MA3403 Algebraic Topology
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Lecture 11

11. Homotopy invariance of singular homology

We are going to prove the Homotopy Axiom for singular homology. The prove will follow from constructing a homotopy between chain complexes.

Let $f,g\colon X \to Y$ be **two homotopic maps** and let $h\colon X \times [0,1] \to Y$ be a homotopy between them. Let $\sigma\colon \Delta^n \to X$ be an n-simplex on X. Then h induces a map

$$\Delta^n \times [0,1] \xrightarrow{\sigma\times \mathrm{id}} X \times [0,1] \xrightarrow{h} Y$$

which defines a homotopy between $f \circ \sigma$ and $g \circ \sigma$.

Our goal is to **turn this into a geometrically induced chain homotopy between $S_n(f)$ and $S_n(g)$**. By our result from the previous lecture, this will imply the Homotopy Axiom.

So let us have a closer look at the space $\Delta^n \times [0,1]$. For $n = 1$, it looks just like a square. Via the diagonal we can divide it into two triangles which look like Δ^2. For $n = 2$, $\Delta^2 \times [0,1]$ looks like a prism which we can divide into three copies of Δ^3.

In general, $\Delta^n \times [0,1]$ looks like a higher dimensional prism which we can divide into $n+1$ copies of Δ^{n+1}. We should make this idea more precise:

Simplices on a prism

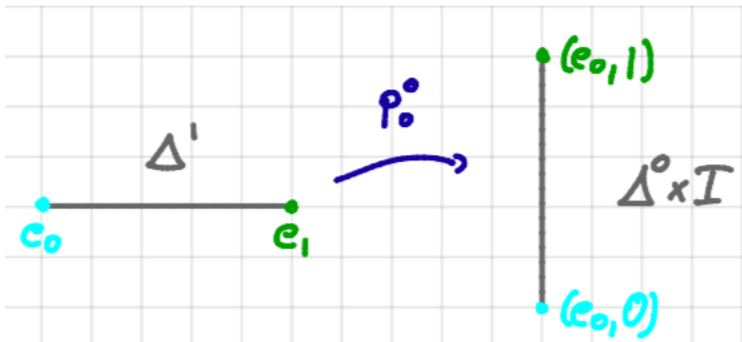
For every $n \geq 0$ and $0 \leq i \leq n$, we define an injective map

$$p_i^n\colon \Delta^{n+1} \to \Delta^n \times [0,1],$$
$$(t_0, \dots, t_{n+1}) \mapsto ((t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}), t_{i+1} + \cdots + t_{n+1}).$$

We can consider each p_i^n as an $n+1$-simplex on the space $\Delta^n \times [0,1]$. When n is clear we will often drop it from the notation.

For $n = 0$, we have only one map

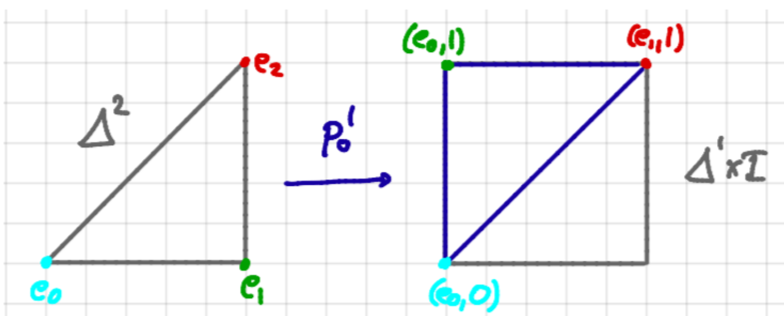
$$p_0^0\colon \Delta^1 \to \Delta^0 \times I = \{e_0\} \times [0,1], (t_0, t_1) \mapsto (0, t_1).$$



Let us have a look at **what happens for $n = 1$**: Then the effect of p_0^1 and p_1^1 is given by

$$p_0\colon \Delta^2 \to \Delta^1 \times [0,1], (t_0,t_1,t_2) \mapsto (t_0 + t_1, t_1 + t_2)$$

