MA3403 Algebraic Topology
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Lecture 13

13. Cell complexes

We return to an important type of topological spaces, called CW- or cell complexes, that is particularly convenient for our purposes in many respects. It will turn out that this type of spaces both appears very frequently and is quite accessible for calculations. In particular, we will learn next week that the homology of a cell complex is quite easy to compute.

The idea of creating a cell complex is to **successively glue cells** to what has already been built. The **general procedure** for doing this is the following:

Gluing a space along a map

Suppose we have a space X and a pair (B,A) of spaces. We **define a space** $X \cup_f B$, often also denoted $X \cup_A B$ if the map f is either understood or just the inclusion, which fits into the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ B \arrow[r, "\varphi"'] & X \cup_f B \end{tikzcd}$$

by

$$X \cup_f B := (X \sqcup B)/(a \sim f(a) \text{ for all } a \in A).$$

We say that $X \cup_f B$ arises from **attaching B to X along f**, or along A, and f is called an **attaching map**.

By its construction, there are two types of equivalence classes in $X \cup_f B$:

- classes which consist of single points of $B - A$,
- classes which consist of sets $\{x\} \sqcup f^{-1}(X)$ for any point $x \in X$.

Note that the **lower horizontal map** $\varphi\colon B \to B \cup_f X$ arises as **part of the construction**. It is given by

$$\varphi\colon B \to B \cup_f X,\ b \mapsto \begin{cases} b & \text{if } b \in B - A \\ [b] & \text{if } b \in A. \end{cases}$$

In particular, this shows that $\varphi_{|B-A}$ **is a homeomorphism**.

The topology of $X \cup_f B$ is the **quotient topology** and is characterized by the **universal property**: whenever there is a diagram of solid arrows of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & X \cup_f B \arrow[dr, dotted] & \\ & & Y \end{tikzcd}$$

then **there is a unique dotted arrow** which makes all triangles commute. We can reformulate this fact by saying that $X \cup_f B$ is the **pushout** of the solid diagram.

For **example**:

- if $X = *$ consists of just a point, then
$$X \cup_f B = * \cup_f B = B/A;$$
- if $A = \emptyset$, then $X \cup_f B = X \sqcup B$ is just a disjoint union.

A more important example is the following:

Attaching a cell

We consider the pair (D^n, S^{n-1}) of an n-disk and its boundary. We are going to think of D^n as an **n-cell**.
Suppose we are given a map $f \colon S^{n-1} \to X$. Then we can **attach an n-cell** to X via f as

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ D^n \arrow[r] & X \cup_f D^n. \end{tikzcd}$$

We could speed up this process by attaching several cells at once:

$$\begin{tikzcd} \coprod_{\alpha \in J} S^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ \coprod_{\alpha \in J} D^n_\alpha \arrow[r] & X \cup_f \coprod_{\alpha \in J} D^n_\alpha. \end{tikzcd}$$

Let us look at some **examples**:

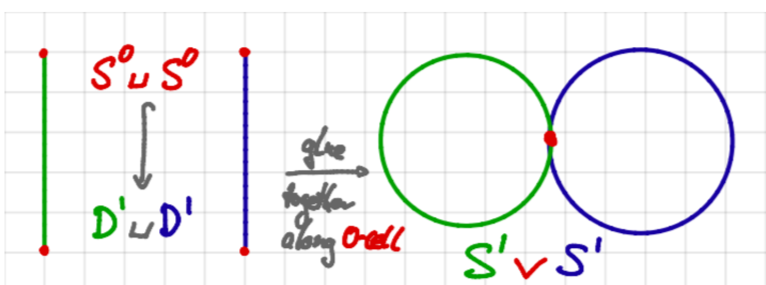
- Let us start with $n = 0$ and write (D^0, S^{-1}) for $(*, \emptyset))$. Attaching 0-cells to a space X just means **adding a set of discrete points** to X:

$$X \cup_f \coprod_{\alpha \in J} D^0_\alpha = X \sqcup J$$

  where J is a set with the discrete topology.
- Now let us attach two 1-cells to a point $X = *$:

$$\begin{tikzcd} S^0 \sqcup S^0 \arrow[r, "f"] \arrow[d, hook] & * \arrow[d] \\ D^1 \sqcup D^1 \arrow[r] & * \cup_f (D^1 \sqcup D^1). \end{tikzcd}$$

