MA3403 Algebraic Topology
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Lecture 19

19. Ext and the Universal Coefficient Theorem for cohomology

In the previous lecture, we introuced the singular cochain complex and defined singular chomology. Along the way we ran into some exact sequences to which applied the Hom-functor. In particular, we constructed the Kronecker map

$$\kappa \colon H^n(X;M) \to \mathrm{Hom}(H_n(X),M).$$

Our goal for this lecture is to study the Hom-functor in more detail and to prove the **Universal Coefficient Theorem** for singular cohomology which will tell us that κ is surjective. However, κ is not injective in general, but the UCT will tell us what the kernel is.

Again, for some this will be a review of known results in homological algebra. Nevertheless, those who have not seen this before, should get a chance to catch up.

We will again focus on the main ideas.

Let M be an abelian group. We would like to understand the effect of the functor $\mathrm{Hom}(-,M)$ on exact sequences.

Before we start, note that Hom is not symmetric in general, i.e., $\mathrm{Hom}(A,M)$ and $\mathrm{Hom}(M,A)$ might be very different indeed. For example,

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/n) \cong \mathbb{Z}/n, \text{ but } \mathrm{Hom}(\mathbb{Z}/n,\mathbb{Z}) = 0.$$

Our next observation tells us that Hom is left-exact:

Lemma: Hom is left-exact

(a) Let M be an abelian group. Suppose we have an exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0.$$

Then applying $\mathrm{Hom}(-,M)$ yields an exact sequence

$$0 \to \mathrm{Hom}(C,M) \xrightarrow{j^*} \mathrm{Hom}(B,M) \xrightarrow{i^*} \mathrm{Hom}(A,M).$$

In other words, the functor $\mathrm{Hom}(-,M)$ is **left-exact** and sends **cokernels to kernels**.
(b) Similarly, applying $\mathrm{Hom}(M,-)$ to an exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C$$

yields an exact sequence

$$0 \to \mathrm{Hom}(M,A) \xrightarrow{j_*} \mathrm{Hom}(M,B) \xrightarrow{i_*} \mathrm{Hom}(M,A).$$

In other words, the functor $\mathrm{Hom}(M,-)$ is **left-exact** and sends **kernels to kernels**.

Proof: (a) To show that j^* is **injective**, assume that $\gamma \in \mathrm{Hom}(C,M)$ satisfies $j^*(\gamma) = 0$. That means

$$j^*(\gamma)(b) = (\gamma \circ j)(b) = \gamma(j(b)) = 0 \text{ for all } b \in B.$$

But j is **surjective**, and hence every element in C is of the form $j(b)$ for some $b \in B$. Hence $\gamma = 0$ is the trivial homomorphism.

The **composition** $i^* \circ j^*$ is clearly 0, since $j \circ i = 0$ by assumption. Thus $\mathrm{Im}\,(j^*) \subseteq \mathrm{Ker}\,(i^*)$.

Now if $\beta \in \mathrm{Hom}(B,M)$ is in $\mathrm{Ker}\,(i^*)$, then

$$0 = i^*(\beta)(a) = \beta(i(a)) \text{ for all } a \in A.$$

In other words, β is trivial on the image of i and hence factors as

$$\beta\colon B \to B/\mathrm{Im}\,(i) \to M.$$

But $B/\mathrm{Im}\,(i) \cong C$, since the intial sequence was exact. Hence β is the composition of a map $B \xrightarrow{j} C \xrightarrow{\gamma} M$ for some $\gamma \in \mathrm{Hom}(C,M)$. Thus, $\beta \in \mathrm{Im}\,(j^*)$.

(b) The proof is of course similar. To show that i_* is **injective**, let $\alpha \in \mathrm{Hom}(M,A)$ be a map such that $i_*(\alpha) = 0$. That means

$$i_*(\alpha(m)) = i(\alpha(m)) = 0 \text{ for all } m \in M.$$

Since i is **injective**, this implies $\alpha(m) = 0$ for all $m \in M$, and hence $\alpha = 0$.