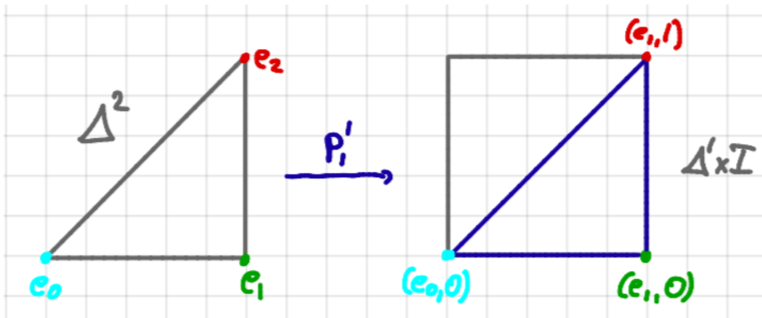
$$\begin{cases} e_0 = (1,0,0) & \mapsto (1,0,0) = (e_0,0) \\ e_1 = (0,1,0) & \mapsto (1,0,1) = (e_0,1) \\ e_2 = (0,0,1) & \mapsto (0,1,1) = (e_1,1) \end{cases}$$



and

$$p_1\colon \Delta^2 \to \Delta^1 \times [0,1], (t_0,t_1,t_2) \mapsto (t_0, t_1 + t_2, t_2)$$

$$\begin{cases} e_0 = (1,0,0) & \mapsto (1,0,0) = (e_0,0) \\ e_1 = (0,1,0) & \mapsto (0,1,0) = (e_1,0) \\ e_2 = (0,0,1) & \mapsto (0,1,1) = (e_1,1). \end{cases}$$



In general, the effect of p_i^n on the vertex e_k of Δ^{n+1} for $0 \leq k \leq n+1$ is given by

$$p_i^n(e_k) = \begin{cases} (e_k,0) & \text{if } 0 \leq k \leq i \\ (e_{k-1},1) & \text{if } k > i. \end{cases} \tag{1}$$

In fact, we could **define** p_i^n as the unique **affine map which satisfies** (1).

Let j_0 and j_1 be the two inclusions

$$j_0 \colon \Delta^n \hookrightarrow \Delta^n \times [0,1],\ x \mapsto (x,0)$$
$$j_1 \colon \Delta^n \hookrightarrow \Delta^n \times [0,1],\ x \mapsto (x,1)$$

determined by the endpoints of $[0,1]$.

For the next result, recall our formulae for the **face maps** on standard simplices:

For $0 \leq i \leq n+1$ which can be described as

$$\phi_i^{n+1}(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n)$$

with the 0 inserted at the ith coordinate.

Using the **standard basis**, ϕ_i^{n+1} can be described as the affine map (a translation plus a linear map)

$$\phi_i^{n+1} \colon \Delta^n \hookrightarrow \Delta^{n+1} \text{determined by } \phi_i^{n+1}(e_k) = \begin{cases} e_k & k < i \\ e_{k+1} & k \geq i. \end{cases} \tag{2}$$

Lemma: Prism and face maps

We have the following identifications of maps:

(3) $$p_0^n \circ \phi_0^{n+1} = j_1,$$

(4) $$p_n^n \circ \phi_{n+1}^{n+1} = j_0,$$

(5) $$p_i^n \circ \phi_i^{n+1} = p_{i-1}^n \circ \phi_i^{n+1} \text{ for } 1 \leq i \leq n,$$

(6) $$p_{j+1}^n \circ \phi_i^{n+1} = (\phi_i^n \times \mathrm{id}) \circ p_j^{n-1} \text{ for } j \geq i.$$

(7) $$p_j^n \circ \phi_{i+1}^{n+1} = (\phi_i^n \times \mathrm{id}) \circ p_j^{n-1} \text{ for } j < i,$$

Proof: (3) We check the effect of $p_0^n \circ \phi_0^{n+1}$

$$\begin{aligned} p_0^n(\phi_0^{n+1}(t_0,\ldots,t_n)) &= p_0(0,t_0,\ldots,t_n) \\ &= \left((t_0,\ldots,t_n), \sum_{i=0}^{n} t_i\right) \\ &= (t_0,\ldots,t_n,1) = j_1(t_0,\ldots,t_n). \end{aligned}$$

(4) Similarly, we calculate

$$\begin{aligned} p_n^n(\phi_{n+1}^{n+1}(t_0,\ldots,t_n)) &= p_n^n(t_0,\ldots,t_n,0) \\ &= (t_0,\ldots,t_n,0) = j_0(t_0,\ldots,t_n). \end{aligned}$$

(5) We calculate and compare:

