**MA3403 Algebraic Topology**
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**Lecture 18**

# 18. Singular cohomology

We are going to define a **new algebraic invariant**, called singular cohomology. At first glance it might look almost the same as homology, but we will see that there is a **striking difference** between homology and cohomology. For, singular cohomology allows us to define an additional algebraic structure: **multiplication**.

As a **motivation**, we start with the following familiar situation. Recall that in **calculus**, we learn to calculate path integrals. Given a path $\gamma\colon [a,b] \to \mathbb{R}^2$ and a 1-form $pdx + qdy$. Then we can form the integral $\int_\gamma pdx + qdy$, and we learned all kinds of things about it.

In particular, we can **consider taking the integral as a map**

$$\gamma \mapsto \int_\gamma pdx + qdy \in \mathbb{R}.$$

Since any path $\gamma$ can be reparametrized to a 1-simplex, we can think of taking the integral of a given 1-form over a path as a map

$$\mathrm{Sing}_1(\mathbb{R}^2) \to \mathbb{R}.$$

This map captures certain geometric and topological information. It is an important example of a 1-cochain, a concept we are now going to define.

## Definition: Singular cochains

Let $X$ be a topological space and let $M$ be an abelian group. An **$n$-cochain on $X$ with values in $M$** is a function

$$\mathrm{Sing}_n(X) \to M.$$

We turn the set

$$S^n(X;M) := \mathrm{Map}(\mathrm{Sing}_n(X), M)$$

of $n$-cochains into a **group** by defining $c + c'$ to be the function which sends $\sigma$ to $c(\sigma) + c'(\sigma)$.

As an **example**, let us look at the case $M = \mathbb{Z}$. Then an $n$-cochain on $X$ is just a **function** which assigns to any $n$-simplex $\sigma\colon \Delta^n \to X$ a number in $\mathbb{Z}$.

We know that simplices in different dimensions are connected via the face maps. As for chains, the face maps induce an operator between cochains in different dimensions. But note that, for cochains, the degree will **increase instead of decrease**.

## Definition: Coboundaries

The **coboundary operator**

$$\delta \colon S^n(X;M) \to S^{n+1}(X;M)\delta(c)(\sigma) = c(\partial\sigma)$$

is defined as follows:
Given an $n$-cochain $c$ and an $n+1$-simplex $\sigma \colon \Delta^{n+1} \to X$. Then we define the $n+1$-cochain $\delta(c)$ as

$$\delta^n(c)(\sigma) = c(\partial_{n+1}(\sigma)) = \sum_{i=0}^{n+1} (-1)^i c(\sigma \circ \phi_i^{n+1})$$

where $\phi_i^{n+1}$ is the $i$th face map. This defines $\delta^n(c)$ as a function on $\mathrm{Sing}_{n+1}(X)$.

• For an **example**, let us look again at the case $m = \mathbb{Z}$. We learned that an $n$-cochain on $X$ is a **function** which assigns to any $n$-simplex $\sigma \colon \Delta^n \to X$ a number in $\mathbb{Z}$. In order to be an $n$**-cocycle**, the numbers assigned to the boundary of an $n+1$-simplex cancel out (with the sign convention).

To get more concrete, let $c \in S^1(X;\mathbb{Z})$ be a 1-cochain. Let $\sigma \colon \Delta^2 \to X$ be a 2-simplex. Then, for $c$ to be a cocycle, we need that it the **numbers** it assigns to the faces of $\sigma$ **cancel out** in the sense that

$$c(d_0\sigma) - c(d_1\sigma) + c(d_2\sigma) = 0.$$

• Let us have another look at the **example from calculus** we started with. A **function**

$$f \colon \mathbb{R}^2 \to \mathbb{R}$$

is a $0$**-cochain** on $\mathbb{R}^2$ with values in $\mathbb{R}$. For it assigns to each zero-simplex, i.e., a point $x \in \mathbb{R}^2$, a real number $f(x)$.

