MA3403 Algebraic Topology
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Lecture 14

14. Homology of cell complexes

We are going to show that there is a relatively simple procedure to determine the homology of a cell complex.

Before we start this endeavour we need an auxiliary result which is a consequence of the excision property of singular homology:

Lemma: Homology after collapsing a subspace

Let $A \subset X$ be a subspace. Suppose there is another subspace B of X such that
(a) $\bar{A} \subseteq B^\circ$ and
(b) $A \hookrightarrow B$ is a **deformation retract**.
Then

$$H_n(X,A) \xrightarrow{\cong} H_n(X/A,*)$$

is an **isomorphism** for all n.

Proof: We have a **commutative diagram**

$$\begin{array}{ccccc} (X,A) & \xrightarrow{i} & (X,B) & \xleftarrow{j} & (X-A, B-A) \\ \downarrow & & \downarrow & & \downarrow k \\ (X/A,*) & \xrightarrow{\bar{i}} & (X/A,B/A) & \xleftarrow{\bar{j}} & (X/A-*, B/A-*). \end{array}$$

Our goal is to show that the left-hand vertical map induces an isomorphism in homology. We will achieve this by showing that all the other maps **induce isomorphisms in homology**:

- The **map** k is a **homeomorphism of pairs** and hence induces an isomorphism in homology.
- The **map** j induces an isomorphism in homology by the assumption **(a)** and **excision**.

- The **map** i induces a homomorphism of long exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & \cdots \\ & & \downarrow \cong & & \| & & \downarrow & & \\ \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,B) & \longrightarrow & \cdots \end{array}$$

  By **assumption (b)**, the left-hand vertical arrow is an isomorphism for all n. By the **Five-Lemma** this implies that i induces an isomorphism in homology.
- For the **map** $\bar{i}$, we observe that the retraction $\rho\colon B \to A \hookrightarrow B$ induces a map $\bar{\rho}\colon B/A \to A/A = * \hookrightarrow B/A$.

  Moreover, the homotopy $B \times I \to B$ between ρ and the identity of B is constant on A. Thus it induces a homotopy $B/A \times I \to B/A$ between $\bar{\rho}$ and the identity of B/A.

  In other words, $* \to B/A$ is a **deformation retract**. Hence the long exact sequence and the Five-Lemma imply that $\bar{i}$ induces an isomorphism in homlogy.
- Finally, we have $\bar{*} \subset (B/A)^\circ$ by definition of the quotient topology. Hence **map** $\bar{j}$ induces an isomorphism in homology by **excision**.

QED

Corollary: Homology of a bouquet of spheres

For any indexing set J, let us write $\bigvee_{\alpha\in J} S^k_\alpha$ for the quotient

$$\coprod_{\alpha\in J} S^{k-1}_\alpha \hookrightarrow \coprod_{\alpha\in J} D^k_\alpha \to \bigvee_{\alpha\in J} S^k_\alpha.$$

The homology of this space, often called bouquet of k-spheres, is given by

$$H_q(\bigvee_{\alpha\in J} S^k_\alpha, *) \cong \begin{cases} \mathbb{Z}[J] & \text{if } q = k \\ 0 & \text{if } q \neq k \end{cases}$$

where $\mathbb{Z}[J]$ denotes the free abelian group on the set J.

(Note that the relative homology group in the statement is an example of a reduced homology that we introduced in last week's exercises.)

Proof: Each summand S^{k-1}_α is a subspace of D^k_α for which there is an open neighborhood U_α such that $S^{k-1}_\alpha \hookrightarrow U_\alpha$ is a deformation retract (we could even

take $U_\alpha = D^n_\alpha - \{0\}$). Hence we can **apply the previous result** to conclude

$$H_*(\coprod_\alpha D^k_\alpha, \coprod_\alpha S^{k-1}_\alpha) \xrightarrow{\cong} H_*(\bigvee_\alpha S^k_\alpha, *).$$

Hence we reduced to calculate the relative homology on the left-hand side.

