MA3403 Algebraic Topology
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Lecture 15

15. Computations of cell homologies and Euler characteristic

• Homology of real projective n-space

As another application of the theorem on the cellular chain complex and the homology of cell complexes we are going to compute the homology of real projective space.

First of all, recall that attaching and characteristic maps assemble to a commutative diagram

$$\begin{array}{ccc} (\coprod D^n, \coprod S^{n-1}) & \longrightarrow & (X_n, X_{n-1}) \\ & \searrow & \downarrow \\ & & (\bigvee S^n, *). \end{array}$$

We have shown that all these maps induce isomorphisms in homology. In particular,

$$H_n(\coprod D^n, \coprod S^{n-1}) \xrightarrow{\cong} C_n(X) = H_n(X_n, X_{n-1}). \tag{1}$$

We are now going to exploit this fact for the computation of $H_*(\mathbb{R}\mathrm{P}^n)$.

Recall that the cell structure of $\mathbb{R}\mathrm{P}^n$ is such that

- $\mathrm{Sk}_k(\mathbb{R}\mathrm{P}^n) = \mathbb{R}\mathrm{P}^k$ and
- there is exactly one k-cell in each dimension $k = 0, \ldots, n$. We denote this k-cell by e^k.

Hence the cellular chain complex looks like this:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_n(\mathbb{R}\mathrm{P}^n) & \xrightarrow{d_n} & \cdots & \xrightarrow{d_2} & C_1(\mathbb{R}\mathrm{P}^n) & \xrightarrow{d_1} & C_0(\mathbb{R}\mathrm{P}^n) & \longrightarrow & 0 \\ & & \| & & & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}[e^n] & \xrightarrow[d_n]{} & \cdots & \xrightarrow[d_2]{} & \mathbb{Z}[e^1] & \xrightarrow[d_1]{} & \mathbb{Z}[e^0] & \longrightarrow & 0 \end{array}$$

In order to compute the homology of this chain complex, we need to determine the differentials d_n:

- We know that $H_0(\mathbb{R}\mathrm{P}^n)$ is $\mathbb{Z} = \mathbb{Z}[e^0]$. That implies that the differential d_1 must be trivial.
- For $k > 1$, the differential d_k is defined as the top row in the following commutative diagram

$$\begin{array}{ccccccc} C_k(\mathbb{R}\mathrm{P}^n) = H_k(\mathbb{R}\mathrm{P}^k, \mathbb{R}\mathrm{P}^{k-1}) & \xrightarrow{\partial_k} & H_{k-1}(\mathbb{R}\mathrm{P}^{k-1}) & \xrightarrow{j_{n-1}} & H_{k-1}(\mathbb{R}\mathrm{P}^{k-1}, \mathbb{R}\mathrm{P}^{k-2}) = C_{k-1}(\mathbb{R}\mathrm{P}^n) \\ \cong\uparrow & & \uparrow{\scriptstyle H_{k-1}(\pi)} & & \cong\uparrow & \\ H_k(D^k, S^{k-1}) & \xrightarrow{\partial_k} & H_{k-1}(S^{k-1}) & \longrightarrow & H_{k-1}(S^{k-1}, *) & \end{array}$$

The map $\pi\colon S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$ is the **attaching map** of the k-cell in $\mathbb{R}\mathrm{P}^n$. The outer vertical maps are isomorphisms by our discussion of diagram (1). We also know that the lower differential ∂_k is an **isomorphism** by our original calculation of the homology of the sphere.

Hence, in order to understand the effect of the differential d_k, we need to uderstand the effect of the maps in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}\mathrm{P}^{k-1} & \xrightarrow{q} & \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} \\ \pi\uparrow & & \downarrow\approx \\ S^{k-1} & \xrightarrow[g]{} & S^{k-1} \end{array} \tag{2}$$

where q is the quotient map. In other words, we need to calculate the **degree of the lower horizonatal map** g.