Then the 1-cochain $\delta(f)$ is the function which assigns to a (smooth) path $\gamma$ the number $f(\gamma(1)) - f(\gamma(0))$:

$$\delta(f) \colon \gamma \mapsto f(\gamma(1)) - f(\gamma(0)).$$

By Green's Theorem, this is also the value of the integral

$$\int_\gamma f_x dx + f_y dy = \int_\gamma df$$

which is the integral of the 1-form $df$ along $\gamma$.

Hence the cochain complex, while it looks very much like homology, also has a **natural connection to calculus**. In fact, there is some justification for saying that **cochains and cohomology are more natural** notions than chains and homology.

Back to the **general case**. The coboundary operator turns $S^*(X;M)$ into a **cochain complex**, since $\delta \circ \delta = 0$ which follows from our previous calculation. For, given an $n+1$-simplex $\sigma$ and an $n-1$-cochain $c$, we get

$$(\delta^n \circ \delta^{n-1}(c))(\sigma) = (\delta^n c)(\partial_n(\sigma)) = c(\partial_n \circ \partial_{n+1}(\sigma)) = 0.$$

An **equivalent way** to obtain this complex, is to look at **homomorphisms** of abelian groups from $S_n(X)$ to $M$, i.e., we have

$$S^n(X;M) = \mathrm{Hom}_{\mathbf{Ab}}(S_n(X), M).$$

The coboundary operator is just the homomorphism induced by the boundary operator on chains:

$$\delta = \mathrm{Hom}(\partial, M) \colon \mathrm{Hom}_{\mathbf{Ab}}(S_n(X), M) \to \mathrm{Hom}_{\mathbf{Ab}}(S_{n+1}(X), M), \; c \mapsto c \circ \partial.$$

In other words, $\delta = \partial^*$ **equals the pullback along** $\partial$.

The subgroup given as the **kernel** of $\delta^n$ is denoted by

$$Z^n(X;M) = \mathrm{Ker}\,(\delta \colon S^n(X;M) \to S^{n+1}(X;M))$$

and called the group of $n$-**cocylces** of $X$.

The **image** of $\delta^{n-1}$ is called the group of $n$-**coboundaries** of $X$ and is denoted by

$$B^n(X;M) = \mathrm{Im}\,(\delta^{n-1} \colon S^{n-1}(X;M) \to S^n(X;M)).$$

Since $\delta \circ \delta = 0$, we have

$$B^n(X;M) \subseteq Z^n(X;M).$$

In other words, every coboundary is a cocycle.

**Definition: Singular cohomology**

Let $X$ be a topological space and let $M$ be an abelian group. The **$n$th singular cohomology group of $X$** is defined as the $n$th cohomology group of the cochain complex $S^*(X;M)$, i.e.,

$$H^n(X;M) = \frac{Z^n(X;M)}{B^n(X;M)}.$$

**Integrals over forms** yield elements in cohomology with coefficients in $\mathbb{R}$. This is in fact the origin of cohomology theory and is connected to **de Rham cohomology**. Though as natural as de Rham cohomology is, it has the drawback that we have to stick to coefficients in $\mathbb{R}$.

This demonstrates why it might be smart to take the detour via singular simplices and taking maps in chains. For **we gain the flexibility** to study singular cohomology with coefficients in an arbitrary abelian group.

As a first example, let us try to understand $H^0(X;M)$.

**Cohomology in dimension zero**

A 0-cochain is a function

$$c\colon \mathrm{Sing}_0(X) \to M.$$

Since $\mathrm{Sing}_0(X)$ is just the **underlying set** of $X$, a 0-cochain corresponds to just an arbitrary, that is **not necessarily continuous**, function

$$f\colon X \to M.$$

Now what does it mean for such a function to be a cocycle? To figure this out we need to calculate $\delta(f)$. Since $\delta(f)$ is defined on 1-simplices, let $\sigma\colon \Delta^1 \to X$ be a 1-simplex on $X$. The effect of $\delta(f)$ is to **evaulate $f$ on the boundary** of $\sigma$:

$$\delta(f)(\sigma) = f(\partial\sigma) = f(\sigma(e_0)) - f(\sigma(e_1)).$$

Since this expression must be 0 for every 1-simplex, we deduce that $f$ is a **cocycle** if and only if it is **constant on the path-components** of $X$.