  Since there is only one choice for f, we get a **figure eight**: we start with two 1-disks D^1 and then we identify all four boundary points with the 0-cell. We **denote this space by** $S^1 \vee S^1$.



- We continue with this space and attach one 2-cell: We can think of $S^1 \vee S^1$ as an empty square where we glue together the horizontal edges and the vertical edges. Then we glue in a 2-cell into the square by attaching its boundary to the edges a, b, a^{-1}, and b^{-1}, i.e., by walking clockwise:

$$\begin{tikzcd} S^1 \arrow[r, "f = aba^{-1}b^{-1}"] \arrow[d, hook] & S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r] & (S^1 \vee S^1) \cup_f D^2 = T^2. \end{tikzcd}$$

  The result of this procedure is a two-dimensional **torus**.

This example motivates the following key concept:

Cell complex

A **cell complex**, or CW-complex, is a space X equipped with a sequence of subspaces

$$\emptyset = \mathrm{Sk}_{-1}X \subseteq \mathrm{Sk}_0 X \subseteq \mathrm{Sk}_1 X \subseteq \mathrm{Sk}_2 X \subseteq \cdots X$$

such that

- X is the union of the $\mathrm{Sk}_n X$s,
- for all n, $\mathrm{Sk}_n X$ arises from Sk_{n-1} by **attaching n-cells**, i.e., there is a **pushout diagram**

$$\begin{tikzcd} \coprod_{\alpha \in J_n} S^{n-1}_\alpha \arrow[r, "f_n"] \arrow[d, hook] & \mathrm{Sk}_{n-1}X \arrow[d] \\ \coprod_{\alpha \in J_n} D^n_\alpha \arrow[r, "\varphi_n"'] & \mathrm{Sk}_n X. \end{tikzcd}$$

The space $\mathrm{Sk}_n X$ is called the **n-skeleton** of X.

In our **example of the torus** T^2 the skeleta are

$$\mathrm{Sk}_0 T^2 = *,\ \mathrm{Sk}_1 T^2 = S^1 \vee S^1,\ \mathrm{Sk}_2 T^2 = T^2.$$

Before we study more examples, we fix more terminology and list some facts which should help clarify the picture:

- The **topology** of a cell complex is determined by its skeleta, i.e., a subset $U \subset X$ is **open** (closed) if and only if $U \cap \mathrm{Sk}_n X$ is open (closed) for all n.

- In fact, the topology on X is determined by its cells, i.e., U is **open (closed)** in X if and only its intersection with each cell is open (closed), or equivalently, if $\varphi_\alpha^{-1}(U)$ is open (closed) in each D^n_α. This topology is called the **weak topology** and explains the **W** in *CW-complex*.

- That implies that a map $g\colon X \to Y$ is continuous if and only if its restriction to each skeleton is continuous, or equivalently, if and only if
$$g \circ \varphi_\alpha \colon D^n_\alpha \to Y \text{ is continuous for all } D^n_\alpha.$$

- For any n-cell D^n_α, the induced map $\varphi_\alpha \colon D^n_\alpha \to X$ is called the **characteristic map** of the cell. As we explained before, the restriction to the **open interior** $(D^n_\alpha)^\circ = D^n_\alpha - S^{n-1}_\alpha$
$$(\varphi_\alpha)_{|(D^n_\alpha)^\circ} \to X$$
is a **homeomorphism onto its image**.

- We will call the image of D^n_α under φ_α in X a **closed n-cell** of X. We will refer to n as the **dimension of the cell**. Since D^n is compact, it is a compact subset.