To do this, we can apply the **long exact sequence of a pair** to deduce that

$$\partial\colon H_q(\coprod_\alpha D^k_\alpha, \coprod_\alpha S^{k-1}_\alpha) \xrightarrow{\cong} H_{q-1}(\coprod_\alpha S^{k-1}_\alpha, *)$$

is an **isomorphism for all** q. Finally, we know that the latter group is isomorphic to $\bigoplus_{\alpha\in J}\mathbb{Z} = \mathbb{Z}[J]$ when $q = k$ and 0 otherwise. **QED**

Now we would like to apply this observation to a cell complex X. If we write $X_k = \mathrm{Sk}_k X$ for the **k-skeleton** of X, then we get the following commutative diagram

$$\begin{array}{ccccc} \coprod_\alpha S^{k-1}_\alpha & \hookrightarrow & \coprod_\alpha D^k_\alpha & \longrightarrow & \bigvee_\alpha S^k_\alpha \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & & \vdots{\scriptstyle \bar\varphi} \\ X_{k-1} & \hookrightarrow & X_k = X_{k-1} \cup_f (\coprod_\alpha D^k_\alpha) & \longrightarrow & X_k/X_{k-1}. \end{array} \tag{1}$$

where the right-hand vertical map is induced by φ and taking quotients. Since the **restriction** of φ to the **open interior** of the n-disks is a **homeomorphism** onto its image, this implies that the **dotted arrow $\bar\varphi$ is a homeomorphism**.

Hence we deduce from the previous result on bouquets of spheres:

$$H_q(X_k, X_{k-1}) \cong H_q(X_k/X_{k-1}, *) \cong \begin{cases} \mathbb{Z}[J_n] & \text{if } q = k \\ 0 & \text{if } q \neq k \end{cases}$$

where J_n denotes the indexing set of the attached k-cells.

In other words, the relative homology group $H_k(X_k, X_{k-1})$ **keeps track of the k-cells** of X.

This group will play a crucial role for us today. Let us analyze some consequences of what we have found out about this group.

Let us look at a piece of the long exact sequence of the pair (X_k, X_{k-1}):

$$H_{q+1}(X_k, X_{k-1}) \to H_q(X_{k-1}) \to H_q(X_k) \to H_q(X_k, X_k - 1).$$

For $q \neq k$, the last term $H_q(X_k, X_k - 1) = 0$ vanishes and hence the map

$$H_q(X_{k-1}) \to H_q(X_k) \text{ is surjective.}$$

For $q \neq k-1$, the first term $H_{q+1}(X_k, X_k - 1) = 0$ vanishes and hence the map

$$H_q(X_{k-1}) \to H_q(X_k) \text{ is injective.}$$

Hence we have shwon that the inclusion $X_{k-1} \hookrightarrow X_k$ induces an **isomorphism**

$$H_q(X_{k-1}) \xrightarrow{\cong} H_q(X_k) \text{ for } q \neq k, k-1. \tag{2}$$

Hence, **for a fixed** $q > 0$, we can observe how $H_q(X_k)$ varies when we **let** X_k **go through all skeleta** of X:

- $H_q(X_0) = 0$ since X_0 is a **discrete set** and the higher homology groups of points vanish.
- For $k = 0, \ldots, q-1$, $H_q(X_k) = 0$ remains trivial by (2).
- As a consequence, we observe that $H_n(X_k) = 0$ whenever $n > k$.
- **For** $k = q$, $H_q(X_q)$ is a subgroup of the free abelian group $H_q(X_q, X_{q-1})$, and therfore it is **free abelian** as well.
- **For** $k = q+1$, $H_q(X_{q+1})$ may not be free anymore, i.e., there might be **relations** induced by the exact sequence
$$H_{q+1}(X_{q+1}, X_q) \to H_q(X_q) \to H_q(X_{q+1}) \to 0.$$
- **For** $k \geq q+1$, $H_q(X_k)$ **remains stable**, i.e., the inclusions of skeleta induce a sequence of isomorphisms
$$H_q(X_{q+1}) \xrightarrow{\cong} H_q(X_{q+2}) \xrightarrow{\cong} \cdots .$$
- If X is **finite-dimensional**, there is a d such that $X = X_d$. The above sequence of isomorphisms then implies the inclusion $X_{q+1} \hookrightarrow X$ induces an **isomoprhism**
$$H_q(X_k) \cong H_q(X) \text{ for } q < k.$$
- Still, for X **finite-dimensional**, since $H_q(X_{q+1}) \xrightarrow{\cong} H_q(X)$ and since
$$H_q(X_q) \to H_q(X_{q+1}) \to H_q(X_{q+1}, X_q) = 0$$
is **exact**, we see that
$$H_q(X_q) \to H_q(X) \text{ is } \textbf{surjective}.$$