If we denote by $\pi_0(X)$ the set of path-components, then we have shown:

$$H^0(X;M) = \mathrm{Map}(\pi_0(X),M).$$

**Cohomology of a point**

If $X$ is just a point, then $\mathrm{Sing}_n(\mathrm{pt})$ consists just of the **constant map** for each $n$. Hence an $n$-cochain $c \in S^n(\mathrm{pt};M)$ is **completely determined by its value** $m_c$ on the constant map, and therefore $\mathrm{Hom}(\mathrm{Sing}_n(\mathrm{pt}),M) \cong M$ for all $n$.

The coboundary operator takes $c \in \mathrm{Hom}(\mathrm{Sing}_n(\mathrm{pt}),M)$ to the alternating sum

$$\delta(c) = \sum_{i=0}^{n+1}(-1)^i c(\text{constant map} \circ \phi_i^n) = \sum_{i=0}^{n+1}(-1)^i m_c.$$

Hence the coboundary is **trivial if $n$ is even** and the **identity if $n$ is odd**. The cochain complex therefore looks like

$$0 \to M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} \cdots$$

Thus the **cohomology of a point** is given by

$$H^n(\mathrm{pt};M) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

Now let us see what else we know about singular cohomology.

**Properties of singular cohomology**

Fix an abelian group $M$. Singular cohomology has the following properties:

- **Cohomology is contravariant**, i.e., a continuous map $f\colon X \to Y$ induces a homomorphism

  $$f^*\colon S^*(Y;M) \to S^*(X;M).$$

  This map works as follows: Let $c \in S^n(Y;M)$ be an $n$-cochain on $Y$. Then $f^*c$ is the map which assigns to $n$-simplex $\sigma\colon \Delta^n \to X$, the value

  $$(f^*c)(\sigma) = c(f \circ \sigma) = c(\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y).$$

  Since $f^*$ is in fact a map of cochain complexes (which is defined in analogy to maps of chain complexes), this induces a homomorphism on cohomology

  $$f^*\colon H^*(Y;M) \to H^*(X;M).$$

This assignment is functorial, i.e., the identity map is sent to the identity homomorphism of cochains and if $g\colon Y \to Z$ is another map, then

$$(g \circ f)^* = f^* \circ g^*.$$

- In our **calculus example**, the **contravariance corresponds to restriction** of a form to an open subspace.

## Why cohomology?

At first glance it seems like cohomology and homology are the same guys, just wrapped up in slightly different cloths and reversing the arrows. In fact, this is kind of true as we will see in the next lecture. However, there is also a striking difference which is due to the innocent looking fact that cohomology is contravariant. We are going to exploit this fact as follows:
Assume that $R$ is a ring, and let

$$X \xrightarrow{\Delta} X \times X,\ x \mapsto (x,x)$$

be the **diagonal map**. It induces a homomorphism in cohomology

$$H^*(X \times X; R) \xrightarrow{\Delta^*} H^*(X; R).$$

Now we only need to construct a suitable map $H^p(X;R) \otimes H^q(X;R) \to H^{p+q}(X \times X;R)$ to get a multiplication on the direct sum $H^*(X;R) = \bigoplus_q H^q(X;R)$:

$$H^p(X;R) \otimes H^q(X;R) \to H^{p+q}(X \times X;R) \xrightarrow{\Delta^*} H^{p+q}(X;R).$$

It will still take some effort to make this idea work. Nevertheless, this gives us a first idea of how contravariance can be useful.

