MA3403 Algebraic Topology
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Lecture 17

17. Tensor products, Tor and the Universal Coefficient Theorem

Our goal for this lecture is to prove the **Universal Coefficient Theorem** for singular homology with coefficients. This will require some preparations in homological algebra. For some this will be a review. Though to keep everybody on board, this is what we have to do.

We will not treat the most general cases, but rather focus on the main ideas. Any text book in homological algebra will provide more general results.

- **Tensor products**

Let A and B be abelian groups. We would like to combine A and B into just one object, denoted $A \otimes B$, such a way that having a bilinear homomorphism

$$f \colon A \times B \to C$$

is the same as having a homomorphism from $A \otimes B$ into C.

That f is bilinear means

$$f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$$
$$f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2).$$

We can achieve this by brute force.

Tensor product

For, we can construct $A \otimes B$ as the quotient of the free abelian group generated by the set $A \times B$ modulo the subgroup generated by $\{(a+a', b) - (a, b) - (a', b)\}$ and $\{(a, b+b') - (a, b) - (a, b')\}$ for all $a, a' \in A$ and $b, b' \in B$. We denote the equivalence class of (a, b) in $A \otimes B$ by $a \otimes b$. We call $A \otimes B$ the **tensor product** of A and B.

Let us collect some immediate **observations**:

- For any $a \in A$, $b \in B$ and any integer $n \in \mathbb{Z}$, the relations imply

$$n(a \otimes b) = (na) \otimes b = a \otimes (nb).$$

- The abelian group $A \otimes B$ is generated by elements $a \otimes b$ with $a \in A$ and $b \in B$.
- Elements in the abelian group $A \otimes B$ are finite sums of equivalence classes $\sum_{i=1}^{m} n_i(a_i \otimes b_i)$.
- The tensor product is symmetric up to isomorphism with isomorphism given by

$$A \otimes B \xrightarrow{\cong} B \otimes A, \ \sum_{i=1}^{m} n_i a_i \otimes b_i \mapsto \sum_{i=1}^{m} n_i b_i \otimes a_i.$$

- The tensor product is associative up to isomorphism:

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C.$$

- For homomorphisms $f\colon A \to A'$ and $g\colon B \to B'$, there is an induced homomorphism

$$f \otimes g\colon A \otimes B \to A' \otimes B', \ (f \otimes g)(a \otimes b) = f(a) \otimes g(b).$$

- The tensor product has the desired universal property:

$$\mathrm{Hom}_{\mathrm{bilinear}}(A \times B, C) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Ab}}(A \otimes B, C),$$

i.e., if we have a bilinear map $A \times B \to C$, then there is a unique (up to isomorphism) dotted map which makes the diagram commutative

$$\begin{tikzcd} A \times B \arrow[r, "q"] \arrow[dr] & A \otimes B \arrow[d, dotted] \\ & C. \end{tikzcd}$$

- The universal property of the tensor product implies that we have an isomorphism

$$(\bigoplus_{\alpha} A_\alpha) \otimes B \cong \bigoplus_{\alpha} (A_\alpha \otimes B).$$

Now it is time to see **some examples**:

- For every abelian group A, we have isomorphisms

$$A \otimes \mathbb{Z} \cong A \cong \mathbb{Z} \otimes A$$

which sends $a \otimes n \mapsto na$ and inverse $a \mapsto a \otimes 1$.
- For every abelian group A and every m, we have an isomorphism

$$A \otimes \mathbb{Z}/m \cong A/mA, \ a \otimes [n] \mapsto [an] \in A/mA.$$

- If M is an **abelian group** and X a space, we can form the tensor product

$$S_n(X;M) := S_n(X) \otimes M \cong \bigoplus_{\sigma \in \mathrm{Sing}_n(X)} M$$
$$= \left\{ \sum_j m_j \sigma_j : \sigma_j \in \mathrm{Sing}_n(X), m_j \in M \right\}.$$

There is a boundary operator defined by

$$\partial_n^M \colon S_n(X;M) \to S_{n-1}(X;M),\ \partial_n^M = \partial_n \otimes 1.$$

This turns $S_*(X;M)$ into a chain complex. The homology of this complex is the homology of X with coefficients in M.