- If X is infinite-dimensional, the group $H_q(X_k)$ still maps **isomorphically into** $H_q(X)$ for $q < k$. For, the image of a standard simplex is **compact** and therefore lands in a **finite subcomplex**. Hence the union of the images of a finite collection of standard simplices is still **compact** and therefore also lands in a **finite subcomplex**. Hence it lands in a **finite skeleton**. Thus any q-chain in X is the image of a chain in a finite skeleton. For the same reason, if $c \in S_q(X)$ is a **boundary**, then it is a boundary in $S_q(X_m)$ for some $m \geq q$.

- In summary, all the q-dimensional homology of X is created in the q-skeleton X_q, and all the relations in $H_q(X)$ occur in the $q+1$-skeleton X_{q+1}.

The key points of this disussion are:

Proposition: The homology is governed by the skeleta

For any $k, q \geq 0$ and cell complex X, we have
- $H_q(X_k) = 0$ for $k < q$ and
- $H_q(X_k) \xrightarrow{\cong} H_q(X)$ for $k > q$.

In particular, $H_q(X) = 0$ if q is bigger than the dimension of the cell compelx X.

Now we would like to find an efficient way to calculate the homology of our cell complex X. Apparently, the group $H_n(X_n, X_{n-1})$ carries crucial information about X. Therefore, we are going to give it a new name:

Cellular n-chains

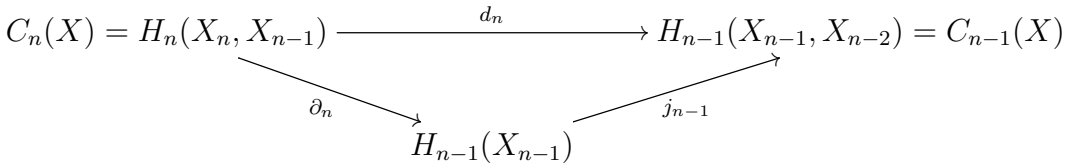
The group of **cellular n-chains** in a cell complex X is defined to be

$$C_n(X) := H_n(X_n, X_{n-1}).$$

We claim that these groups sit inside a sequence of homomorphisms who form a **chain complex**. The differential

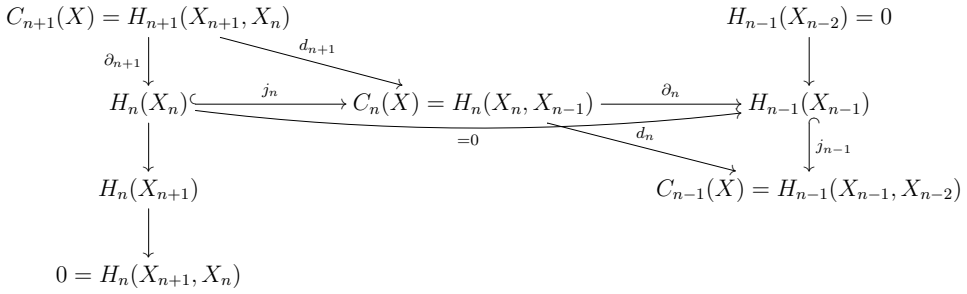
$$d_n \colon C_n(X) \to C_{n-1}(X)$$

is defined as the composite



where ∂_n is the connecting homomorphism in the long exact sequence of pairs and j_{n-1} is the homomorphism induced by the inclusion $(X_{n-1}, \emptyset) \hookrightarrow (X_{n-1}, X_{n-2})$.