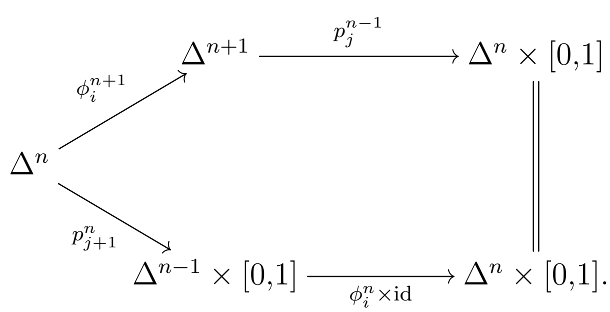
$$\begin{aligned} &p_i^n(\phi_i^{n+1}(t_0,\ldots,t_n)) \\ &= p_i^n(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_n) \\ &= ((t_0,\ldots,t_{i-1},0+t_i,t_{i+1},\ldots,t_n), 0+t_i+\cdots+t_n) \\ &= \left((t_0,\ldots,t_n), \sum_{j=i}^{n} t_j\right), \end{aligned}$$

and

$$
\begin{aligned}
&p^n_{i-1}(\phi^{n+1}_i(t_0,\dots,t_n)) \\
&= p^n_{i-1}(t_0,\dots,t_{i-1},0,t_i,\dots,t_n) \\
&= ((t_0,\dots,t_{i-2},t_{i-1}+0,t_i,t_{i+1},\dots,t_n), t_i+\cdots+t_n) \\
&= \left((t_0,\dots,t_n), \sum_{j=i}^{n} t_j\right).
\end{aligned}
$$

Hence both maps agree.

(6) **For** $j \geq i$, the assertion amounts to showing that the following diagram commutes:



To check this, we are going to use formulae (1) and (2). Since the affine maps involved are determined by their effect on the e_ks, this will suffice to prove the formulae.

For $k < i$, we get

$$
\begin{aligned}
p^n_{j+1} \circ \phi^{n+1}_i(e_k) &= p^n_{j+1}(e_k) = (e_k,0) \\
&= (\phi^n_i \times \mathrm{id})(e_k,0) = (\phi^n_i \times \mathrm{id}) \circ p^{n-1}_j(e_k).
\end{aligned}
$$

For $i \leq k \leq j$, we get

$$
\begin{aligned}
p^n_{j+1} \circ \phi^{n+1}_i(e_{k+1}) &= p^n_{j+1}(e_k) = (e_{k+1},0) \\
&= (\phi^n_i \times \mathrm{id})(e_k,0) = (\phi^n_i \times \mathrm{id}) \circ p^{n-1}_j(e_k).
\end{aligned}
$$

And for $k > j$, we get

$$
\begin{aligned}
p^n_{j+1} \circ \phi^{n+1}_i(e_k) &= p^n_{j+1}(e_{k+1}) = (e_k,1) \\
&= (\phi^n_i \times \mathrm{id})(e_{k-1},1) = (\phi^n_i \times \mathrm{id}) \circ p^{n-1}_j(e_k).
\end{aligned}
$$

(7) follows by a similar argument.

QED

Definition: Induced prism operator

- For every $n \geq 0$ and $0 \leq i \leq n$, the map p_i^n induces a group homomorphism
$$P_i^n \colon S_n(X) \to S_{n+1}(X \times [0,1])$$
which is defined on generators by composition with p_i^n
$$P_i^n(\sigma) = (\sigma \times \mathrm{id}) \circ p_i^n \colon \Delta^{n+1} \xrightarrow{p_i^n} \Delta^n \times [0,1] \xrightarrow{\sigma \times \mathrm{id}} X \times [0,1]$$
and extended $\mathbb{Z}$-linearly.
- This construction descends to a map P_i^n on **relative chains** for any pair (X,A).
- We define a group homomorphism, often called **prism operator**,
$$P^n \colon S_n(X,A) \to S_{n+1}(X \times I, A \times I),\ P^n = \sum_{i=0}^{n} (-1)^i P_i^n.$$

Let j_t^X denote the inclusion $X \hookrightarrow X \times [0,1]$, $x \mapsto (x,t)$. The prism operator is the desired chain homotopy:

Chain homotopy lemma

The homomorphisms P^n provide a **chain homotopy** between the two morphisms of chain complexes
$$S_*(j_0^X) \simeq S_*(j_1^X) \colon S_*(X,A) \to S_*(X \times I, A \times I).$$

