MA3403 Algebraic Topology
Lecturer: Gereon Quick
Lecture 12

12. LOCALITY AND THE MAYER-VIETORIS SEQUENCE

We are going to discuss the Excision Axiom for singular homology and some consequences. Let us first recall what it says:

Excision Axiom of singular homology

Let (X,A) be a pair of spaces and let $Z \subset A$ be a subspace the closure of which is contained in the interior of A, in formulae $\bar{Z} \subseteq A^{\circ}$. Then the inclusion map $k\colon (X-Z, A-Z) \hookrightarrow (X,A)$ induces an **isomorphism**

$$H_n(k)\colon H_n(X-U, A-U) \to H_n(X,A) \text{ for all } n.$$

We are going to deduce the excision property of homology from the following **locality principle**.

Let X be a topological space and let $\mathcal{A} = \{A_j\}_{j\in J}$ be a **cover** of X, i.e., a collection of subsets $A_j \subseteq X$ such that X is the **union of the interiors** of the A_js.

$\mathcal{A}$-small chains

- An n-simplex $\sigma\colon \Delta^n \to X$ is called **$\mathcal{A}$-small** if the image of σ is contained in one of the A_js.
- An n-chain $c = \sum_i n_i\sigma_i$ if X is called **$\mathcal{A}$-small** if, for every i, there is a A_j such that $\sigma_i(\Delta^n) \subset A_j$.
- We are going to denote the subgroup of $\mathcal{A}$-small n-chains by

$$S_n^{\mathcal{A}}(X) := \{c \in S_n(X) : \ \sigma \text{ is } \mathcal{A}-\text{small}\}.$$

- For a subspace $A \subset X$, we write

$$S_n^{\mathcal{A}}(X,A) := \frac{S_n^{\mathcal{A}}(X)}{S_n^{\mathcal{A}}(A)}.$$

If, for each j, $\iota_j\colon A_j \hookrightarrow X$ denotes the inclusion map, then we can describe $S_n^{\mathcal{A}}(X)$ also as

$$S_n^{\mathcal{A}}(X) = \operatorname{Im}\left(\bigoplus_{j\in J} S_n(A_j) \xrightarrow{\oplus_j S_n(\iota_j)} S_n(X)\right).$$

The point of $\mathcal{A}$-small chains is that we can use their chain complex to compute singular homology:

Locality Principle/Small Chain Theorem

For any cover $\mathcal{A}$ of X, the inclusion of chain complexes

$$S_*^{\mathcal{A}}(X,A) \subset S_*(X,A)$$

induces an **isomorphism in homology**.

The **proof** of this theorem takes quite an effort and we will postpone it for a moment. Instead we will now explain how the excision property follows from the theorem.

- **Proof of the Excision Axiom using small chains:**

Since $\bar{Z} \subseteq A^\circ$, we have $(X-Z)^\circ \cup A^\circ = X$. Thus, if we set $B := X - Z$, $\mathcal{A} = \{A, B\}$ is a cover of X.

Moreover, we can rewrite

$$(X-Z, A-Z) = (B, A\cap B).$$

Hence our **goal** is to show that

$$S_*(B,A\cap B) \to S_*(X,A)$$

induces an **isomorphism in homology**.

The inclusion of chain complexes $S_*^{\mathcal{A}}(X) \subset S_*(X)$ induces a morphism of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & S_*(A) \arrow[r] \arrow[d, equal] & S_*^{\mathcal{A}}(X) \arrow[r] \arrow[d] & S_*^{\mathcal{A}}(X)/S_*(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & S_*(X)/S_*(A) \arrow[r] & 0.
\end{tikzcd}$$

The **middle vertical map** induces an isomorphism in homology by the **Small Chain Theorem**. The induced long exact sequences in homology and the **Five-Lemma** imply that the **right-hand vertical map** induces an isomorphism in homology as well. Thus we are **reduced to compare** $S_*(B,A \cap B)$ and $S_*^{\mathcal{A}}(X)/S_*(A)$.

