MA3403 Algebraic Topology
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Lecture 09

9. Local vs global degrees

Last time we defined the local degree of a map. The situation was as follows:

For $n \geq 1$, let $f \colon S^n \to S^n$ be a map with the property that there is a point $y \in S^n$ such that $f^{-1}(y) = \{x_1, \dots, x_m\}$ consists of finitely many points.

We choose small **disjoint** open neighborhoods $U_1, \dots, U_m$ of each x_i such that each U_i is mapped into an open neighborhood V of y in S^n. (We could choose V first, and then intersect $f^{-1}(V)$ with small open disks around x_i...).

Since $x_i \in U_i$ and the different U_js are disjoint, we have

$$f(U_i \setminus \{x_i\}) \subset V \setminus \{y\} \text{ for each } i.$$

For any given i, the obvious inclusions of pairs induce the following diagram:

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus \{x_i\}) \arrow[dl, "\cong"'] \arrow[d, "k_i"] \arrow[r, dotted, "H_n(f_{|U_i})", "\deg(f|x_i)"'] & H_n(V, V \setminus \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_i\}) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "H_n(f)"] & H_n(S^n, S^n \setminus \{y\}) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "H_n(f)"] & H_n(S^n). \arrow[u, "\cong"']
\end{tikzcd} \tag{1}$$

By the **excision axiom** applied as in the proof of the lemma below and by an exercise, we know that the diagonal maps on the left and the vertical maps on the right are **isomorphisms**, as indicated in (1). Then we made the following definition:

Definition: Local degree

The source and target of the dotted top horizontal arrow in (1) are identified with $\mathbb{Z}$. Hence the effect of this homomorphism is given by mutliplication by an integer. We denote this integer by $\deg(f|x_i)$ and call it the **local degree of f at x_i**.

We have the following result which connects global and local degrees:

Proposition: Global is the sum of local

With the above assumptions we have

$$\deg(f) = \sum_{i=1}^{m} \deg(f|x_i).$$

Let us finally prove this result.

Proof: • As explained in the lemma below, excision implies that

$$\oplus_i k_i \colon \oplus_i \mathbb{Z} = H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus f^{-1}(y)) = \oplus_i \mathbb{Z}$$

is an **isomorphism**. Henceforth we are going to identify the groups in diagram (1) with $\mathbb{Z}$ or the direct sum $\sum_i \mathbb{Z}$, respectively.

• We know that $p_i \circ k_i$ is the diagonal isomorphism. Hence k_i corresponds to the **inclusion** of and p_i corresponds to the **projection to the ith summand**.

• Since the **lower triangle commutes**, the composite $p_i \circ j$ satisfies

$$p_i \circ j(1) = 1.$$

Since we also know

$$p_i \circ k_i(1) = 1 \text{ and } (\sum_i k_i)(1) = (1, \ldots, 1)$$

we must have

$$j(1) = (1, \ldots, 1)$$

as well.

• Since the **upper square in diagram (1) commutes**, we know

$$H_n(f)(k_i(1)) = \deg(f|x_i).$$

• Together with the above, this shows

$$H_n(f)(j(1)) = \sum_i \deg(f|x_i).$$

• Since the **lower square in diagram (1) commutes** and since the lower horizontal map is given by the degree of f, the asserted formula follows. **QED**

We have used the following lemma in diagram (1) the above proof:

Lemma

(a) Let $U \subset S^n$ be an open subset and $x \in U$. Then there is an isomorphism

$$H_n(U, U \setminus \{x\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x\}) \cong \mathbb{Z}.$$

(b) Let $x_1, \ldots, x_m$ be m distinct points in S^n and $U_1, \ldots, U_m$ disjoint open neighborhoods with $x_i \in U_i$. Then there is an isomorphism

$$\oplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x_1, \ldots, x_m\}) \cong \oplus_{i=1}^m \mathbb{Z}.$$

Proof: **(a)** Let Z be the complement of U in $\mathbb{R}^n$. Since U is open, Z is closed. Hence $\bar{Z} = Z \subset S^n \setminus \{x\} = (S^n \setminus \{x\})^\circ = S^n \setminus \{x\}$. Hence we can apply **excision** to the inclusion of pairs $(U, U \setminus \{x\}) \hookrightarrow (S^n, S^n \setminus \{x\})$ and get the above isomorphism.

(b) Let $U := \cup_i U_i$. Then $Z := S^n \setminus U$ is closed. As above, we can apply **excision** to the inclusion of pairs

$$(U, U \setminus \{x_1, \ldots, x_m\}) \hookrightarrow (S^n, S^n \setminus \{x_1, \ldots, x_m\})$$

and get an isomorphism

$$H_n(U, U \setminus \{x_1, \ldots, x_m\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x_1, \ldots, x_m\}).$$

Since U is actually a **disjoint** union and each $x_i \in U_i$, we know the inclusions induce an isomorphism

$$\oplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(U, U \setminus \{x_1, \ldots, x_m\}).$$

Together with (a) this proves the assertion. **QED**

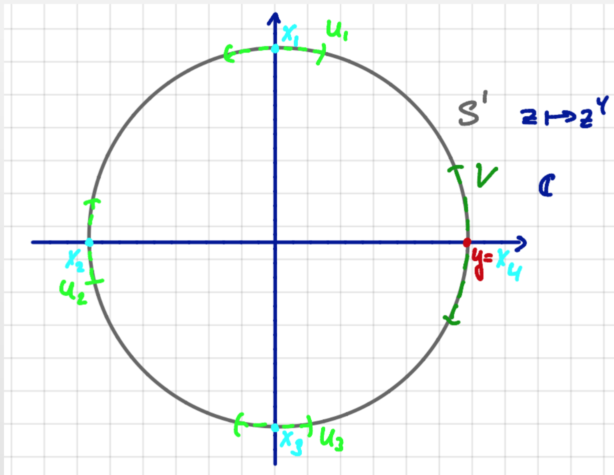
Now let us apply this in an example:

Example: Degree on the unit circle

Let $S^1 \subset \mathbb{C}$ be the unit circle, $k \in \mathbb{Z}$, and let

$$f_k \colon S^1 \to S^1,\ z \mapsto z^k.$$

We claim $\deg(f_k) = k$.



• We know this is true for **for** $k = 0$ when f_0 is the constant map and **for** $k = 1$ when f_1 is the identity.
• We know it also **for** $k = -1$, since $z \mapsto z^{-1}$ is a **reflection** at the real axis.
• It suffices to check the remaining cases for $k > 0$, since the cases **for** $k < 0$ follow from composition with $z \mapsto z^{-1}$ and the **multiplicativity of the degree**.
• So **let** $k > 0$. For any $y \in S^n$, $f_k^{-1}(y)$ consists of k distinct points $x_1, \dots, x_k$. Each point x_i has an open neighborhood U_i which is **mapped homeomorphically** by f to an open neighboorhood V of y. This local homeomorphism is given by **stretching** (by the factor k) and a **rotation in positive direction**.
• Stretching by a factor is homotopic to the identity near x_i. Hence the local degree of the stretching is $+1$.
A rotation is a homeomorphism and its global and local degree at any point agree.
Since the rotation is in the positive direction, it is homotopic to the identity and has therefore degree $+1$.
• Hence $\deg(f|x_i) = 1$.
• Thus we can **conclude by the proposition** that

$$\deg(f) = \sum_{i=1}^{k} \deg(f|x_i) = k.$$