- **Cohomology is homotopy-invariant**, i.e., if the maps $f$ and $g$ are **homotopic** $f \simeq g$, then they **induce the same map in cohomology** $f^* = g^*$.

  In fact, the **proof** we used for homology **dualizes to cohomology**.

  For, recall that a homotopy between $f$ and $g$ induces a chain homotopy between $h$ between the maps $f_*$ and $g_*$ on singular chain complexes. Now we use that the singular cochain complex is the value of the singular chain complex under the functor $\mathrm{Hom}(-,M)$.

  Then $\mathrm{Hom}(h,M)$ is a homotopy between the maps of cochain complexes

$$f^* = \mathrm{Hom}(f_*,M) \text{ and } g^* = \mathrm{Hom}(g_*,M).$$

For the relation

$$f_* - g_* = h \circ \partial + \partial \circ h$$

implies the relation

$$f^* - g^* = \delta \circ \mathrm{Hom}(h,M) + \mathrm{Hom}(h,M) \circ \delta.$$

- If $A$ is a **subspace** of $X$, then there are also **relative cohomology** groups. Let $i\colon A \hookrightarrow X$ denote the inclusion map. We consider the cochain complex

$$S^*(X,A;M) = \mathrm{Ker}\,(S^*(X;M) \xrightarrow{i^*} S^*(A;M))$$

consisting of **those maps** $\mathrm{Sing}_n(X) \to M$ **which vanish on the subset** $\mathrm{Sing}_n(A)$.

The $n$th **relative cohomolgy** group is defined as the cohomology of this cochain complex

$$H^n(X,A;M) = H^n(S^*(X,A;M)).$$

By definition, there is a **short exact sequence**

$$0 \to S^*(X,A;M) \to S^*(X;M) \to S^*(A;M) \to 0$$

**which induces a long exact sequence** of the cohomolgy groups of the complexes in the same way as this was the case for chain complexes and homology:

$$\cdots \to H^n(X,A;M) \to H^n(X;M) \to H^n(A;M) \xrightarrow{\partial^n} H^{n+1}(X,A;M) \to \cdots$$

- There is also a **reduced version** of cohomology. Let $\epsilon\colon S_0(X;M) \to M$ be the **augmentation map** sending $\sum_i m_i\sigma_i$ to $\sum_i m_i \in M$. Since we know $\partial_0 \circ \epsilon = 0$, we observe that applyig the functor $\mathrm{Hom}(-,M)$ yields the **augmented singular cochain complex**

$$0 \to M \xrightarrow{\epsilon^*} S^0(X;M) \xrightarrow{\delta^0} S^1(X;M) \xrightarrow{\delta^1} \cdots$$

The **reduced cohomology** of $X$ with coefficients in $M$ is the cohomology of the augmented singular cochain complex.

- **Cohomology satisfies Excision**, i.e., if $Z \subset A \subset X$ with $\bar{Z} \subset A^\circ$, then the inclusion $k\colon (A-Z,X-Z) \hookrightarrow (X,A)$ induces an isomorphism

$$k^*\colon H^*(X,A;M) \xrightarrow{\cong} H^*(X-Z,A-Z;M).$$

- **Cohomology sends sums to products**, i.e.,

$$H^*(\coprod_\alpha X_\alpha; M) \cong \prod_\alpha H^*(X_\alpha; M).$$

- **Cohomology has Mayer-Vietoris sequences**, i.e., if $\{A,B\}$ is a **cover** of $X$, then, for every $n$, there are connecting homomorphisms $d$ which fit into a long exact sequence

$$\cdots \xrightarrow{d} H^n(X;M) \xrightarrow{\begin{bmatrix} i_A^* \\ -i_B^* \end{bmatrix}} H^n(A;M) \oplus H^n(B;M) \xrightarrow{\begin{bmatrix} j_A^* & j_B^* \end{bmatrix}} H^n(A\cap B;M) \xrightarrow{d} H^{n+1}(X;M) \to \cdots$$

Note that the maps go in the **other direction** and the **degree** of the connecting homomorphism **increases**.