Tensor products are great. Except for the following:

- **Tor functor**

Suppose we have an abelian group M and a surjective homomorphism of abelian groups

$$B \twoheadrightarrow C.$$

Then we can check by looking at the generators that

$$B \otimes M \twoheadrightarrow C \otimes M$$

is also surjective.

More generally, we can show that tensor products preserve cokernels:

Lemma: Tensor products preserve cokernels

Let M be an abelian group. Suppose we have an exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0.$$

Then taking the tensor product $- \otimes M$ yields an exact sequence

$$A \otimes M \xrightarrow{i \otimes 1} B \otimes M \xrightarrow{j \otimes 1} C \otimes M \to 0$$

where 1 denotes the identity map on M. In other words, the functor $- \otimes M$ is **right exact** and **preserves cokernels**.

Proof: We are going to show that $- \otimes M$ preserves cokernels. This is in fact equivalent to the other statements.

Let $f\colon B \otimes C \to Q$ be a homomorphism. We need to show that there is a unique factorization as indicated by the dotted arrow in the diagram

$$\begin{tikzcd} A \otimes M \arrow[r, "i\otimes 1"] \arrow[dr, "0"'] & B \otimes M \arrow[r, "j\otimes 1"] \arrow[d, "f"] & C \otimes M \arrow[r] \arrow[dl, dotted] & 0 \\ & Q. & & \end{tikzcd}$$

By the universal property and the fact that $C \times M$ generates $C \otimes M$, this is equivalent to a unique factorization of the diagram of **bilinear maps**

$$\begin{tikzcd} A \times M \arrow[r, "i\times 1"] \arrow[dr, "0"'] & B \times M \arrow[r, "j\times 1"] \arrow[d, "F"] & C \times M \arrow[r] \arrow[dl, dotted] & 0 \\ & Q. & & \end{tikzcd}$$

But now we only need to find an approproate extension $C \to Q$ the existence of which is implied by assumption. **QED**

However, suppose we have an injective homomorphism

$$A \hookrightarrow B.$$

Then it is in general **not the case** that

$$A \otimes M \to B \otimes M$$

is **injective**.

For example, take the map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ given by multiplication by 2. It is clearly **injective**. But if we tensor with $\mathbb{Z}/2$, we get the map

$$\mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2$$

which is **not injective**.

Thus, **tensor products do not preserve exact sequences**, in general.

We would like to remedy this defect. And, in fact, the tensor product is **not so far from being exact**. For, if M is a **free** abelian group, then the functor $M \otimes -$ is **exact**, i.e., it preserves all exact sequences.

We can see this as follows: Assume M is the free abelian group on the set S. Then $M \otimes N = \oplus_S N$, since tensoring distributes over direct sums, as we remarked above.

To exploit this fact we make use of the following observation:

Lemma: Direct sums of exact sequences

If $M_i' \to M_i \to M_i''$ is exact for every $i \in I$, then

$$\bigoplus_i M_i' \to \bigoplus_i M_i \to \bigoplus_i M_i''$$

is exact.

Proof: The composition is zero and if $(x_i)_i$ is sent to 0 in $\bigoplus_i M_i''$, then each x_i must be sent to 0 in M_i''. Hence each x_i comes from some x_i', and hence $(x_i)_i$ comes from $(x_i')_i$. We just need to remember to choose $x_i' = 0$ whenever $x_i = 0$.
QED

Now let A be any abelian group. As we did in the previous lecture, we **choose a free abelian group F_0** mapping surjectively onto A

$$F_0 \twoheadrightarrow A.$$

The **kernel** F_1 of this map is also **free** abelian as a subgroup of a free abelian group. Hence we get an **exact sequence** of the form

$$0 \to F_1 \hookrightarrow F_0 \twoheadrightarrow A.$$

Free resolutions

Such an exact sequence with F_1 and F_0 free abelian groups, is called a **free resolution of A** of length two.

(Note that the fact that we can always choose such a free resolution of length two is particular to the case of abelian groups, i.e., $\mathbb{Z}$-modules. For R-modules over other rings, one might only be able to find projective resolutions of higher length. The fact that $\mathbb{Z}$ is a principal ideal domain, a PID, does the trick.)

For any abelian group M, tensoring these maps with M yields an exact sequence

$$F_1 \otimes M \to F_0 \otimes M \to A \otimes M \to 0.$$

The **kernel** of the left-hand map **is not necessarily zero**, though.