The **composition** $j_* \circ i_*$ is clearly 0, since $j \circ i = 0$ by assumption. Thus $\mathrm{Im}\,(i_*) \subseteq \mathrm{Ker}\,(j_*)$.

If $\beta \in \mathrm{Hom}(M,B)$ is in $\mathrm{Ker}\,(j_*)$, then

$$0 = j_*(\beta)(m) = j(\beta(m)) \text{ for all } m \in M.$$

In other words, $\beta(m) \in \mathrm{Ker}\,(j)$ for all $m \in M$. Since $\mathrm{Ker}\,(j) = \mathrm{Im}\,(i)$, we get $\beta(m) \in \mathrm{Im}\,(i)$ for all $m \in M$. Hence β factors as

$$\beta\colon M \xrightarrow{\alpha} A \xrightarrow{i} B$$

for some $\alpha \in \mathrm{Hom}(M,A)$. Thus, $\beta \in \mathrm{Im}\,(i_*)$. **QED**

However, suppose we have an injective homomorphism

$$A \hookrightarrow B.$$

Then it is in general **not the case** that the induced map

$$\mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$$

is **surjective**.

For example, take the map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ given by multiplication by 2. It is clearly **injective**. But if we apply $\mathrm{Hom}(-,\mathbb{Z}/2)$, we get the map

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2) \cong \mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2 \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2)$$

which is **not surjective**.

We would like to remedy this defect. And we can already guess how this can be achieved. As we have seen in the previous lecture, $\mathrm{Hom}(-,M)$ is **not so far from being exact**. For, if we apply $\mathrm{Hom}(-,M)$ to a short exact sequence of **free** abelian groups, then the induced sequence is still short exact.

So let A be an abelian group and let us choose a free resolution of A as in a previous lecture

$$0 \to F_1 \hookrightarrow F_0 \twoheadrightarrow A.$$

Applying $\mathrm{Hom}(-,M)$ to this equence yields an exact sequence

$$0 \to \mathrm{Hom}(A,M) \to \mathrm{Hom}(F_0,M) \to \mathrm{Hom}(F_1,M).$$

The right-hand map is not necessarily surjective, or in other words, the **cokernel** of the right-hand map **is not necessarily zero**.

This leads to the following important definition:

Definition: Ext

The **cokernel** of the map $\mathrm{Hom}(F_0,M) \to \mathrm{Hom}(F_1,M)$ is called $\mathrm{Ext}(A,M)$. Hence by definition we have an **exact sequence**

$$0 \to \mathrm{Hom}(A,M) \to \mathrm{Hom}(F_0,M) \to \mathrm{Hom}(F_1,M) \to \mathrm{Ext}(A,M) \to 0.$$

Roughly speaking, the group $\mathrm{Ext}(A,-)$ measures how far $\mathrm{Hom}(A,-)$ is from being exact.

Let us calculate **some examples**:

- Let $A = \mathbb{Z}/p$. Then we can take $F_0 = F_1 = \mathbb{Z}$ and

  $$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

  as a free resolution of $\mathbb{Z}/p$. For an abelian group M, the sequence defining Ext looks like

  $$0 \to \mathrm{Hom}(\mathbb{Z}/p,M) \to \mathrm{Hom}(\mathbb{Z},M) \xrightarrow{p} \mathrm{Hom}(\mathbb{Z},M) \to \mathrm{Ext}(\mathbb{Z}/p,M) \to 0.$$

  Since $\mathrm{Hom}(\mathbb{Z},M) = M$, this sequence equals

  $$0 \to \text{p-torsion in } M \to M \xrightarrow{p} M \to \mathrm{Ext}(\mathbb{Z}/p,M) \to 0.$$

  Thus

  $$\mathrm{Ext}(\mathbb{Z}/p,M) = \mathrm{Coker}(M \xrightarrow{p} M) = M/pM.$$

- For a concrete case, let us calculate $\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2)$. We use the free resolution

  $$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

  Applying $\mathrm{Hom}(-,\mathbb{Z}/2)$ yields

  $$0 \to \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}/2) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2) \xrightarrow{2} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2).$$