Proof: We need to show $S_n(j_0^X) - S_n(j_1^X) = \partial_{n+1} \circ P^n + P^{n-1} \circ \partial_n$. Let σ be an n-simplex. Then we calculate

$$\begin{aligned}
P^{n-1} \circ \partial_n(\sigma) &= P_{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma \circ \phi_i^n \right) \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} (\sigma \times \mathrm{id}) \circ (\phi_i^n \times \mathrm{id}) \circ p_j^{n-1} \\
&+ \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} (\sigma \times \mathrm{id}) \circ (\phi_i^n \times \mathrm{id}) \circ p_j^{n-1} \\
&= - \sum_{0 \leq j < i \leq n} (-1)^{i+j+1} (\sigma \times \mathrm{id}) \circ p_{j+1}^n \circ \phi_{i+1}^{n+1} \text{ by (7)} \\
&- \sum_{0 \leq i \leq j \leq n} (-1)^{i+j+1} (\sigma \times \mathrm{id}) \circ p_j^n \circ \phi_{i+1}^{n+1} \text{ by (6)}.
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
\partial_{n+1} \circ P^n(\sigma) &= \partial_{n+1} \left(\sum_{j=0}^{n} (-1)^j (\sigma \times \mathrm{id}) \circ p_j^n \right) \\
&= \sum_{j=0}^{n} \sum_{i=0}^{n+1} (-1)^{i+j} (\sigma \times \mathrm{id}) \circ p_j^n \circ \phi_i^{n+1} \\
&= \sum_{0 \le i < j \le n} (-1)^{i+j} (\sigma \times \mathrm{id}) \circ p_j^n \circ \phi_i^{n+1} \ (i < j) \\
&+ \sum_{i=0}^{n} (\sigma \times \mathrm{id}) \circ p_i^n \circ \phi_i^{n+1} \ (i = j) \\
&- \sum_{i=1}^{n+1} (\sigma \times \mathrm{id}) \circ p_{i-1}^n \circ \phi_i^{n+1} \ (i = j+1) \\
&+ \sum_{1 \le j+1 \le i \le n+1} (-1)^{i+j} (\sigma \times \mathrm{id}) \circ p_j^n \circ \phi_i^{n+1} \ (i > j+1) \\
&= \sum_{0 \le i \le j' \le n-1} (-1)^{i+j'+1} (\sigma \times \mathrm{id}) \circ p_{j'+1}^n \circ \phi_i^{n+1} \\
&+ (\sigma \times \mathrm{id}) \circ j_1 - (\sigma \times \mathrm{id}) \circ j_0 \\
&+ \sum_{0 \le j \le i' \le n} (-1)^{i'+j+1} (\sigma \times \mathrm{id}) \circ p_j^n \circ \phi_{i'+1}^{n+1}.
\end{aligned}
$$

For the final step we used again the trick to relabel the indices and wrote $j' = j-1$ and $i' = i-1$. By comparing the two calculations, we see that all summands cancel out except for $(\sigma \times \mathrm{id}) \circ j_1 - (\sigma \times \mathrm{id}) \circ j_0$.

Thus we can conclude:

$$
\begin{aligned}
\partial_{n+1} \circ P^n(\sigma) + P^{n-1} \circ \partial_n(\sigma) &= (\sigma \times \mathrm{id}) \circ j_1^X - (\sigma \times \mathrm{id}) \circ j_0^X \\
&= j_1^X \circ \sigma - j_0^X \circ \sigma \\
&= S_n(j_1^X)(\sigma) - S_n(j_0^X)(\sigma).
\end{aligned}
$$

QED

As a consequence we get the Homotopy Axiom:

Theorem: Homotopy Invariance

If $f \simeq g\colon (X,A) \to (Y,B)$, then

$$S_n(f) \simeq S_n(g)\colon S_n(X,A) \to S_n(Y,B)$$

for all n. Hence $f \simeq g$ implies $H_n(f) = H_n(g)$.

Proof: Let h be a homotopy between f and g. We can write this as

$$f = h \circ j_1^X \text{ and } g = h \circ j_0^X.$$

Then the **previous lemma** yields

$$\begin{aligned}
& S_n(f) - S_n(g) \\
&= S_n(h) \circ S_n(j_1^X) - S_n(h) \circ S_n(j_0^X) \\
&= S_n(h) \circ (\partial_{n+1} \circ P^n) + S_n(h) \circ (P^{n-1} \circ \partial_n) \\
&= \partial_{n+1} \circ (S_{n+1}(h) \circ P^n) + (S_n(h) \circ P^{n-1}) \circ \partial_n \text{ since } S_*(h) \text{ is a } \textbf{chain map}.
\end{aligned}$$

Thus the sequence of homomorphisms $S_{n+1}(h) \circ P^n$ is a **chain homotopy** between $S_n(f)$ and $S_n(g)$.

