 \simeq f_1$ implies $\deg(f_0) = \deg(f_1)$, it induces a function

$$\deg\colon [S^n, S^n]_* \to \mathbb{Z},\ f \mapsto \deg(f).$$

This function is actually an **isomorphism of abelian groups**. In fancier language, we write $\pi_n(S^n) = [S^n, S^n]_*$, call it the **$n$th homotopy group of S^n** and say that **the degree completely determines $\pi_n(S^n)$**.

Now let us see what kind of maps between spheres there. Actually, such maps arise quite naturally. For, every invertible real $n \times n$-matrix A defines a homeomorphism between $\mathbb{R}^n \xrightarrow{\approx} \mathbb{R}^n$. It extends to a homeomorphism on the **one-point compactification** S^n of $\mathbb{R}^n$ and therefore defines a map

$$A\colon S^n \to S^n.$$

A more direct way to produce a map is to assume we have an orthogonal matrix:

Orthogonal matrices

Let $O(n)$ denote the group of orthogonal (real) $n \times n$-matrices, i.e.,

$$O(n) = \{A \in M(n \times n, \mathbb{R}) : A^T A = I\}$$

where I is the identity matrix. The restriction to S^{n-1} of any A in $O(n)$ defines a map

$$A \colon S^n \to S^n,\ x \mapsto Ax.$$

The degree of this map is $\det A$, i.e., $\deg(A) = \det(A) = \pm 1$.

Proof: This follows from the fact that every othogonal matrix is the **product of reflections** (at appropriate hyperplanes in $\mathbb{R}^n$). A reflection has determinant -1, but it also has degree -1 as we have shown before. Since both deg and det are **multiplicative**, the result follows. **QED**

Now let $A \in \mathrm{GL}_n(\mathbb{R})$ be an **invertible** $n \times n$-matrix. It defines a map

$$f \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\},\ f(x) := Ax.$$

It induces a map

$$H_{n-1}(f) \colon H_{n-1}(\mathbb{R}^n \setminus \{0\}) \to H_{n-1}(\mathbb{R}^n \setminus \{0\}).$$

Since $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a **deformation retract**, we know

$$\mathbb{Z} \cong H_{n-1}(S^{n-1}) \cong H_{n-1}(\mathbb{R}^n \setminus \{0\}).$$

Hence the effect of $H_{n-1}(A)$ is given by multiplication by an integer.

Proposition: It's the sign

$H_{n-1}(A) = \mathrm{sign}(\det(A))$ where $\mathrm{sign}(\det(A))$ denotes the sign, i.e., 1 or -1, of $\det(A)$.

Proof: Recall from linear algebra that any invertible matrix A has a **polar decomposition** $A = BC$ with B a symmetric matrix **with only positive eigenvalues** and $C \in O(n)$. Since we already know that the assertion is true if $A \in O(n)$, it suffices to show that B is homotopic to the identity as maps $\mathbb{R}^n \to \mathbb{R}^n$.

Since all eigenvalues of B are positive, we know $\det(B) > 0$. Hence B and I lie both in the component $\mathrm{GL}_n(\mathbb{R})^+$ of the matrices with $\det > 0$. The continuous map

$$\Gamma\colon [0{,}1] \to \mathrm{GL}_n(\mathbb{R})^+,\ t \mapsto tI + (1-t)B$$

defines a **homotopy between I and B**.

To check that $\Gamma(t)$ is in $\mathrm{GL}_n(\mathbb{R})^+$ for all t, we observe that the eigenvalues of $\Gamma(t)$ are all strictly positive. For, let λ be an eigenvalue of B. Then $t + (1-t)\lambda$ is an eigenvalue of $\Gamma(t)$, since all nonzero vectors are eigenvectors of tI. This implies $\det(\Gamma(t)) > 0$. **QED**

For $n > 1$, we know $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$, since $(D^n, S^{n-1}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is a deformation retract. For A as above, we obtain a commutative diagram from the long exact sequences of pairs

$$\begin{array}{ccc} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{H_n(A)} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \downarrow & & \downarrow \\ H_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xrightarrow[H_{n-1}(A)]{} & H_{n-1}(\mathbb{R}^n \setminus \{0\}). \end{array}$$

The vertical connecting homomorphisms are isomorphisms, since they are isomorphisms for the pair (D^n, S^{n-1}). Since the diagram commutes, we deduce the following consequence from the previous result:

Corollary

The effect of the map

$$H_n(A)\colon H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

is given by multiplication with $\mathrm{sign}(\det(A))$.

Now we would like to apply this to a situation familiar from Calculus. First a brief observation:

Lemma

Let $U \subset \mathbb{R}^n$ be an open subset and $x \in U$. Then

$$H_n(U, U \setminus \{x\}) \cong \mathbb{Z}.$$

Proof: Let Z be the complement of U in $\mathbb{R}^n$. Since U is open, Z is closed. Hence $\bar{Z} = Z \subset \mathbb{R}^n \setminus \{x\} = (\mathbb{R}^n \setminus \{x\})^\circ$. Hence we can apply **excision** to the inclusion of pairs $(U, U \setminus \{x\}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ and get

$$H_n(U, U \setminus \{x\}) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}.$$

QED

Proposition: Degree of smooth maps

Let $U \subset \mathbb{R}^n$ open with $0 \in U$. Let

$$f \colon U \to \mathbb{R}^n$$

be a smooth map (or say twice differentiable with continuous second derivatives) with $f^{-1}(0) = 0$ and $Df(0) \in \mathrm{GL}_n(\mathbb{R})$.
For such a map f, the effect of the homomorphism

$$H_n(f) \colon H_n(U, U \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

is given by multiplication with $\mathrm{sign}(\det(Df(0)))$.