Here we used the inclusion maps

$$\begin{tikzcd} A \cap B \arrow[r, hook, "j_A"] \arrow[d, hook, "j_B"'] & A \arrow[d, hook, "i_A"] \\ B \arrow[r, hook, "i_B"'] & X. \end{tikzcd}$$

To **prove**, for example, the statement about **Mayer-Vietoris sequences**, let us go back to the proof in homology.

Let $\mathcal{A} = \{A,B\}$ be our cover. We used a short exact sequence of chain complexes

$$0 \to S_*(A\cap B) \to S_*(A) \oplus S_*(B) \to S_*^{\mathcal{A}}(X) \to 0$$

where $S_*^{\mathcal{A}}(X)$ denotes the $\mathcal{A}$-small chains.

We would like to **turn this** into an exact sequence **in cohomology**. As we have learned last time, **not all functors preserve exactness**.

And, in fact, $\mathrm{Hom}(-,M)$ is **unfortunately no exception**. We will study the behaviour of Hom next time, but for the present purpose we observe a fact which saves our day.

For, the singular chain complexes involved in the above short exact sequence consist of **free** abelian groups in each dimension. And exactness is indeed preserved by Hom for such sequences.

More precisely, we would like to use the following fact:

## Lemma: Exactness of Hom-functor on free complexes

Let $M$ be an abelian group and let

$$0 \to A_* \to B_* \to C_* \to 0$$

be an **exact** sequence of chain complexes of **free** abelian groups. Then the induced sequence of cochain complexes

$$0 \to \mathrm{Hom}(C_*,M) \to \mathrm{Hom}(B_*,M) \to \mathrm{Hom}(A_*,M) \to 0$$

is **exact**.

**Proof:** By definition of exactness for sequences of complexes, it **suffices** to show the assertion for a short exact sequence of **free abelian groups**.

The key is that any short exact sequence of free abelian groups **splits**. The splitting induces a splitting on the induced sequence of Hom-groups.

More concretely, let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of **free abelian groups**.

Since $C$ is **free** and $p$ is **surjective**, there is a **dotted lift** in the solid diagram

$$\begin{tikzcd} & B \arrow[d, "p"] \\ C \arrow[ur, dotted, "s"] \arrow[r, equal] & C \end{tikzcd}$$

which makes the diagram commute, i.e., $p \circ s = 1_C$.

This implies

$$s^* \circ p^* = 1_{\mathrm{Hom}(C,M)},$$

and hence $s^*$ **is a secton of** $p^*$ in

$$0 \to \mathrm{Hom}(C,M) \xrightarrow{p^*} \mathrm{Hom}(B,M) \xrightarrow{i^*} \mathrm{Hom}(A,M) \to 0.$$

This implies that $\mathrm{Hom}(B,M) = \mathrm{Hom}(A,M) \oplus \mathrm{Hom}(C,M)$ and that $i^*$ is surjective. That the sequence is exact at $\mathrm{Hom}(B,M)$ is now obvious as well. **QED**

**Warning:** Note that if $A = \mathbb{Z}[S]$ is a free abelian group, then

$$\mathrm{Hom}(A,M) = \mathrm{Hom}(\bigoplus_S \mathbb{Z},M) = \prod_S \mathrm{Hom}(\mathbb{Z},M)$$

which **might be an uncountable product**. This leads to the annoying fact that $\mathrm{Hom}(A,M)$ not a free abelian group, in general.

**Back to the proof of the MVS**, with this fact at hand we get an induced short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}(S_*^{\mathcal{A}}(X),M) \to \mathrm{Hom}(S_*(A),M) \oplus \mathrm{Hom}(S_*(B),M) \to \mathrm{Hom}(S_*(A \cap B),M) \to 0$$

where we also use that Hom commutes with direct sums.

Again, such a short exact sequence of cochain complexes **induces a long exact sequence** of the cohomolgy groups.