This leads to the following important definition:

Definition: Tor

The **kernel** of the map $A \otimes F_1 \to A \otimes F_0$ is called $\mathrm{Tor}(A,M)$. Hence by definition we have an **exact sequence**

$$0 \to \mathrm{Tor}(A,M) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes M \to 0.$$

This group measures how far $- \otimes M$ is from being exact.

Note that if we replace abelian groups with R-modules over other rings than $\mathbb{Z}$ and take tensor products over R, we might have to consider higher Tor-terms. Hence we should really write $\mathrm{Tor}_1^{\mathbb{Z}}(A,M)$ for $\mathrm{Tor}(A,M)$. But we are going to keep things simple and focus on the idea rather than general technicalities.

It is again time to see **some examples**:

- If M is a **free** abelian group, then $\mathrm{Tor}(A,M) = 0$ for any abelian group A. That follows from the lemma above.

- Let $M = \mathbb{Z}/m$. Then we can take $F_0 = F_1 = \mathbb{Z}$ and
$$\mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$$
as a free resolution of $\mathbb{Z}/m$. For an abelian group A, the sequence defining Tor looks like
$$0 \to \mathrm{Tor}(A,\mathbb{Z}/m) \to A \otimes \mathbb{Z} \xrightarrow{1\otimes m} A \otimes \mathbb{Z} \to A \otimes \mathbb{Z}/m \to 0.$$
Since we know $A \otimes \mathbb{Z}/m = A/mA$, we get
$$\mathrm{Tor}(A,\mathbb{Z}/m) = \mathrm{Ker}\,(m \colon A \to A) = \text{m-torsion in } A.$$
Hence $\mathrm{Tor}(A,\mathbb{Z}/m)$ is the subgroup of m-torsion elements in A.

- For a concrete case, let us try to calculate $\mathrm{Tor}(\mathbb{Z}/4, \mathbb{Z}/6)$. We use the free resolution
$$\mathbb{Z} \xrightarrow{6} \mathbb{Z} \to \mathbb{Z}/6 \to 0.$$
Tensoring with $\mathbb{Z}/4$ yields
$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to \mathbb{Z}/4 \otimes \mathbb{Z}/6 \to 0$$
where we use $6 = 2$ in $\mathbb{Z}/4$. The kernel of $\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$ is $\mathbb{Z}/2$. Thus
$$\mathrm{Tor}(\mathbb{Z}/4, \mathbb{Z}/6) = \mathbb{Z}/2.$$

- More generally, we get
$$\mathrm{Tor}(\mathbb{Z}/n,\mathbb{Z}/m) = \mathbb{Z}/\gcd(n,m)$$

where $\gcd(n,m)$ denotes the greatest common divisor of n and m.

The last three examples explain the name Tor.

We should hold our breath for a moment and check a couple of things. For example, that Tor does not depend on the choice of free resolution, that it is a functor etc. So let us get to work:

Lemma: Lifting resolutions

Let $f\colon M \to N$ be a homomorphism and $0 \to E_1 \xrightarrow{i} E_0 \xrightarrow{p} M$ and $0 \to F_1 \xrightarrow{j} F_0 \xrightarrow{q} N$ be free resolutions. Then we can lift f to a chain map $f_*\colon E_* \to F_*$, i.e, to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r, "i"] \arrow[d, dotted, "f_1"] & E_0 \arrow[r, "p"] \arrow[d, dotted, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "j"] & F_0 \arrow[r, "q"] & N \arrow[r] & 0. \end{tikzcd}$$

Moreover, this lift is unique up to chain homotopy, i.e., for another lift f'_* of f, there is a chain homotopy h between f_* and f'_*:

$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r] \arrow[d, bend right, "f'_1"'] \arrow[d, bend left, "f_1"] & E_0 \arrow[r] \arrow[dl, "h"'] \arrow[d, bend right, "f'_0"'] \arrow[d, bend left, "f_0"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & 0. \end{tikzcd}$$

Proof: • Since E_0 is a free abelian group, we know there is some set S of generators such that $E_0 = \mathbb{Z}S$. Now we can map the elements in S to M via the map $E_0 \xrightarrow{p} M$, and further to N via $M \xrightarrow{f} N$. Since $F_0 \xrightarrow{q} N$ is **surjective**, **we can choose lifts in F_0** of the elements in $f(p(S))$. Since a homomorphism on a free abelian group is determined by the image of the generators, we can extend this process to get a homomorphism

$$E_0 \xrightarrow{f_0} F_0 \text{ such that } f \circ p = q \circ f_0.$$

Now can define f_1 to be the restriction of f_0 to the kernel of p which is E_1 by definition. This yields the desired commutative diagram.