- The image of the **interior** $(D^n_\alpha)^\circ$ of D^n_α in X is often called an **n-cell** or **open n-cell** of X and will be denoted by e^n_α. Note that this subset is **not** necessarily an open subset of X.

- The **C** in *CW-complex* stands for **closure finite** which means that, for every cell, $\varphi_\alpha(S^{n-1}_\alpha)$ is contained in **finitely** many cells (of dimension at most $n-1$).

- A cell complex X is called **finite-dimensional** if there is an n such that $X = \mathrm{Sk}_n X$. The smallest such n is called the **dimension of X**, i.e., the unique n such that $\mathrm{Sk}_n X = X$ and $\mathrm{Sk}_{n-1} X \subsetneq X$.

- A cell complex is called **of finite type** if each indexing set J_n is finite, i.e., if only finitely many cells are attached in each step.

- A cell complex is called **finite** if it is finite-dimensional and of finite type, i.e., if it has only finitely many cells.

- The dimension of a cell complex is a topological invariant, i.e., it is invariant under homeomorphisms. Moreover, every cell complex is **Hausdorff**.

- However, a cell complex is **compact** if and only if it is **finite**.

- Note that every nonempty cell complex must have at least one 0-cell.

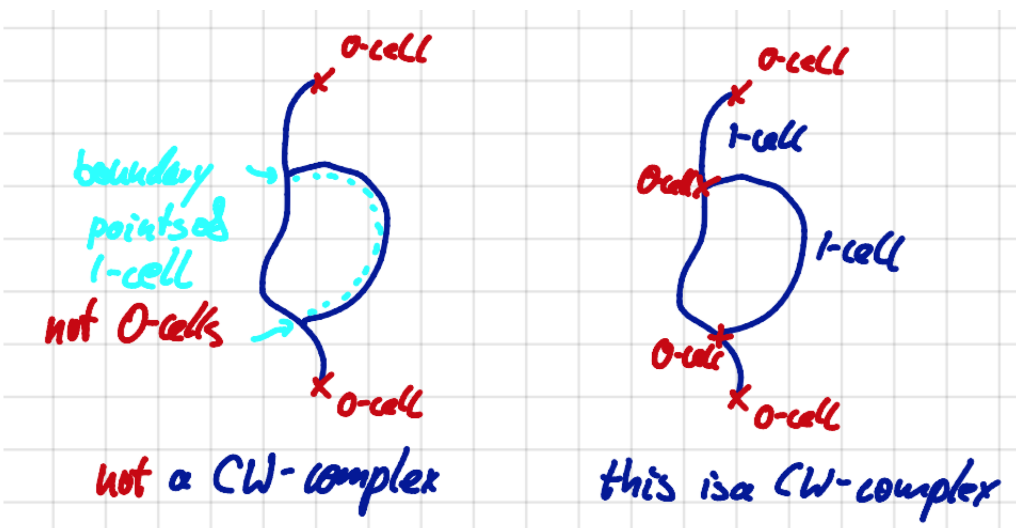
- The cell structure of a cell complex is in general not unique. Often there are many different cell structures. We will observe this for example for the n-sphere.

Here is an important theorem which demonstrates the wide range and importance of cell complexes:

Compact smooth manifolds are cell complexes

Every compact smooth manifold can be given the structure of a cell complex.