The **final step** of the proof is that we **need to check** that the induced map of cochain complexes

$$\mathrm{Hom}(S_*(X),M) \to \mathrm{Hom}(S_*^{\mathcal{A}}(X),M)$$

**induces an isomorphism in cohomology**.

In fact, this follows from the Small Chain Theorem and the following fact:

## Proposition: From isos in homology to isos in cohomology

Let $C_*$ and $D_*$ be two chain complexes of **free** abelian groups.
Assume that there is a map $C_* \xrightarrow{\varphi} D_*$ which induces an **isomorphism in homology**

$$\varphi_* \colon H_*(C_*) \xrightarrow{\cong} H_*(D_*).$$

Then, for any abelian group $M$, the map $\varphi$

$$\varphi^* \colon H^*(D;M) \to H^*(C;M)$$

induces an **isomorphism in cohomology** with coefficients in $M$ as well.
Here we wrote $H^*(C;M) = H^*(\mathrm{Hom}(C_*,M))$ and $H^*(D;M) = H^*(\mathrm{Hom}(D_*,M))$ for the cohomology of the induced cochain complexes.

We are going to deduce this result from the **Universal Coefficient Theorem in cohomology** which we will prove in the next lecture. Roughly speaking, it will tell us how homology and cohomology are related.

As a first approach, we observe the following phenomenon.

**The Kronecker pairing**

Let $M$ be an abelian group. For a chain complex $C_*$ and cochain complex $C^* := \mathrm{Hom}(C_*,M)$ there is a natural pairing given by evaluating a cochain on chains:

$$\langle -,- \rangle \colon C^n \otimes C_n \to M,\ (\varphi,a) \mapsto \langle \varphi,a \rangle := \varphi(a).$$

This is called the **Kronecker pairing**.

The boundary and coboundaries are compatible with this pairing, i.e.,

$$\langle \delta\varphi,a \rangle = \delta(\varphi)(a) = \varphi(\partial(a)) = \langle \varphi,\partial a \rangle.$$

## Lemma: Kronecker pairing

The Kronecker pairing induces a well-defined pairing on the level of cohomology and homology, i.e., we get an induced pairing

$$\langle -,- \rangle \colon H^n(C^*) \otimes H_n(C_*) \to M.$$

**Proof:** Let $\varphi$ be a **cocycle**, i.e., $\delta\varphi = 0$. Then we get

$$\langle \varphi, a + \partial b \rangle = \langle \varphi, a \rangle + \langle \varphi, \partial b \rangle = \langle \varphi, a \rangle + \langle \delta\varphi, b \rangle = \langle \varphi, a \rangle.$$

Thus, the map $\langle \varphi, - \rangle$ **descends to homology** if $\varphi$ is a cocycle.

It remains to check that this map **vanishes** if $\varphi$ is a **coboundary**. So assume $\varphi = \delta\psi$ and $a$ is a **cycle**, i.e., $\partial a = 0$. Then we get

$$\langle \varphi, a \rangle = \langle \delta\psi, a \rangle = \langle \psi, \partial a \rangle = 0.$$

This show that the pairing is well-defined on $H^n(C^*)$ and $H_n(C_*)$. **QED**

## Kronecker homomorphism

Thus, applied to the integral singular chain complex and the cochain complex with coefficients in $M$, this pairing yields a natural homomorphism

$$\kappa \colon H^n(X;M) \to \mathrm{Hom}(H_n(X),M),$$

which sends the class $[c]$ of a cocycle to the homomorphism $\kappa([c])$ defined by

$$\kappa([c]) \colon H_n(X) \to M,\ [\sigma] \mapsto \langle c, \sigma \rangle = c(\sigma).$$

This leads to the important question:

## From homology to cohomology?

If we know the singular homology of a space, what can we deduce about its cohomology?
More concretely, what can we say about the map $\kappa$? Is $\kappa$ injective? Is $\kappa$ surjective?