- Now let f_0' and f_1' be another choice of maps which lift f. The differences $g_0 := f_0 - f_0'$ and $g_1 := f_1 - f_1'$ are then maps which lift $f - f = 0\colon M \to N$:

$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r, "i"] \arrow[d, "g_1"] & E_0 \arrow[r, "p"] \arrow[d, "g_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "j"] & F_0 \arrow[r, "q"] & N \arrow[r] & 0. \end{tikzcd}$$

Since the diagram commutes, we get $q \circ g_0 = p \circ 0 = 0$. Therefore, the **universal property of kernels** implies that we can lift g_0 to a map $h\colon E_0 \to F_1$ such that $j \circ h = g_0$:

$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r, "i"] \arrow[d, "g_1"] & E_0 \arrow[r, "p"] \arrow[d, "g_0"] \arrow[dl, dotted, "h"'] & M \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "j"] & F_0 \arrow[r, "q"] & N \arrow[r] & 0. \end{tikzcd}$$

Moreover, since E_1 is the kernel of i, we must have $h \circ i = g_1$. Thus h is a chain homotopy between f_* and f_*' (the next map $E_1 \to 0$ being trivial). **QED**

With this result at hand we can finally prove:

Corollary: Tor is independent of resolutions

Tor is independent of the choice of free resolutuon: For any free resolution $0 \to E_1 \xrightarrow{i} E_0 \to M$ of M, there is a unique isomorphism

$$\operatorname{Ker}(i \otimes 1) \xrightarrow{\cong} \operatorname{Tor}(A,M).$$

Proof: We just apply the previous result to the **identity** of M to get that, with whatever resolution we calculate Tor, there is an isomorphism between any two different ways. And this isomorphism is unique by the theorem on chain homotopies and their induced maps on homologies. **QED**

There are other **properties of Tor** the proof of which we are going to omit:

- Tor is **functorial**: For any homomorphisms of abelian groups $A \to A'$ and $M \to M'$, there is a homomorphism
$$\operatorname{Tor}(A,M) \to \operatorname{Tor}(A',M').$$
- Tor is **symmetric**, i.e., $\operatorname{Tor}(A,M) \cong \operatorname{Tor}(M,A)$.
- If M is **free**, then $\operatorname{Tor}(A,M) = 0$ for any abelian group A.

- Since the direct sum of free resolutions of each A_i is a free reolution of $\bigoplus_i A_i$, we know that Tor commutes with direct sums:
$$\mathrm{Tor}(\bigoplus_i A_i, M) \cong \bigoplus_i \mathrm{Tor}(A_i, M),$$
- Let $T(M)$ be the subgroup of **torsion elements** of M. Then
$$\mathrm{Tor}(A, M) \cong \mathrm{Tor}(A, T(M))$$
for any abelian group A.

Now we can prove the main result in this story:

Theorem: Universal Coefficient Theorem

Let C_* be a chain complex of **free** abelian groups and let M be an abelian group. Then there are natural short exact sequences
$$0 \to H_n(C_*) \otimes M \to H_n(C_* \otimes M) \to \mathrm{Tor}(H_{n-1}(C_*), M) \to 0$$
for all n. These sequences split, but the splitting is not natural.

Proof: We write Z_n for the **kernel** and B_{n-1} for the **image** of the differential $d\colon C_n \to C_{n-1}$. Since C_n and C_{n-1} are **free**, both Z_n and B_{n-1} are **free as well**.

Together with the differential in C_*, this yields a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "d_n"] & C_n \arrow[r, "d_n"] \arrow[d, "d_n"] & B_{n-1} \arrow[r] \arrow[d, "d_{n-1}"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] & C_{n-1} \arrow[r, "d_{n-1}"] & B_{n-2} \arrow[r] & 0.
\end{tikzcd}$$

By definition of Z_n and B_n, the restriction of the **differentials** to these groups **vanish**. This implies that (Z_*, d) and (B_*, d) are chain complexes (with trivial differentials).