Appyling the theorem about chain homotopic maps and their induced maps on homology implies the last statement. **QED**

We have already used homotopy invariance of singular homology at numerous occasions. Here is yet another one:

Proposition: Homology of weak retracts

Let $i\colon A \hookrightarrow X$ be a **weak retract**, i.e., assume there is continuous map $\rho\colon X \to A$ such that $\rho \circ i \simeq \mathrm{id}_A$. Then

$$H_n(X) \cong H_n(A) \oplus H_n(X,A) \text{ for all } n.$$

Proof: Functoriality and homotopy invariance tell us that $\rho \circ i \simeq \mathrm{id}_A$ implies

$$H_n(\rho) \circ H_n(i) = H_n(\rho \circ i) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)} \text{ for all } n.$$

Hence $H_n(i)$ is **injective** for all n. That means that the sequence

$$0 \to H_n(A) \xrightarrow{H_n(i)} H_n(X)$$

is **exact**. Since $H_{n-1}(i)$ is injective, the exactness of the sequence

$$\cdots \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(X) \to \cdots$$

implies that the connecting homomorphism $\partial\colon H_n(X,A) \to H_{n-1}(A)$ is the zero map. Thus we get a **short exact sequence**

$$0 \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \to 0. \tag{8}$$

Since $H_n(\rho)$ is a **left-inverse** of $H_n(i)$, this sequence **splits**, i.e.,

$$H_n(X) \xrightarrow{\cong} H_n(A) \oplus H_n(X,A),\ [c] \mapsto (H_n(\rho)([c]), H_n(j)([c]).$$

is an **isomorphism**.

We can describe an **inverse** of this map as

$$\begin{aligned} &H_n(A) \oplus H_n(X,A) \to H_n(X), \\ &\qquad ([a],[b]) \mapsto H_n(i)([a]) + [c'] - H_n(i \circ \rho)([c']) \end{aligned}$$

where $[c']$ is any class with $H_n(j)([c']) = [b]$.

We need to check that **the choice of $[c']$ does not matter**. So let $[c'']$ be another class with $H_n(j)([c'']) = [b]$. Then we have

$$H_n(j)([c'] - [c'']) = 0, \text{ i.e., } [c'] - [c''] \in \mathrm{Ker}\,(H_n(j)).$$

By the exactness of (8), this implies that there exists a class $[\tilde{a}] \in H_n(A)$ with $H_n(i)([\tilde{a}]) = [c'] - [c'']$. Thus

$$\begin{aligned} &[c'] - H_n(i \circ \rho)([c']) - ([c''] - H_n(i \circ \rho)([c''])) \\ &= [c'] - [c''] - H_n(i \circ \rho)([c'] - [c'']) \\ &= H_n(i)([\tilde{a}]) - H_n(i \circ \rho)(H_n(i)([\tilde{a}])) \\ &= H_n(i)([\tilde{a}]) - (H_n(i) \circ H_n(\rho) \circ H_n(i))([\tilde{a}]) \\ &= H_n(i)([\tilde{a}]) - H_n(i)([\tilde{a}]) \text{ since } H_n(\rho) \circ H_n(i) = \mathrm{id}_{H_n(A)} \\ &= 0. \end{aligned}$$

QED

Homotoy invariance revisited

Let **hoTop** denote the **homotopy category** of **Top**, i.e., the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps:

$$\mathrm{Mor}_{\mathbf{hoTop}}(X,Y) = [X,Y] = \mathrm{Map}_{\mathbf{Top}}(X,Y)/\simeq$$

where $\mathrm{Map}_{\mathbf{Top}}(X,Y)$ denotes the set of continuous maps from X to Y. The result we have just shown implies that singular homology descends to a functor on **hoTop**:

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{H_n} & \mathbf{Ab} \\ \downarrow & \nearrow_{H_n} & \\ \mathbf{hoTop}. & & \end{array}$$

This observation applies to almost all algebraic invariants in Topology. In other words, invariants in Algebraic Topology distinguish neither between homotopic maps nor between homotopy equivalent spaces.
However, many topological properties are not invariant under homotopy. For example, **compactness** is not invariant under homotopy. In other words, if X is **compact**, then it may well be the case that a space which is homotopy equivalent to X is not compact. To convince ourselves of this fact, it suffices to take $X = \{0\}$ and $Y = \mathbb{R}^n$. A bit more interesting is $X = S^n$ and $Y = \mathbb{R}^{n+1} \setminus \{0\}$. While X **is compact**, Y **is not**, but the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence.
This observation should make us aware of the scope of our abilities. The **tools** we develop in this class **are great**. But they are **not the end of the story...**