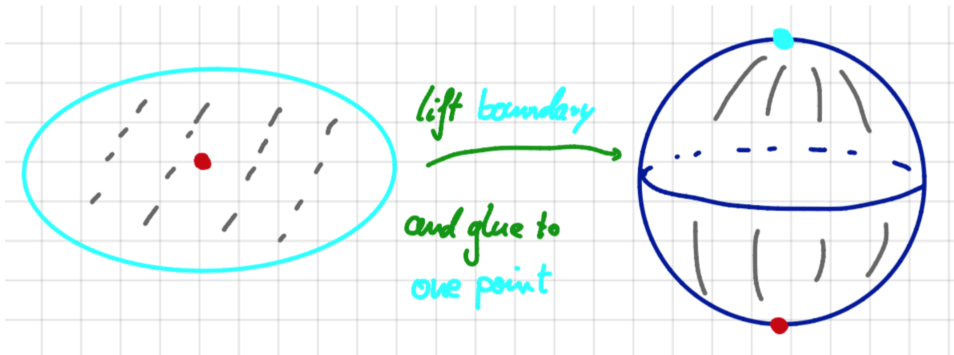
Here some important examples:



- A simple **example** is given by surfaces of a three-dimensional **cube**: it has eight 0-cells, twelve 1-cells, six 2-cells.

- Similarly, every **n-simplex** is a cell complex. For example, Δ^3 has four 0-cells, six 1-cells, four 2-cells, and one 3-cell.

- The **sphere** S^n is a cell complex with just two cells: **one 0-cell** e^0 (that is a point) and **one n-cell** which is attached to e^0 via the constant map $S^{n-1} \to e^0$. Geometrically, this corresponds to expressing S^n as $D^n/\partial D^n$: we take the open n-disk $D^n \setminus \partial D^n$ and collapse the boudnary ∂D^n to a single point which is, say, the north pole $N = e^0$.



- The n-sphere $X = S^n$ can also be equipped with a **different cell structure**:

  We start with two 0-cells which give us the 0-skeleton

$$\mathrm{Sk}_0 X \xrightarrow{\approx} S^0.$$

  Now we attach two 1-cells via the homeomorphism $f\colon S^0 \xrightarrow{\approx} \mathrm{Sk}_0 X$. This gives us one 1-cell as the **upper half-circle** and one 1-cell as the **lower half-circle** and

$$S^1 \xrightarrow{\approx} \mathrm{Sk}_1 X.$$

  Then we attach two 2-cells as the **upper and lower hemisphere** along the map $S^1 \xrightarrow{\approx} \mathrm{Sk}_1$, i.e., this gives us

$$\mathrm{Sk}_2 X \xrightarrow{\approx} S^2$$

  with $\mathrm{Sk}_1 X \approx S^1$ as the equator of S^2. Now we continue this procedure until we reach S^n.

  Hence, in this cell structure on S^n, there are **exactly two k-cells in each dimension** $k = 0, \ldots, n$.

- **Real projective space** $\mathbb{RP}^n$ is a cell complex with **one cell in each dimension** up to n. To show this we proceed inductively. We know that $\mathbb{RP}^0$ consists of a single point, since it is S^0 whose two antipodal points are identified.

  Now we would like to understand how $\mathbb{RP}^n$ can be constructed from $\mathbb{RP}^{n-1}$:

  We embed D^n as the **upper hemisphere** into S^n, i.e., we consider D^n as $\{(x_0, \ldots, x_n) \in S^n : x_0 \geq 0\}$. Then

$$\mathbb{RP}^n = S^n/(x \sim -x) = D^n/(x \sim -x \text{ for } \textbf{boundary} \text{ points } x \in \partial D^n).$$

But ∂D^n is just S^{n-1}. Hence the quotient map

$$S^{n-1} \to S^{n-1}/\sim = \mathbb{R}\mathrm{P}^{n-1}$$

attaches an n-cell e^n, the open interior of D^n, at $\mathbb{R}\mathrm{P}^{n-1}$.

Thus we obtain $\mathbb{R}\mathrm{P}^n$ from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching one n-cell via the quotient map $S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$.

Summarizing, we have shown that $\mathbb{R}\mathrm{P}^n$ is a cell complex with one cell in each dimension from 0 to n.

- We can continue this process and build the **infinite projective space** $\mathbb{R}\mathrm{P}^\infty := \bigcup_n \mathbb{R}\mathrm{P}^n$. It is a cell complex with one cell in each dimension. We can think of $\mathbb{R}\mathrm{P}^\infty$ as the space of lines in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

- **Complex projective space** $\mathbb{C}\mathrm{P}^n$ is a cell complex.