Now we observe

$$S_*^{\mathcal{A}}(X) = S_*(A) + S_*(B) \subset S_*(X)$$

and hence

$$\frac{S_*(B)}{S_*(A \cap B)} = \frac{S_*(B)}{S_*(A) \cap S_*(B)} \xrightarrow{\cong} \frac{S_*(A) + S_*(B)}{S_*(A)} = \frac{S_*^{\mathcal{A}}(X)}{S_*(A)}$$

where the middle isomorphism follows from the **general comparison** of quotients of sums and intersections of abelian groups.

Thus the chain map

$$S_*(B,A \cap B) \to S_*^{\mathcal{A}}(X)/S_*(A)$$

induces an isomorphism in homology and the excison axiom holds. **QED**

• The Mayer-Vietoris sequence

The above proof inspires us to look at the following situation which will lead to an important computational tool.

Assume that $\mathcal{A} = \{A,B\}$ is a **cover** of X. Consider the diagram

$$\begin{tikzcd}
A \cap B \arrow[r, "j_A"] \arrow[d, "j_B"'] & A \arrow[d, "i_A"] \\
B \arrow[r, "i_B"'] & X.
\end{tikzcd}$$

For every n, these maps induce homomorphisms in homology

$$\alpha_n \colon H_n(A \cap B) \to H_n(A) \oplus H_n(B), \alpha_n = \begin{bmatrix} H_n(j_A) \\ -H_n(j_B) \end{bmatrix}$$
$$x \mapsto (H_n(j_A)(x), -H_n(j_B)(x))$$

and

$$\beta_n \colon H_n(A) \oplus H_n(B) \to H_n(X), \beta_n = \begin{bmatrix} H_n(i_A) & H_n(i_B) \end{bmatrix}$$
$$(a,b) \mapsto H_n(i_A)(a) + H_n(i_B)(b).$$

Theorem: Mayer-Vietoris sequence

For any cover $\mathcal{A} = \{A,B\}$ of X, there are natural homomorphisms

$$\partial_n^{MV} \colon H_n(X) \to H_{n-1}(A \cap B) \text{ for all } n$$

which fit into an **exact sequence**

$$\begin{tikzcd}
 & \cdots \arrow[r, "\beta_{n+1}"] & H_{n+1}(X) \arrow[dll, "\partial^{MV}_{n+1}"'] \\
H_n(A \cap B) \arrow[r, "\alpha_n"'] & H_n(A) \oplus H_n(B) \arrow[r, "\beta_n"] & H_n(X) \arrow[dll, "\partial^{MV}_n"] \\
H_{n-1}(A \cap B) \arrow[r, "\alpha_{n-1}"'] & \cdots &
\end{tikzcd}$$

Proof: From the proof of the Excision Axiom we remember that there is a short exact sequence of chain complexes

$$0 \to S_*(A \cap B) \xrightarrow{\begin{bmatrix} S_*(j_A) \\ -S_*(j_B) \end{bmatrix}} S_*(A) \oplus S_*(B) \xrightarrow{\begin{bmatrix} S_*(i_A) & S_*(i_B) \end{bmatrix}} S_*^{\mathcal{A}}(X) \to 0.$$

Note that the exactness at the right-hand term was part of the proof of the Excision Axiom and the exacntess at the middle term can be easily checked by looking long enough at the **commutative diagram**

$$\begin{tikzcd}
\Delta^n \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & A \cap B \arrow[r, "j_A"] \arrow[d, "j_B"'] & A \arrow[d, "i_A"] \\
 & B \arrow[r, "i_B"'] & X.
\end{tikzcd}$$

It induces a long exact sequence in homology

$$\begin{tikzcd}
 & \cdots \arrow[r, "\beta^{\mathcal{A}}_{n+1}"] & H^{\mathcal{A}}_{n+1}(X) \arrow[dll, "\partial^{\mathcal{A}}_{n+1}"'] \\
H_n(A \cap B) \arrow[r, "\alpha_n"'] & H_n(A) \oplus H_n(B) \arrow[r, "\beta^{\mathcal{A}}_n"] & H^{\mathcal{A}}_n(X) \arrow[dll, "\partial^{\mathcal{A}}_n"] \\
H_{n-1}(A \cap B) \arrow[r, "\alpha_{n-1}"'] & \cdots &
\end{tikzcd}$$