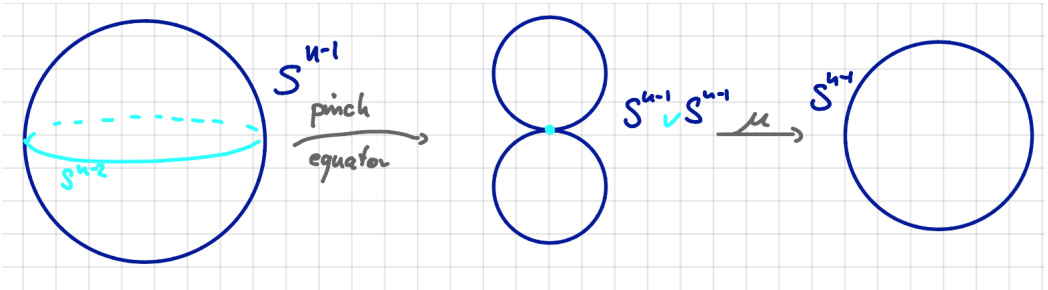
The composite

$$S^{k-1} \xrightarrow{\pi} \mathbb{R}\mathrm{P}^{k-1} \xrightarrow{q} \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2}$$

pinches the subspace $S^{k-2} \subset S^{k-1}$ **to a point**.

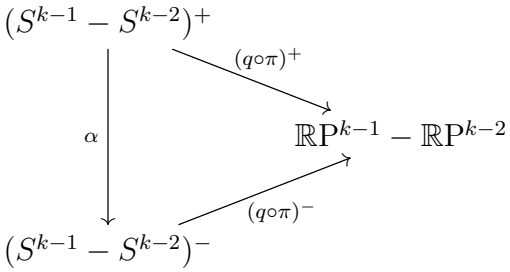
Hence the lower horizontal map g in (2) is given by

$$\begin{array}{ccccc} S^{k-1} & \xrightarrow{\pi} & \mathbb{R}\mathrm{P}^{k-1} & \xrightarrow{q} & S^{k-1} \\ & \searrow^{\text{pinch equator}} & & \nearrow_{\mu} & \\ & & S^{k-1}/S^{k-2} = S^{k-1}_- \vee S^{k-1}_+. & & \end{array}$$



To determine the effect of μ, we observe that the subspace $S^{k-1} - S^{k-2}$ consists of **two components**. Let us denote these two components by $(S^{k-1} - S^{k-2})^+$ and $(S^{k-1} - S^{k-2})^-$, respectively. The **restriction** of $q \circ \pi$ **to each component** of $S^{k-1} - S^{k-2}$ is a **homeomorphism** onto $\mathbb{R}\mathrm{P}^{k-1} - \mathbb{R}\mathrm{P}^{k-2}$.

Let us write $(q \circ \pi)^+$ and $(q \circ \pi)^-$ for the restrictions of $q \circ \pi$ to the subspaces $(S^{k-1} - S^{k-2})^+$ and $(S^{k-1} - S^{k-2})^-$, respectively:



By definition of $\mathbb{R}\mathrm{P}^{k-1}$, both $(q \circ \pi)^+$ and $(q \circ \pi)^-$ are homeomorphisms and they **differ by precomposing with the antipodal map**.

Hence the map μ is the **identity** on one copy of S^{k-1} and the **antipodal map** α on the other copy of S^{k-1}.

Thus the effect of g on homology is given by

$$H_{k-1}(g)\colon H_{k-1}(S^{k-1}) \to H_{k-1}(S^{k-1}),\ \sigma \mapsto \sigma + H_{k-1}(\alpha)(\sigma).$$

But we know what the effect of $H_{k-1}(\alpha)$ is. Namely, it is given by $H_{k-1}(\alpha) = (-1)^{k-1}$. Hence

$$H_{k-1}(g) = 1 + (-1)^k = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Summarizing, we have shown that the cellular chain complex of $\mathbb{R}\mathrm{P}^n$ looks like:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is even}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is odd}$$

where the left-hand copy of $\mathbb{Z}$ is in deimension n and the right-hand one is in dimension 0.

And in words: in real projective space, **odd cells create new generators**, whereas **even cells create torsion** (except for the zero-cell) in the previous dimension.

Homomology of $\mathbb{R}\mathrm{P}^n$

The homology of real projective n-space is given by

$$H_k(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n \text{ is odd} \\ \mathbb{Z}/2 & 0 < k < n \text{ and } k \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

- **What homology sees and does not see**

The example of $\mathbb{R}\mathrm{P}^n$ indicates what kind of structure of a cell complex singular homology can detect and what it cannot detect and also how we can calculate the differential in the cellular chain complex.

Let X be a cell complex. Its cell structure is determined by attaching maps

$$\coprod S_\alpha^{n-1} \to X_{n-1}.$$

Knowing thses maps, up to homotopy, determines the homotopy type of the cell complex X.

