MA3403 Algebraic Topology
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Lecture 16

16. Designing homology groups and homology with coefficients

• Designing cell complexes

We announced last time that cell complexes enable us to design spaces with prescribed homology groups. We are going to prove this result today.

We will need a construction on spaces that we have already used in special cases.

Recall that the **wedge** $X \vee Y$ of two **pointed spaces** (X,x) and (Y,y) is defined as the quotient of $X \sqcup Y$ modulo $x \sim y$, i.e., the disjoint union with x and y identified. We can think of $X \vee Y$ glued together at the joint point $[x] = [y]$. This generalizes the wedge of spheres that we have seen before. This construction generalizes to **infinite wedges**.

If each point x_α is a deformation retract of a neighborhood U_α in X_α, then the wedge satisfies a formula for **reduced homology** that we are used to for the homology of disjoint unions:

$$\tilde{H}_*(\bigvee_\alpha X_\alpha) \cong \bigoplus_\alpha \tilde{H}_*(X_\alpha).$$

Now we can prove the following result:

Theorem: Moore spaces

Let A_* be any graded abelian group with $A_n = 0$ for $n < 0$. Then there exists a cell complex X with $\tilde{H}_*(X) = A_*$.

Proof: Let us start with just a single abelian group A. By choosing generators for A, we can define a surjective homomorphism

$$F_0 \to A$$

from a **free abelian group** F_0. The **kernel** of this homomorphism, denoted by F_1, is also **free**, since it is a subgroup of a free abelian group.

We write J_0 for a minimal set of generators of F_0 such that we have a surjection $F_0 \to A$ and J_1 for a minimal set of generators of F_1.

For $n \geq 1$, we define X_n to be

$$X_n := \bigvee_{\alpha \in J_0} S^n_\alpha.$$

The nth homology of X_n is $H_n(X_n) = \mathbb{Z}[J_0]$.

Now we are going to define an attaching map

$$f\colon \coprod_{\beta \in J_1} S^n_\beta \to X_n$$

by specifying it on each summand S^n_β.

In F_0, we can write the generator β of F_1 as a linear combination of the generators of F_0

$$\beta = \sum_{i=1}^{s} n_i \alpha_i.$$

We can reproduce this relation in topology. For, let

$$S^n \to \bigvee_{i=1}^{s} S^n_{\alpha_i}$$

be the map obtained by pinching $s-1$ circles on S^n to points. The effect of this map in homology is to send the generator in $H_n(S^n)$ to the s-tuple of generators in $H_n(S^n_{\alpha_i})$:

$$H_n(S^n) \to H_n(\bigvee_{i=1}^{s} S^n_{\alpha_i}) = \bigoplus_{i=1}^{s} H_n(S^n_{\alpha_i}),\ 1 \mapsto (1, \ldots, 1).$$

For each i, we choose a map $S^n_{\alpha_i} \to S^n_{\alpha_i}$ of **degree** n_i.

The map on the summand S^n_β is now defined as the composite

$$S^n_\beta \to \bigvee_{i=1}^{s} S^n_{\alpha_i} \to \bigvee_{\alpha} S^n_\alpha.$$

Taking the disjoint union of all these maps as attaching maps, we get a cell complex X whose cellular chain complex looks like

$$0 \to F_1 \to F_0 \to 0$$

with F_0 in dimension n and F_1 in dimension $n+1$, and whose homology is

$$\tilde{H}_q(X) = \begin{cases} A & \text{for } q = n \\ 0 & \text{for } q \neq n. \end{cases}$$

We write $M(A,n)$ for the CW-complex produced this way and call it a **Moore space** of type A and n.

Finally, for a **graded abelian group** A_* as in the theorem, we define X to be the **wedge** of all the $M(A_n,n)$. **QED**

Moore spaces are not functorial

It is important to note that the construction of Moore spaces **cannot** be turned into a **functor Ab $\to$ hoTop**. This might surprise at first glance. For given a homomorphism $g\colon A \to B$ we can construct a continuous map $\gamma\colon M(A,n) \to M(B,n)$ such that $H_n(\gamma) = g$.
However, this construction **depends on the various choices** we make. That means that for homomorphisms

$$A \xrightarrow{g_1} B, \text{ and } B \xrightarrow{g_2} C$$

we cannot guarantee that $\gamma_2 \circ \gamma_1$ is the same map as the one we would have constructed by starting with $g_2 \circ g_1\colon A \to C$ directly.