By definition of small chains, the homomorphism β_n factors through small chains, in other words, it is induced by the composition

$$S_n(A) \oplus S_n(B) \xrightarrow{\begin{bmatrix} S_n(i_A) & S_n(i_B) \end{bmatrix}} S^{\mathcal{A}}_n(X) \hookrightarrow S_n(X).$$

Thus we can apply the inverse of the isomorphism of the **Small Chain Theorem** and define ∂^{MV}_n to be

$$\partial^{MV}_n : H_n(X) \xrightarrow{\cong} H^{\mathcal{A}}_n(X) \xrightarrow{\partial^{\mathcal{A}}_n} H_{n-1}(A \cap B).$$

Then the following sequence

$$\begin{tikzcd}
H_n(A) \oplus H_n(B) \arrow[r, "\beta_n"] \arrow[dr, "\beta^{\mathcal{A}}_n"'] & H_n(X) \arrow[r, "\partial^{MV}_n"] & H_{n-1}(A \cap B) \\
 & H^{\mathcal{A}}_n(X) \arrow[u, "\cong"] \arrow[ur, "\partial^{\mathcal{A}}_n"'] &
\end{tikzcd}$$

is exact at $H_n(X)$, since the triangles commute.

This yields the sequence of homomorphisms and the desired long exact sequence. **QED**

The MVS is an extremely useful tool

The Mayer-Vietoris sequence (MVS) is an important computational tool. Its power relies on the simple idea: If you want to understand a big space, split it up into smaller spaces you understand and then put the information back together.
The MVS tells us how the homology of X is built out of homologies of the cover by A and B.

Let us apply this new insight to some **concrete examples**:

• Let us calculate the homology of $X = S^1$ yet another time. Let $x = (0,1)$ and $y = (0,-1)$ on S^1. We set $A = S^1 - \{y\}$ and $B = S^1 - \{y\}$. Then A and B are **two open subsets** which cover S^1. We observe that both A and B are contractible.

The intersection $A \cap B$ contains the points $p = (-1,0)$ and $q = (1,0)$. In fact, the inclusion

$$\{p,q\} \hookrightarrow A \cap B$$

is a **deformation retract**.

Since $\mathcal{A} = \{A,B\}$ is a cover of S^1, we can write down the corresponding MVS. For $n \geq 2$, all the homology groups hitting and being hit by $H_n(S^1)$ are zero, since $H_n(A) \oplus H_n(B) = H_n(\{x\}) \oplus H_n(\{y\}) = 0$ and $H_{n-1}(A \cap B) = H_{n-1}(\{p,q\}) = 0$. Thus

$$H_n(S^1) = 0 \text{ for all } n \geq 2.$$

Since S^1 is path-connected, we know $H_0(S^1) = \mathbb{Z}$. It remains to check $n = 1$.

The MVS for $n = 1$ looks like

$$\begin{tikzcd}
0 \arrow[r] & H_1(S^1) \arrow[r] \arrow[d, equal] & H_0(A \cap B) \arrow[r] \arrow[d, equal] & H_0(A) \oplus H_0(B) \arrow[d, equal] \\
0 \arrow[r] & H_1(S^1) \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}}"'] & \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}$$

where we obtain the lower right-hand map by observing that all summands are of the form $H_0(\mathrm{pt})$ and hence each generator in $H_0(A \cap B)$ is sent to $(1,-1)$ by $[H_0(j_A), -H_0(j_B)]$. Thus $H_1(S^1)$ is the kernel of this map:

$$H_1(S^1) \cong \mathrm{Ker}\left(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right) = \{(x,-x) \in \mathbb{Z} \oplus \mathbb{Z}\} \cong \mathbb{Z}.$$

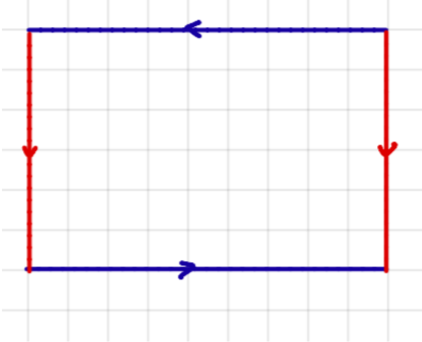
• For $n \geq 2$, let $A = S^n - \{S\}$ and $B = S^n - \{N\}$ where N and S are the **north- and south-pole** of S^n, respectively,. We observe that both A and B are **contractible**. Moreover, the inclusion of $j \colon S^{n-1} \hookrightarrow A \cap B$ as the equator is a strong deformation retract. In particular, j is a homotopy equivalence.

