MA3403 Algebraic Topology
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Lecture 10

10. Homotopies of chain complexes

We still need to prove the Homotopy Axiom and Excision Axiom for singular homology. The prove will follow from constructing a homotopy between chain complexes, a concept we are now going to explore.

Recall that a **chain complex** $K_* = (K_*, \partial^K)$ constists of a sequence of abelian groups

$$\cdots \xrightarrow{\partial^K_{n+2}} K_{n+1} \xrightarrow{\partial^K_{n+1}} K_n \xrightarrow{\partial^K_n} K_{n-1} \xrightarrow{\partial^K_{n-1}} \cdots$$

together with homomorphisms $\partial^K_n \colon K_n \to K_{n-1}$ with the property that $\partial_{n-1} \circ \partial_n = 0$. Our main example is the singular chain complex.

Just to make sure that we understand the definition, let us look at an example of a sequence of groups that is **not** a chain complex. Consider the sequence of maps

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots$$

where each map consists of multiplication by 2. This is not a chain complex, since $2 \cdot 2 = 4$, i.e., $\partial_{n-1} \circ \partial_n = 4 \neq 0$.

Recall the definition of a map of chain complexes from Lecture 5:

Maps of chain complexes

Let $K_* = (K_*, \partial^K)$ and $L_* = (L_*, \partial^L)$ be two chain complexes. A **morphism of chain complexes** $f_* \colon K_* \to L_*$, also called **chain map**, is a sequence of homomorphisms $\{f_n\}_{n \in \mathbb{Z}}$

(1) $\quad f_n \colon K_n \to L_n$ such that $f_{n-1} \circ \partial^K_n = \partial^L_n \circ f_n$ for all $n \in \mathbb{Z}$.

A homomorphism of chain complexes induces a homomorphism on homology

$$H_n(f) \colon H_n(K_*) \to H_n(L_*),\ [x] \mapsto [f_n(x)].$$

We need to check that this is **well-defined**. Since we hopped over this point in Lecture 5, let us do it now.

There are two things we should check. We need to know that f sends cycles to cycles and boundaries to boundaries.

First, let $x \in K_n$ be a cycle in K, i.e., $\partial_n^K(x) = 0$. Then (1) implies

$$\partial_n^L(f_n(x)) = f_{n-1}(\partial_n^K((x)) = f_{n-1}(0) = 0.$$

Thus $f_n(x)$ is a cycle in L and we get $f_n(Z_n(K_*)) \subset Z_n(L_*)$.

Second, let a $x \in K_n$ be a boundary, say $\partial_{n+1}^K(y) = x$. Then (1) implies

$$f_n(x) = f_n(\partial_{n+1}^K(y)) = \partial_{n+1}^L(f_{n+1}(y)).$$

Thus $f_n(x)$ is a boundary in L and we get $f_n(B_n(K_*)) \subset B_n(L_*)$. This shows that f induces a well-defined homomrphism between the homologies of K_* and L_*.

We would like to transfer the notion of homotopies between maps of spaces to the homotopies between maps of chain complexes. This follows the general slogan: **Homotopy is a smart thing to do.**

Why? The notion of an **isomorphism** in a category, e.g. the category of topological spaces or the category of chain complexes, is often **too rigid**. There are too few isomorphism such that **classifying objects** up to isomorphism is too **difficult**. Therefore, one would like to relax the conditions. For many situations, homotopy turns out to provide the **right amount of flexibility** and rigidity at the same time. Moreover, many invariants, in fact all invariants in Algebraic Topology, do not change if we alter a map by a homotopy.

In other words, **our invariants only see the homotopy type**.

Actually, this is exactly what we are going to show for singular homology today. It is also true in Homological Algebra. The homology of a chain complex only depends on the homotopy type of the complex.