Hence we get a short exact sequence of chain complexes

$$0 \to Z_* \to C_* \to B_{*-1} \to 0. \tag{1}$$

Since all groups in these chain complexes are **free**, tensoring with M yields again a **short exact sequence of chain complexes**

$$0 \to Z_* \otimes M \to C_* \otimes M \to B_{*-1} \otimes M \to 0.$$

This can be checked as in the above lemma on direct sums of exact sequences.

Since the **differentials** in Z_* and B_* are **trivial**, the associated long exact sequence in homology looks like

$$\cdots \to B_n \otimes M \xrightarrow{\partial_n} Z_n \otimes M \to H_n(C_* \otimes M) \to B_{n-1} \otimes M \xrightarrow{\partial_{n-1}} Z_{n-1} \otimes M \to \cdots$$

The connecting homomorphism $B_n \otimes M \xrightarrow{\partial_n} Z_n \otimes M$ in this sequence is $i_n \otimes 1$, where $i_n \colon B_n \hookrightarrow Z_n$ denotes the inclusion and 1 denotes the identity on M. This can be easily checked using the definition of the connecting homomorphism.

A long exact sequence can always be **cut into short exact sequences** of the form

$$0 \to \mathrm{Coker}(i_n \otimes 1) \to H_n(C_* \otimes M) \to \mathrm{Ker}\,(i_{n-1} \otimes 1) \to 0.$$

Since the **tensor product preserves cokernels**, the cokernel on the left-hand side is just

$$\mathrm{Coker}(i_n \otimes 1) = \mathrm{Coker}(i_n) \otimes M = Z_n/B_n \otimes M = H_n(C_*) \otimes M.$$

For $\mathrm{Ker}\,(i_{n-1} \otimes 1)$, we observe that

$$B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C_*) \to 0$$

is a free resolution of $H_{n-1}(C_*)$. Hence after tensoring with M we get an **exact sequence**

$$0 \to \mathrm{Ker}\,(i_{n-1} \otimes 1) \to B_{n-1} \otimes M \xrightarrow{i_{n-1}\otimes 1} Z_{n-1} \otimes M \to H_{n-1}(C_*) \otimes M \to 0.$$

Thus, since Tor is **independent of the chosen free resolution**,

$$\mathrm{Ker}\,(i_{n-1} \otimes 1) = \mathrm{Tor}(H_{n-1}(C_*), M).$$

Finally, to obtain the asserted **splitting** we use that subgroups of free abelian groups are free. That implies that sequence (1) splits and we have

$$C_n \cong Z_n \oplus B_{n-1}.$$

Tensoring with M yields

$$C_n \otimes M \cong (Z_n \otimes M) \oplus (B_{n-1} \otimes M).$$

Now one has to work a little bit more to get that this induces a direct sum decomposition in homology. We skip this here. **QED**

Since the **singular chain complex** $S_*(X,A)$ is an example of a chain complex of **free** abelian groups, the theorem implies:

Corollary: UCT for singular homology

For each pair of spaces (X,A) there are split short exact sequences

$$0 \to H_n(X,A) \otimes M \to H_n(X,A;M) \to \mathrm{Tor}(H_{n-1}(X,A),M) \to 0$$

for all n, and these sequences are natural with respect to maps of pairs $(X,A) \to (Y,B)$.

One of the goals of introducing coefficients is to simplify calculations. The simplest case is often when we consider a field as coefficients. For example, the finite fields $\mathbb{F}_p$ or the rational numbers $\mathbb{Q}$. The UCT tells how we can recover integral homology from these pieces. We will figure out how this works in the **exercises**.

Since we put so much work into defining Tor, let us mention another important theorem. It tells us how the homology of the product of two spaces depends on the homology of the individual spaces. For that relation is not as straight forward as one might hope:

Künneth Theorem

For any pair of spaces X and Y and every n, there is a split short exact sequence

$$0 \to \bigoplus_{p+q=n} (H_p(X) \otimes H_q(Y)) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0.$$

This sequence is natural in X and Y. But the splitting is not natural.

The maps $H_p(X) \to H_n(X \times Y)$ and $H_q(Y) \to H_n(X \times Y)$ arise from the cross product construction on singular chains. We will not have time to discuss this in class though.