However, homology does not record all of the information of the attaching maps. For, homology only sees the effect of the composite obtained by pinching out X_{n-1}:

$$\begin{array}{ccccc} \coprod_{\alpha\in J_n} S_\alpha^{n-1} & \longrightarrow & X_{n-1} & \longrightarrow & X_{n-1}/X_{n-2} \\ & \searrow & \downarrow & \swarrow \approx & \\ & & \bigvee_{\beta\in J_{n-1}} S_\beta^{n-1}. & & \end{array}$$

In other words, **homology only records what is going on modulo subskeleta**. However, we will see now that homolgy does a pretty good job at this recording.

Let us try to understand this picture a bit better. For each α, the left-hand diagonal map can be described as the **composite**

$$S_\alpha^{n-1} \xrightarrow{f_\alpha} X_{n-1} \xrightarrow{q_{\alpha\beta}} X_{n-1}/(X_{n-1} - e_\beta^n) =: S_\beta^{n-1}$$

where $q_{\alpha\beta}$ is the quotient map. Moreover, we identify the quotient $X_{n-1}/(X_{n-1} - e^n_\beta)$ with the boundary S^{n-1}_β of the cell e^n_β. Note that this map might be trivial from some (or all) β.

The sum of the effect of these maps in homology is actually the differential d_n in the cellular chain complex. For we have a **commutative diagram**

$$\begin{array}{ccccc}
H_n(\bigvee_\alpha S^{n-1}_\alpha,*) & \xrightarrow{\partial_n} & H_{n-1}(\coprod_\alpha S^{n-1}_\alpha) & \xrightarrow{\sum_{\alpha,\beta} H_{n-1}(q_{\alpha\beta}\circ f_\alpha)} & H_n(\bigvee_\beta S^{n-1}_\beta,*) \\
\downarrow{\scriptstyle H_n(\bar{\varphi})}\cong & & \downarrow & & \downarrow\cong \\
H_n(X_n, X_{n-1}) & \xrightarrow[\partial_n]{} & H_{n-1}(X_{n-1}) & \xrightarrow[j_{n-1}]{} & H_{n-1}(X_{n-1}, X_{n-2}).
\end{array}$$

$$H_n(X_n, X_{n-1}) \xrightarrow{d_n} H_{n-1}(X_{n-1}, X_{n-2}).$$

We conclude from this discussion:

Cellular differentials are sums of degrees

With the above notations, the **effect of the cellular differential** on the generator in $C_n(X)$ which corresponds to the cell e^n_α in X is given by the sum of degrees

$$d_n([e^n_\alpha]) = \sum_\beta H_{n-1}(q_{\alpha\beta} \circ f_\alpha)([e^n_\alpha]) = \sum_\beta \deg(q_{\alpha\beta} \circ f_\alpha) \cdot [e^n_\alpha].$$

In other words, in order to compute the cellular differential we need to calculate the degrees of various maps.

- **Euler characteristic of cell complexes**

Euler characteristic of finite CW-complexes

Let X be a **finite** cell complex. Let c_k denote the number of k-cells in X. Then the **Euler characteristic** of X is defined to be the integer given by the finite sum

$$\chi(X) = \sum_k (-1)^k c_k.$$

The **main result** on $\chi(X)$ we are going to prove today is that it **only depends on the homotopy type** of X and is, in particular, **independent of the given cell structure** of X. We are going to prove this by showing that $\chi(X)$ can be computed using the singular homology of X.

Recall that we have seen an Euler number for polyhedra in the first lecture. It was defined as the number of vertices minus the number of edges plus the number of faces. This fits well with the above definition for a finite cell complex.

For, if we assume the invariance of χ for a moment, then we get $\chi(S^2) = 2$ using the **standard cell structure** on S^2, i.e., one 0-cell and one 2-cell. This implies that Euler's polyhedra formula holds.

Corollary: Euler's polyhedra formula

For any cell structure on the 2-sphere S^2 with F 2-cells, E 1-cells and V 0-cells, we have the formula

$$F - E + V = 2.$$

As a preparation, we recall some facts about abelian groups.

Let A be an abelian group. Recall that the set of **torsion elements** is defined as

$$\text{Torsion}(A) = \{a \in A : na = 0 \text{ for some } n \neq 0\}.$$

This set is in fact a subgroup of A. A group is called **torsion-free** if $\text{Torsion}(A) = 0$. The quotient $A/\text{Torsion}(A)$ is always torsion-free.