So let us define homotopies between chain maps:

Definition: Homotopies of chain maps

Let $f, g\colon K_* \to L_*$ be two morphisms of chain complexes. A **chain homotopy** between f and g is a sequence of homomorphisms

$$h_n\colon K_n \to L_{n+1}$$

such that

(2) $$f_n - g_n = \partial_{n+1}^L \circ h_n + h_{n-1} \circ \partial_n^K \text{ for all } n \in \mathbb{Z}.$$

$$\begin{array}{ccc} K_{n+1} & \longrightarrow & L_{n+1} \\ \partial^K_{n+1}\downarrow & \overset{h_n}{\nearrow} & \downarrow\partial^L_{n+1} \\ K_n & \xrightarrow{f_n - g_n} & L_n \\ \downarrow\partial^K_n & \underset{h_{n-1}}{\nearrow} & \downarrow\partial^L_n \\ K_{n-1} & \longrightarrow & L_{n-1} \end{array}$$

If such a homotopy exists, we are going to say that f and g are homotopic and write $f \simeq g$.
We say that f is **null-homotopic** if $f \simeq 0$.

As for topological spaces, this yields an equivalence relation:

Lemma: Homotopy is an equivalence relation

(1) Chain homotopy is an equivalence relation on the set of all morphisms of chain complexes.
(2) If $f \simeq f' \colon K_* \to L_*$ and $g \simeq g' \colon L_* \to M_*$, then $g \circ f \simeq g' \circ f'$.

Proof: **(1)** We need to show that homotopy is reflexive, symmetric and transitive:

- We obtain $f \simeq f$ with $h = 0$ being the zero map.
- If h is a homotopy which gives $f \simeq g$, then $-h$ is a homotopy which shows $g \simeq f$.
- If h is a homotopy which gives $f \simeq g \colon K_* \to L_*$ and h' is a homotopy which shows $g \simeq k \colon K_* \to L_*$, then $h + h'$ is a homotopy which shows $f \simeq k$. For

$$\begin{aligned} f_n - k_n &= f_n - g_n + g_n - k_n \\ &= \partial^L_{n+1} \circ h_n + h_{n-1} \circ \partial^K_n + \partial^L_{n+1} \circ h'_n + h'_{n-1} \circ \partial^K_n \\ &= \partial^L_{n+1} \circ (h_n + h'_n) + (h_{n-1} + h'_{n-1}) \circ \partial^K_n. \end{aligned}$$

(2) Let h be a homotopy which shows $f \simeq f'$ and k be a homotopy which shows $g \simeq g'$. Composition with g on the left and using that g is a chain map yields

$$\begin{aligned} g_n \circ (f_n - f'_n) &= g_n \circ (\partial^L_{n+1} \circ h_n + h_{n-1} \circ \partial^K_n) \\ &= \partial^M_{n+1} \circ (g_{n+1} \circ h_n) + (g_n \circ h_{n-1}) \circ \partial^K_n. \end{aligned}$$

This shows that the sequence of maps $g_{n+1} \circ h_n$ defines a homotopy $g \circ f \simeq g \circ f'$.

Composition with f' on the right and using that f' is a chain map yields

$$\begin{aligned}(g_n - g'_n) \circ f'_n &= (\partial^M_{n+1} \circ k_n + k_{n-1} \circ \partial^L_n) \circ f'_n \\ &= \partial^M_{n+1} \circ (k_n \circ f'_n) + (k_{n-1} \circ f'_{n-1}) \circ \partial^K_n.\end{aligned}$$

This shows that the sequence of maps $k_n \circ f'_n$ defines a homotopy $g \circ f' \simeq g' \circ f'$.

summarizing we have shown

$$g \circ f \simeq g \circ f' \simeq g' \circ f'.$$

By transitivity, this shows the desired result. **QED**

Now we are ready to show an important fact in homological algebra:

Homology identifies chain homotopies

If $f \simeq g \colon K_* \to L_*$ are **homotopic** morphisms of chain complexes, then

$$H_n(f) = H_n(g) \text{ for all } n \in \mathbb{Z}.$$

Proof: This follows immediately from the fact that $f_n - g_n$ is just given by boundaries which, by definition, vanish in homology.

More concretely, let $x \in K_n$ be an arbitrary cycle in K_n and let h be a homotopy which gives $f \simeq g$. Then we get by using the definition of homotopies

$$H_n(f)([x]) = [f_n(x)] = [g_n(x) + \partial^L_{n+1}(h_n(x)) + h_{n-1}(\partial^K_n(x))] = [g_n(x)] = H_n(g)([x])$$

where we use that $\partial^L_{n+1}(h_n(x))$ is obviously a **boundary in L_n** and that $h_{n-1}(\partial^K_n(x)) = 0$, since x is a **cycle in K_n** by assumption. **QED**

Now we can also mimick the notion of homotopy equivalences.