Together with the inverse of the isomorphism $H_{q-1}(j)$, we get

$$H_{q-1}(j)^{-1} \circ \partial_q^{MV} \colon H_q(S^n) \xrightarrow{\cong} H_{q-1}(S^{n-1})$$

is an isomorphism for all $q \geq 2$. Since we know $H_q(S^1)$ for all q, this yields $H_q(S^n)$ by induction.

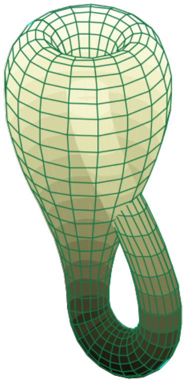
- Let K be the **Klein bottle** which can be constructed from a square by gluing the edges as indicated in the following picture:



- glue together red edges and get a cylinder
- then glue blue edges with reversed orientation
- get two Möbius bands glued together

along the two boundary circle $\approx$ red edge

The outcome of this procedure is the **twisted surface** whose 3-dimensional shadow we see in the next picture which is taken from wikipedia.org:



(Note that we should really think of K as an object in $\mathbb{R}^4$ where it does not self-intersect.)

We observe that K can be constructed by taking **two Möbius bands** A and B and gluing them together by a homeomorphism between their boundary circles. Hence $K = A \cup B$ and $A \cap B \approx S^1$. In the exercises we are going to caculate the homology of the Möbius strip. It is given by $H_0(M) = H_1(M) = \mathbb{Z}$ and $H_2(M) = 0$.

We would like to use this information to calculate the homology of K.

Since K is **path-connected** as a quotient of a path-connected space, we know $H_0(K) = \mathbb{Z}$.

Now we apply the **MVS**: We observe that $H_n(A)$, $H_n(B)$ and $H_n(A \cap B)$ vanish for $n \geq 2$. Hence $H_n(K) = 0$ for all $n \geq 3$.

The remaining MVS looks like this:

$$0 \to H_2(K) \xrightarrow{\partial^{MV}} H_1(A \cap B) \xrightarrow{\varphi_1} H_1(A) \oplus H_1(B) \to H_1(K) \to 0.$$

The 0 on the right-hand side is justified by the fact that

$$H_0(A \cap B) \cong \mathbb{Z} \xrightarrow{\varphi_0} \mathbb{Z} \oplus \mathbb{Z} \cong H_0(A) \oplus H_0(B)$$

is **injective**.

The map φ_1 is given by

$$\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z} \oplus \mathbb{Z},\ 1 \mapsto (2, -2),$$

since

$$H_1(A \cap B) = H_1(S^1) \to H_1(M) = H_1(A)$$

wraps the circle around the boundary of M **twice**, and

$$H_1(A \cap B) = H_1(S^1) \to H_1(M) = H_1(B)$$

does that too, but with **reversed orientation**. (We will understand this fact better after we have done the **exercises**.) Hence on the second factor we use the map $z \mapsto z^{-2}$ to produce a Möbius band.

In particular, φ_1 is **injective** and hence

$$\mathbf{H_2(K) = 0}.$$

Moreover, $H_1(K)$ is the **cokernel** of φ_1. If we choose the **basis**

$$\{b_1 := (1,0), b_2 := (1, -1)\} \text{ for } \mathbb{Z} \oplus \mathbb{Z},$$

then we see that φ_1 maps $1 \in \mathbb{Z}$ to $2b_2$ in $\mathbb{Z}b_1 \oplus \mathbb{Z}b_2$. Hence the cokernel of φ_1 is isomorphic to $\mathbb{Z}b_1 \oplus \mathbb{Z}b_2/2b_2$. Thus

$$\mathbf{H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2}.$$