Despite this caveat, we have witnessed an important pheonmenon that still motivates a l ot of exciting research:

From Algebra to Topology

The proof demonstrates a common phenomenon in Algebraic Topology. Whereas our initial goal was to to translate topology into algebra, now we went in the opposite direction. Starting with an algebraic structure **we modeled a space whose homology reproduces the algebra**. Since being an abelian group is not the only the algebraic structure and homology not the only invariant out there, we can imagine that the above theorem is only a first glance at the makings of a **huge mathematical industry**.
The proof also shows why **Topology is particularly well suited** for this endavour. The gluing construction we used for **building cell complexes is unique for topological spaces**. Requiring any additional structure usually stops us from producing cell complexes.

For example, **there are no cell complexes of smooth manifolds or algebraic varieties**. Nevertheless, there are some inventive procedures to remedy this defect...

• Homology with coefficients

Now it is time to move on and to develop new algebraic invariants which add to the information we get from singular homology, or possibly simplify computations.

Recall that homology produces abelian groups. As nice as abelian groups are, it would be good to have additional structure, for example as vector spaces over a field. So one might wonder if there is a **version of singular homology** with values in the **category of vector spaces** over a field, or more generally the **category of modules** over a ring.

Actually, if R is a ring (with unit and commutative), there is an obvious candidate for such a theory: We define

$$S_n(X;R) := R\mathrm{Sing}_n(X)$$

to be the free R-module over the set $\mathrm{Sing}_n(X)$ of n-simplices. What we have done so far, was the special case $R = \mathbb{Z}$.

Now we can use the face maps and the same formula we had before for defining a boundary operator

$$\partial_n \colon S_n(X;R) \to S_{n-1}(X;R),\ \sum_j r_j\sigma_j \mapsto \sum_j\sum_i (-1)^i r_j(\sigma_j \circ \phi_i^n). \tag{1}$$

which is now a homomorphism of R-modules. The same calculations as before yield $\partial \circ \partial = 0$.

Now we can form the **homology** as usual

$$H_n(X;R) := \frac{\mathrm{Ker}\,(\partial_n \colon S_n(X;R) \to S_{n-1}(X;R))}{\mathrm{Im}\,(\partial_{n+1} \colon S_{n+1}(X;R) \to S_n(X;R))}.$$

For each $n \geq 0$, $H_n(X;R)$ is an R-module and is called the **singular homology of X with coefficients in R**.

More generally, if M is **any abelian group**, we can form the tensor product

$$S_n(X;M) = S_n(X) \otimes_{\mathbb{Z}} M = \bigoplus_{\sigma \in \mathrm{Sing}_n(X)} M$$
$$= \left\{ \sum_j m_j \sigma_j : \sigma_j \in \mathrm{Sing}_n(X), m_j \in M \right\}.$$

The boundary operator is defined as before by

$$\partial_n^M \colon S_n(X;M) \to S_{n-1}(X;M),\ \partial_n^M = \partial_n \otimes 1.$$

More explicitly, ∂_n^M is given by the formula in (1) with r_js replaced with m_js. Since $\partial \circ \partial = 0$, we get $\partial^M \circ \partial^M = 0$.

Homoglogy with coefficients

For a pair of spaces X,A), we define **singular homology of (X,A) with coefficients in M** $H_n(X,A;M)$ to be the nth homology of the chain complex

$$S_*(X,A;M) := \frac{S_*(X;M)}{S_*(A;M)}.$$

Homology with coefficients is **functorial**: That is, a map of pairs

$$f \colon (X,A) \to (Y,B)$$

induces, by composing simplices with f, a homomorphism

$$f_* \colon H_n(X,A;M) \to H_n(Y,B;M) \text{ for all } n \geq 0$$

which we denote just by f_* to keep the notation simple. Moreover, we have $(g \circ f)_* = g_* \circ f_*$.

Note if $M = R$ is a ring, this is the same definition as above, and for $M = \mathbb{Z}$ we recover $H_n(X,A;\mathbb{Z}) = H_n(X,A)$. We will often refer to these groups as **integral homoglogy groups**.

If M is an R-module, then the groups $H_n(X,A;M)$ have the additional structure as an R-module itself.

Eilenberg-Steenrod Axioms are satisfied

Singular homology with coefficients in M satisfies the **Eilenberg-Steenrod axioms** with the only **modification**

$$H_n(\mathrm{pt}; M) = \begin{cases} M & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

Since everything we proved for singular homology was based on these properties, we can **transfer basically all our work to homology with coefficients**. Let us point out **two crucial facts**:

- The calculations for **spheres** can be transfered and we get

$$\tilde{H}_k(S^n; M) = \begin{cases} M & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

- If X is a **cell complex**, there is a ceullular chain complex

$$C_*(X; M) \text{ with } C_n(X; M) = \bigoplus_{e^n_\alpha} M$$

  where the sum is taken over the n-cells of X. As for $M = \mathbb{Z}$, the nth homology of $C_*(X; M)$ is isomorphic to $H_n(X; M)$.