Proof: • By the **Taylor expansion** of a differentiable map, we can write f as

$$f(x) = Ax + g(x) \text{ with } A = Df(0) \text{ and } g(x)/|x| \to 0 \text{ for } x \to 0.$$

• In particular, we can assume $|g(x)| < |x|/2$ for x small enough. By **excision**, we can **shrink** U to become small enough such that still $0 \in U$ and $|g(x)| < |x|/2$ for all $x \in U$.

• We can further assume that $A = I$ is the identity. For if not, we can replace f with $A^{-1}f$ and use the functoriality of H_n.

• Now we have $|f(x) - x| < x/2$ for all $x \in U$. Hence the map

$$h \colon U \times [0,1] \to \mathbb{R}^n, \ (x,t) \mapsto tf(x) + (1-t)x$$

satisfies $h(x,t) \neq 0$ for all (x,t). This implies that $Dh(0,t)$ is in $GL_n(\mathbb{R})^+$ for all t. Thus h defines a homotopy between f and the identity map and the effect of $H_n(f)$ is the same as the oneof the identity map. **QED**

Local degree

Often the effect of a map can be studied by focussing on the neighborhood of certain interesting points. We would like to exploit this idea for studying the degree.

For $n \geq 1$, let $f\colon S^n \to S^n$ be a map with the property that there is a point $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points. (Note that almost all maps have this property.)

We label these points by $x_1, \ldots, x_m$. Now we choose small **disjoint** open neighborhoods $U_1, \ldots, U_m$ of each x_i such that each U_i is mapped into an open neighborhood V of y in S^n. (We could choose V first, and then intersect $f^{-1}(V)$ with small open disks around x_i...).

Since $x_i \in U_i$ and the different U_js are disjoint, we have

$$f(U_i \setminus \{x_i\}) \subset V \setminus \{y\} \text{ for each } i.$$

For any given i, the obvious inclusions of pairs induce the following diagram:

(1)
$$\begin{array}{ccccc}
 & & H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow[\deg(f|x_i)]{H_n(f_{|U_i})} & H_n(V, V \setminus \{y\}) \\
 & \swarrow^{\cong} & \downarrow k_i & & \downarrow \cong \\
H_n(S^n, S^n \setminus \{x_i\}) & \xleftarrow{p_i} & H_n(S^n, S^n \setminus f^{-1}(y)) & \xrightarrow{H_n(f)} & H_n(S^n, S^n \setminus \{y\}) \\
 & \nwarrow_{\cong} & \uparrow j & & \uparrow \cong \\
 & & H_n(S^n) & \xrightarrow{H_n(f)} & H_n(S^n).
\end{array}$$

By the **excision axiom** applied as in the proof of the lemma below and by an exercise, we know that the diagonal maps on the left and the vertical maps on the right are **isomorphisms**, as indicated in (1).

Definition: Local degree

The source and target of the dotted top horizontal arrow in (1) are identified with $\mathbb{Z}$. Hence the effect of this homomorphism is given by mutliplication by an integer. We denote this integer by $\deg(f|x_i)$ and call it the **local degree of f at x_i**.

Let us calculate some **examples**:

- If f is a homeomorphism, then any y has a unique preimage x. In this case, all maps in daigram (1) are isomorphisms and we have
$$\deg(f) = \deg(f|x) = \pm 1.$$
- If f maps each U_i homeomorphically to V, then we have $\deg(f|x_i) = \pm 1$ for each i.

The latter observation can be used to calculate the degree of f in many interesting situations. For we have the following result which connects global and local degrees:

Proposition: Global is the sum of local

With the above assumptions we have

$$\deg(f) = \sum_{i=1}^{m} \deg(f|x_i).$$

We are going to prove this result in the next lecture. In the diagram above we claimed that some maps are isomorphisms. Here is an explanation why:

Lemma

(a) Let $U \subset S^n$ be an open subset and $x \in U$. Then there is an isomorphism

$$H_n(U, U \setminus \{x\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x\}) \cong \mathbb{Z}.$$

(b) Let $x_1, \ldots, x_m$ be m distinct points in S^n and $U_1, \ldots, U_m$ disjoint open neighborhoods with $x_i \in U_i$. Then there is an isomorphism

$$\oplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x_1, \ldots, x_m\}) \cong \oplus_{i=1}^m \mathbb{Z}.$$

Proof: **(a)** Let Z be the complement of U in $\mathbb{R}^n$. Since U is open, Z is closed. Hence $\bar{Z} = Z \subset S^n \setminus \{x\} = (S^n \setminus \{x\})^\circ = S^n \setminus \{x\}$. Hence we can apply **excision** to the inclusion of pairs $(U, U \setminus \{x\}) \hookrightarrow (S^n, S^n \setminus \{x\})$ and get the above isomorphism.

(b) Let $U := \cup_i U_i$. Then $Z := S^n \setminus U$ is closed. As above, we can apply **excision** to the inclusion of pairs $(U, U \setminus \{x_1, \ldots, x_m\}) \hookrightarrow (S^n, S^n \setminus \{x_1, \ldots, x_m\})$ and get an isomorphism

$$H_n(U, U \setminus \{x_1, \ldots, x_m\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x_1, \ldots, x_m\}).$$

Since U is actually a **disjoint** union and each $x_i \in U_i$, we know the inclusions induce an isomorphism

$$\oplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(U, U \setminus \{x_1, \ldots, x_m\}).$$

Together with (a) this proves the assertion. **QED**