Now we assume that A is **finitely generated**. Then $\text{Torsion}(A)$ is a finite abelian group and $A/\text{Torsion}(A)$ is a finitely generated free abelian group and therefore isomorphic to $\mathbb{Z}^r$ for some integer r. The number r is called the **rank** of A denoted by $\text{rank}(A)$.

In fact, by choosing generators of $A/\text{Torsion}(A)$, we can construct a homomorphism $A/\text{Torsion}(A) \to A$ which splits the projection map $A \to A/\text{Torsion}(A)$. Thus if A is finitely generated abelian, then

$$A \cong \text{Torsion}(A) \oplus \mathbb{Z}^r.$$

We are going to use the following lemma from elementary algebra without proving it:

Lemma: Ranks in exact sequences

- Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated abelian groups. Then the ranks of these groups satisfy

$$\text{rank}(B) = \text{rank}(A) + \text{rank}(C).$$

- More generally, for a long exact sequence of finitely generated abelian groups

$$0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to 0$$

the ranks satisfy

$$0 = \sum_{i=0}^{n} (-1)^i \text{rank}(A_i).$$

Now we are euqipped for the proof of the above mentioned result:

Theorem: Euler characteristic via homology

Let X be a **finite** cell complex. Then the Euler characteristic of X satisfies

$$\chi(X) = \sum_{k} (-1)^k \text{rank}(H_k(X)).$$

Proof: Let c_k be again the number of k-cells in the given finite cell structure of X. Let $C_* := C_*(X)$ denote the **cellular chain complex** of X. To simplify the notation let us denote by Z_*, B_*, and H_* the cycles, boundaries and homology, respectively, in this complex.

By their definition, they fit into **two short exact sequences**:

$$0 \to Z_k \to C_k \to B_{k-1} \to 0$$

and

$$0 \to B_k \to Z_k \to H_k \to 0,$$

By our previous study of the cellular chain complex, we know

$$c_k = \text{rank}(C_k).$$

Hence, using the above discussion, we can rewrite $\chi(X)$ as follows:

$$\begin{aligned}\chi(X) &= \sum_k (-1)^k \text{rank}(C_k) \\ &= \sum_k (-1)^k (\text{rank}(Z_k) + \text{rank}(B_{k-1})) \\ &= \sum_k (-1)^k (\text{rank}(B_k) + \text{rank}(H_k) + \text{rank}(B_{k-1})).\end{aligned}$$

When we take the sum over all k, **the summands** $\text{rank}(B_k)$ **and** $\text{rank}(B_{k-1})$ **will cancel out**. Thus we get

$$\chi(X) = \sum_k (-1)^k \text{rank}(H_k).$$

But by the theorem on the homology of the cellular chain complex, H_k is exactly the singular homology group $H_k(X)$ of X. **QED**

Note that the numbers $\text{rank}(H_k(X)$ are called the **Betti numbers** of X. They had already played an important role in mathematics, before homology groups had been systematically developed. As the theorem shows, these numbers are an interesting invariant of a space.

The description of the Euler number in the theorem now generalizes easily:

Definiton: Euler characteristic revisited

Let X be a topological space such that each $H_n(X)$ has finite rank and that there is an d such that $H_n(X) = 0$ for all $n > d$. Then the **Euler characteristic** of X is defined to be the integer given by the finite sum

$$\chi(X) = \sum_k (-1)^k \text{rank}(H_k(X)).$$

- **Designing cell complexes**

For example, for $m \in \mathbb{Z}$, we can easily construct a space X with $H_n(X) = \mathbb{Z}/m$ and $\tilde{H}_i(X) = 0$ for $i \neq n$. We start with S^n and attach an $n+1$-cell to it via a map $f\colon S^n \to S^n$ of degree m. The cellular chain complex of this space is

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$$

with the copies of $\mathbb{Z}$ in dimensions $n+1$ and n, respectively. The homology of this space is exactly what we wanted.

This procedure can easily be generalized.

Theorem: Moore spaces

Let A_* be any graded abelian group with $A_n = 0$ for $n < 0$. Then there exists a cell complex X with $\tilde{H}_*(X) = A_*$.

We are going to prove this result in the next lecture.