  Let $(z_0 : \ldots : z_n)$ denote the homogeneous coordinates of a point in $\mathbb{C}\mathrm{P}^n$. Let $\varphi\colon D^{2n} \to \mathbb{C}\mathrm{P}^n$ be given by

$$(z_0, \ldots, z_{n-1}) \mapsto (z_0 : z_1 : \ldots : z_{n-1} : 1 - (\sum_{i=0}^{n-1} |z_i^2|)^{1/2}).$$

Then φ sends ∂D^{2n} to the points with $z_n = 0$, i.e., into $\mathbb{C}\mathrm{P}^{n-1}$.

Let f denote the restriction of φ to $S^{2n-1} = \partial D^{2n}$. Then φ **factors** through $D^{2n} \cup_f \mathbb{C}\mathrm{P}^{n-1}$, i.e., we get a commutatuve diagram with an induced dotted arrow

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{C}\mathrm{P}^{n-1} \arrow[d] \arrow[ddr, bend left] & \\
D^{2n} \arrow[r] \arrow[drr, bend right, "\varphi"'] & \mathbb{C}\mathrm{P}^{n-1} \cup_f D^{2n} \arrow[dr, dotted, "g"] & \\
 & & \mathbb{C}\mathrm{P}^n.
\end{tikzcd}$$

The induced map

$$g\colon D^{2n} \cup_f \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$$

Since we can rescale the nth coordinate, this map is **bijective**. Hence it is a continuous bijection defined on a **compact** space. We learned earlier that this implies that g is a **homeomorphism**.

We conclude that $\mathbb{C}\mathrm{P}^n$ is a cell complex with exactly **one i-cell in each even dimension** up to $2n$.

- Again we could continue this process and build **infinite complex projective space** $\mathbb{C}\mathrm{P}^\infty$ which is a cell complex with one i-cell in each even dimension.

Finally, we would like to have a **good notion of subspace** in a cell complex which respects the cell structure. It turns out that it is not sufficient to just require to have a subspace. Though not much more is actually required. For, a subspace $A \subset X$ is **subcomplex**, or sub-CW-complex, if it is **closed** and a **union of cells** of X.

These conditions imply that A is a cell complex on its own. For, since A is closed the characteristic maps of each cell of A has image in A and so does each attaching map. Hence the cells with their characteristic maps which lie in A provide A with a cell structure.

A more technical definition sounds like this:

Subcomplexes

Let X be a cell complex with attaching maps $\{f_\alpha \colon S_\alpha^{n-1} \to \mathrm{Sk}_{n-1}X : \alpha \in J_n, n \geq 0\}$.
A **subcomplex** A of X is a closed subspace $A \subseteq X$ such that for all $n \geq 0$, there is a **subset** $J_n' \subset J_n$ so that $\mathrm{Sk}_n A := A \cap \mathrm{Sk}_n X$ turns A into a cell complex with attaching maps $\{f_\beta : \beta \in J_n', n \geq 0\}$.
A pair (X,A) which consists of a CW-complex X and a **subcomplex** A is called a **CW-pair**.

Examples of CW-pairs are given by

- each skeleton $\mathrm{Sk}_n X$ of a cell complex X;
- $\mathbb{R}\mathrm{P}^k \subset \mathbb{R}\mathrm{P}^n$ for every $k \leq n$;
- $\mathbb{C}\mathrm{P}^k \subset \mathbb{C}\mathrm{P}^n$ for every $k \leq n$;
- the spheres $S^k \subset S^n$ for every $k \leq n$ **but only** for the **second cell structure** with two i-cells in each dimension.

With the **first cell structure on S^n** with one 0-cell and one n-cell, S^k is **not a subcomplex** of S^n.

The next step is to study the homology of cell complexes...