To show that $d_n \circ d_{n+1} = 0$ we consider the commutative diagram:



Since j and ∂ are part of long exact sequences, we know $j \circ \partial = 0$ and get

$$d_n \circ d_{n+1} = (j_{n-1} \circ \partial_n) \circ (j_n \circ \partial_{n+1}) = 0.$$

Cellular chain complex

Thus $(C_*(X), d)$ is a chain complex. It is called the **cellular chain complex**.

Now we would like to determine the homology of this chain complex.

- To do this we need to understand the kernel of d:

$$\operatorname{Ker}(d_n) = \operatorname{Ker}(j_{n-1} \circ \partial_n).$$

Since j_{n-1} is **injective**, we get

$$\operatorname{Ker}(d_n) = \operatorname{Ker}(\partial_n) = \operatorname{Im}(j_n) = H_n(X_n)$$

where the middle identity is implied by the **exactness of the long exact sequence** these maps are part of, and the last identity is implied by the fact that $j_n \colon H_n(X_n) \to H_n(X_n, X_{n-1})$ is **injective**.

- For the image of d, we use again that j_n is injective and get

$$\mathrm{Im}\,(d_{n+1}) = j_n(\mathrm{Im}\,(\partial_{n+1})) \cong \mathrm{Im}\,(\partial_{n+1}) \subseteq H_n(X_n).$$

Since the left-hand column in the above big diagram is exact, we know

$$H_n(X_n)/\mathrm{Im}\,(\partial_{n+1}) \cong H_n(X_{n+1}).$$

In other words, we just proved:

$$H_n(C_*(X)) = H_n(X_n)/\mathrm{Im}\,(\partial_{n+1}) \cong H_n(X_{n+1}).$$

But we had already showed $H_n(X_{n+1}) \cong H_n(X)$. Hence we proved the following important result:

Theorem: Cellular Homology

For a **cell complex** X, there is an **isomorphism**

$$H_*(C_*(X)) \cong H_*(X)$$

which is **functorial** with respect to **filtration-perserving maps** between cell complexes.

In this theorem we are referring to maps which preserve the skeleton structure of cell complexes. We should better make this concept precise:

Maps between cell complexes

- A **filtration** on a space X is a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq X_{n+1} \subseteq \ldots \subseteq X.$$

such that X can be written as the union of these subspaces. If X is a space together with a filtration, we call X a **filtered space**.
- For example, every **cell complex** has a filtration by its **skeleta**.
- Let X and Y be filtered spaces. A continuous map $f\colon X \to Y$ is called **filtration-preserving** if $f(X_p) \subset Y_p$ for all p.
- A map between cell complexes is called **cellular** if it preserves the filtration by skeleta.

In other words, if we are given two cell complexes and care about their cell structure, we should only consider filtration-preserving maps between them.

An immediate and very useful consequence of the above theorem is:

Corollary: Homology of even cell complexes

Let X be a cell complex with only even cells, i.e., the inclusion $X_{2k} \hookrightarrow X_{2k+1}$ is an isomorphism for all k. Then

$$H_*(X) \cong C_*(X).$$

In particular, $H_n(X)$ is free ableian for all n, $H_n(X) = 0$ for odd n, and the rank of $H_n(X)$ for even n is the number of n-cells.

For **example**, recall that complex projective n-space $\mathbb{C}\mathrm{P}^n$ has exactly one cell in each even dimension up to $2n$. Hence as an application we can read off the **homology of complex projective space**:

$$H_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & \text{for } 0 \leq k \leq 2n \text{ and } k \text{ even} \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

Note to the theorem and corollary

We should keep in mind that the homology of X is independent of any cell structures. We defined it long before we knew that cell complexes exist.
The theorem shows that knowing a cell structure on X can nevertheless be very helpful for computing $H_*(X)$.
Moreover, we learned that the cell structure on any given cell complex may not be unique. We saw for example two different cell structures on S^n.
However, the theorem tells us that any cell strucure one can construct on X has to obey certain constraints what are induced by the fact the homology of the cellular chain complex is $H_*(X)$.