Chain homotopy equivalences

A morphism of chain complexes $f \colon K_* \to L_*$ is called a **homotopy equivalence** if there exists a morphism of chain complexes $g \colon L_* \to K_*$ such that $g \circ f \simeq \mathrm{id}_{K_*}$ and $f \circ g \simeq \mathrm{id}_{L_*}$.
If such a homotopy equivalence exists, we write $K_* \simeq L_*$ and say that K_* and L_* are **homotopy equivalent**.

In particular, by adopting language from algebraic topology, if the identity map on a chain complex K_* is homotopy equivalent to the zero map, then we say that

K_* is **contractible**. For example, if X is a **contractible space**, then its singular chain complex $S_*(X)$ is a **contractible chain complex**. Note that a chain complex K_* with at least one nonzero homology group cannot be contractible.

As a consequence of what we proved we get:

Chain homotopy equivalences

• If $K_* \simeq L_*$, then $H_n(K_*) \cong H_n(L_*)$.
• Given two chain complexes K_* and L_* we denote the set of morphisms of chain complexes by $\mathrm{Mor}(K_*,L_*)$. Let $[K_*,L_*] := \mathrm{Mor}(K_*,L_*)/\simeq$ denote the set of equivalence classes under the relation given by chain homotopies. Then we can define a new category whose objects are chain complexes and whose sets of morphisms from $K_* \to L_*$ are homotopy classes of chain maps, i.e., the sets $[K_*,L_*]$. Let us call this category $\mathbf{K}$.
Since the homotopy relation respects composition, we obtain that homology defines a functor

$$\mathbf{K} \to \mathbf{Ab},\, K_* \mapsto H_n(K_*)$$

where $\mathbf{Ab}$ denotes the category of abelian groups.

Let us look at **some examples**:

- Let K_* be the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \cdots.$$

  Since all maps are trivial, we have $H_n(K_*) = K_n$ for all n. Hence K_* has exactly two nonzero homology groups, both being $\mathbb{Z}$. In particular, it is **not contractible**.
- Let K_* be the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to 0 \to \cdots.$$

This complex is actually an exact sequence. Thus $H_n(K_*) = 0$ for all n. Moreover it is **contractible**. We can write down a homotopy by

$$
\begin{array}{ccc}
0 & \xrightarrow{\text{id}} & 0 \\
\downarrow & \overset{0}{\nearrow} & \downarrow \\
\mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\
{\scriptstyle 1}\downarrow & \overset{1}{\nearrow} & \downarrow{\scriptstyle 1} \\
\mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\
\downarrow & \overset{0}{\nearrow} & \downarrow \\
0 & \xrightarrow{\text{id}} & 0.
\end{array}
$$

- Let K_* be the chain complex
$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \cdots .$$
This complex has one nonzero homology group $H_1(K_*) = \mathbb{Z}/2$. It is therefore **not contractible**.
- Let K_* be the chain complex
$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots .$$
The homology of K_* vanishes, since, at each stage, the image and the kernel of the differential is $2\mathbb{Z}/4$. Nevertheless, K_* is **not contractible**. For if there was a homotopy between id_{K_*} and the zero map, it would like this
$$
\begin{array}{ccc}
\mathbb{Z}/4 & \xrightarrow{\text{id}} & \mathbb{Z}/4 \\
{\scriptstyle 2}\downarrow & \overset{h_n}{\nearrow} & \downarrow{\scriptstyle 2} \\
\mathbb{Z}/4 & \xrightarrow{\text{id}} & \mathbb{Z}/4 \\
\downarrow{\scriptstyle 2} & \underset{h_{n-1}}{\nearrow} & \downarrow{\scriptstyle 2} \\
\mathbb{Z}/4 & \xrightarrow[\text{id}]{} & \mathbb{Z}/4
\end{array}
$$
and satisfy $\text{id} = 2h_n + h_{n-1}2$. But $2h_n + h_{n-1}2$ can only produce even numbers modulo 4. Hence it cannot be the identity map on $\mathbb{Z}/4$.

After all this abstract stuff we should better demonstrate that the notion of chain homotopies is useful for our purposes. We are going to do this by showing that homotopies between maps of spaces induces a chain homotopy. By what we have just seen, this will prove the Homotopy Axiom for singular homology.