The **reduced homology** groups $\tilde{H}_n(X; M)$ with coefficients in M are defined as the homoogy groups of the augmented chain complex

$$\ldots \to S_1(X; M) \to S_0(X; M) \xrightarrow{\epsilon} M \to 0$$

where ϵ is the homomorphism which sends $\sum_j m_j \sigma_j$ to $\sum_j m_j \in M$.

For a **homomorphism of groups** $\varphi\colon M \to N$ there is an induced morphism of chain complexes $S_*(X,A; M) \to S_*(X,A; N)$ which induces a homomorphism in homology

$$\varphi_*\colon H_*(X,A; M) \to H_*(X,A; N).$$

This homomorphism is compatible with f_* for maps of pairs and with long exact sequences of pairs.

For the calculations using the cellular chain complex, we need to check the following lemma:

Lemma: Degrees with coefficients

Let $f\colon S^n \to S^n$ be a map of **degree** k. Then $f_*\colon \tilde{H}_n(S^n;M) \to \tilde{H}_n(S^n;M)$ is given by **multiplication with** k, where k denotes the image of k in M.

Proof: Let $\varphi\colon \mathbb{Z} \to M$ be a homomorphism of groups ($0_{\mathbb{Z}} \mapsto 0_M$) which sends $1 \in \mathbb{Z}$ to an element $m \in M$. Then the assertion follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & = & H_n(S^n;\mathbb{Z}) & \xrightarrow{f_*} & H_n(S^n;\mathbb{Z}) & = & \mathbb{Z} \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi} \\
M & = & H_n(S^n;M) & \xrightarrow[f_*]{} & H_n(S^n;M) & = & M.
\end{array}$$

That the outer diagram commutes follows from the way we compute the homology groups of S^n with coefficients $\mathbb{Z}$ and M via the Eilenberg-Steenrod axioms. **QED**

- **Why coefficients?**

The coefficients that are most often used are the **fields** $\mathbb{F}_p$, for a prime number p, and the field $\mathbb{Q}$, $\mathbb{R}$ and sometimes $\mathbb{C}$.

In order to get an idea of what happens when we use different coefficients, let us look at the homology of $\mathbb{R}\mathrm{P}^n$ for $R = \mathbb{F}_2$. We use the cellular chain complex which looks like this

$$0 \to \mathbb{F}_2 \to \mathbb{F}_2 \to \ldots \to \mathbb{F}_2 \to \mathbb{F}_2 \to 0.$$

We showed that the differentials alternated between multiplication by 2 and 0. But in $\mathbb{F}_2$, $2 = 0$ which means that all differentials vanish and we get

$$H_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{for } 0 \le k \le n \\ 0 & \text{otherwise.} \end{cases}$$

We learn from this example that

- The calculation of $\mathbb{F}_2$-homology can be particularly easy, and it might see more nontrivial groups than integral homology.
- Nevertheless, $\mathbb{F}_2$-homology is often sufficient to distinguish between trivial and nontrivial spaces or maps.

The situation is quite different if we take R to be any field of **characteristic different from two**. Then the cellular chain complex of $\mathbb{R}\mathrm{P}^n$ looks like (with the left-hand copy of R in dimension n)

$$0 \to R \xrightarrow{\cong} R \xrightarrow{0} R \xrightarrow{\cong} \dots \xrightarrow{\cong} R \xrightarrow{0} R \to 0$$

for n even, and

$$0 \to R \xrightarrow{0} R \xrightarrow{\cong} R \xrightarrow{0} \dots \xrightarrow{\cong} R \xrightarrow{0} R \to 0$$

for n odd.

Thus, for n even, we get

$$H_k(\mathbb{R}\mathrm{P}^n; R) = \begin{cases} R & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and, for n odd,

$$H_k(\mathbb{R}\mathrm{P}^n; R) = \begin{cases} R & \text{for } k = 0 \\ R & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

In other words, **away from** 2, real projective n-space looks for R-homology like a point if n is even and like an n-sphere if n is odd.

This teaches us already that **different coefficients can tell quite different stories**.

This notwithstanding one might wonder whether integral homology is the finest invariant and all other homologies are just coarser variations. This is not the case, and it is indeed **possible that homology with coefficients detects more than integral homology**. Let us look at an example:

Example: When $\mathbb{Z}/m$-homology sees more

Let $X = M(\mathbb{Z}/m,n)$ be a Moore space we constructed in the previous lecture: We start with an n-sphere S^n and form X be attaching an $n+1$-dimensional cell to it via a map $f\colon S^n \to S^n$ of degree m

$$X = S^n \cup_f D^{n+1}.$$

Let

$$q\colon X \to X/S^n \approx S^{n+1}$$

be the quotient map. **It induces a trivial homomorphism** in reduced integral homology. For, the only nontrivial homology occurs in degrees n and $n+1$ where we have

$$\tilde{H}_{n+1}(X;\mathbb{Z}) = 0 \xrightarrow{q_*} \tilde{H}_{n+1}(X/S^n;\mathbb{Z})$$

and

$$\tilde{H}_n(X;\mathbb{Z}) \xrightarrow{q_*} \tilde{H}_{n+1}(X/S^n;\mathbb{Z}) = \tilde{H}_n(S^{n+1};\mathbb{Z}) = 0.$$

Hence **integral homology cannot distinguish** between the quotient map and a constant map.
However, **$\mathbb{Z}/m$-homology does see the difference between q and a constant map**.
For, the $\mathbb{Z}/m$-cellular chain complex of X is

$$0 \to \mathbb{Z}/m \xrightarrow{m} \mathbb{Z}/m \to 0$$

with copies of $\mathbb{Z}/m$ in dimensions $n+1$ and n. Thus, the $\mathbb{Z}/m$-homology of X is

$$\tilde{H}_k(X;\mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & \text{if } k = n+1 \\ \mathbb{Z}/m & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

The long exact sequence of the pair (X,S^n) in dimension $n+1$ then yields

$$0 = \tilde{H}_{n+1}(S^n;\mathbb{Z}/m) \to \tilde{H}_{n+1}(X;\mathbb{Z}/m) \xrightarrow{q_*} \tilde{H}_{n+1}(X/S^n;\mathbb{Z}/m).$$

Since the left-hand group is 0, exactness implies that q_* **is injective and hence nontrivial**, since both $\tilde{H}_{n+1}(X;\mathbb{Z}/m)$ and $\tilde{H}_{n+1}(X/S^n;\mathbb{Z}/m)$ are isomorphic to $\mathbb{Z}/m$.

This example demonstrates that homology groups with coefficients are similar, but often a bit different than integral homology groups. This raises the question how different they can be. More generally, we could ask:

Question

Given an R-module M $H_*(X;R)$, what can we deduce about $H_*(X;M)$?

For **example**, let M be the $\mathbb{Z}$-module $\mathbb{Z}/m$. One might wonder if $H_n(X;\mathbb{Z}/m)$ is just the quotient $H_n(X;\mathbb{Z})/mH_n(X;\mathbb{Z})$, since the latter is isomorphic to the tensor product $H_n(X;\mathbb{Z}) \otimes \mathbb{Z}/m$.

But we have to be **careful**. For, we do have a short exact sequence of chain complexes

$$0 \to S_*(X;\mathbb{Z}) \xrightarrow{m} S_*(X;\mathbb{Z}) \to S_*(X;\mathbb{Z}/m) \to 0.$$

Such a short exact sequence induces a long exact sequence of the respective homology groups a part of which looks like

$$H_n(X;\mathbb{Z}) \xrightarrow{m} H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Z}/m) \to H_{n-1}(X;\mathbb{Z}) \xrightarrow{m} H_{n-1}(X;\mathbb{Z}).$$

Using the exactness of this sequence yields a **short exact sequence**

$$(2)\quad 0 \to H_n(X;\mathbb{Z})/mH_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Z}/m) \to \text{m-Torsion}(H_{n-1}(X;\mathbb{Z})) \to 0$$

where m-Torsion$(H_{n-1}(X;\mathbb{Z}))$ denotes the m-torsion, i.e., the kernel of the map $H_{n-1}(X;\mathbb{Z}) \xrightarrow{m} H_{n-1}(X;\mathbb{Z})$ given by multiplication by m.

In fact, the short exact sequence (2) provides a tool to determine $H_n(X;\mathbb{Z}/m)$ when we know both $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$. **However**, in gerneral, we will **need a more sophisticated method** to understand the relationship of $H_*(X;\mathbb{Z}) \otimes M$ and $H_*(X;M)$.

As a first generalization, we have the following result:

Long exact sequence of coefficients

Assume we have a short exact sequence of abelian groups

$$0 \to M' \to M \to M'' \to 0.$$

For any pair of spaces (X,A), there is an induced short exact sequence of chain complexes

$$0 \to S_*(X,A;M') \to S_*(X,A;M) \to S_*(X,A;M'') \to 0.$$

Such a short exact sequence induces a **long exact sequence**

$$\begin{array}{ccccc} & & \cdots & \longrightarrow & H_{n+1}(X,A;M'') \\ & \overset{\partial}{\swarrow} & & & \\ H_n(X,A;M') & \longrightarrow & H_n(X,A;M) & \longrightarrow & H_n(X,A;M'') \\ & \overset{\partial}{\swarrow} & & & \\ H_{n-1}(X,A;M') & \longrightarrow & \cdots & & \end{array}$$