  This sequence is isomorphic to

  $$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2.$$

  Since $2 = 0$ in $\mathbb{Z}/2$, the second map is trivial. Hence the cokernel of this map is just $\mathbb{Z}/2$. Thus

  $$\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2.$$

- More generally, one can show

  $$\mathrm{Ext}(\mathbb{Z}/n,\mathbb{Z}/m) = \mathbb{Z}/\gcd(n,m)$$

  where $\gcd(n,m)$ denotes the greatest common divisor of n and m.

Now we should study Ext in more detail. As a first step we show that it can be viewed as a cohomology group:

Lemma: Ext and Hom as cohomology groups

Let A and M be abelian groups and $0 \to F_1 \xrightarrow{j} F_0 \to A \to 0$ be a free resolution of A. Consider the cochain complex $\mathrm{Hom}(F_*,M)$ given by

$$0 \to \mathrm{Hom}(F_1,M) \xrightarrow{j^*} \mathrm{Hom}(F_0,M) \to 0$$

with $\mathrm{Hom}(F_1,M)$ in dimension zero and $\mathrm{Hom}(F_0,M)$ in dimension one. Then we have

$$H^0(\mathrm{Hom}(F_*,M)) = \mathrm{Hom}(A,M) \text{ and } H^1(\mathrm{Hom}(F_*,M)) = \mathrm{Ext}(A,M).$$

Proof: By definition, $\mathrm{Ext}(A,M)$ is the cokernel of j^*. Since the differential out of $\mathrm{Hom}(F_0,M)$ is trivial, the first cohomology is just

$$H^1(\mathrm{Hom}(F_*,M)) = \mathrm{Hom}(F_0,M)/\mathrm{Im}\,(j^*) = \mathrm{Coker}(j^*) = \mathrm{Ext}(A,M).$$

For H^0 we remember that the augmented sequence

$$0 \to \mathrm{Hom}(A,M) \to \mathrm{Hom}(F_1,M) \xrightarrow{j^*} \mathrm{Hom}(F_0,M)$$

is exact.

Hence $\mathrm{Hom}(A,M)$ is isomorphic to its image in $\mathrm{Hom}(F_1,M)$ which is, by exactness of the sequence, the kernel of j^*. But this kernel is the cohomology group of $\mathrm{Hom}(F_*,M)$ in dimension 0:

$$H^0(\mathrm{Hom}(F_*,M)) = \mathrm{Ker}\,(j^*) = \mathrm{Hom}(A,M)$$

QED

We should check that Ext does not depend on the choice of free resolution. To do this, we are going to apply the lemma we proved for the Tor-case which states that maps can be lifted to resolutions and any two lifts are chain homotopic in a suitable sense.

Proposition: Ext is independent of resolutions

Ext is independent of the choice of free resolutuon: If $0 \to E_1 \xrightarrow{i} E_0 \to A$ and $0 \to F_1 \xrightarrow{j} F_0 \to A$ are two free resolutions of A, there is a unique isomorphism

$$\mathrm{Coker}(\mathrm{Hom}(i,M)) \xrightarrow{\cong} \mathrm{Coker}(\mathrm{Hom}(j,M)).$$

Proof: We know from the result on lifting resolutions that we can **lift the identity map on A** to a map of resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i} & E_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{j} & F_0 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

The other way around we get a lift

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i} & E_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \| & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{j} & F_0 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

We write E_* for the complex $0 \to E_1 \to E_0 \to 0$ and F_* for the complex $0 \to F_1 \to F_0 \to 0$.

Composition yields maps $f_* \circ g_* \colon E_* \to E_*$ and $g_* \circ f_* \colon F_* \to F_*$ which lift the identity map on A. But since the identity maps on E_* and F_*, respectively, also lift the identity on A, the lemma of a previous lecture implies that there is a chain homotopy h_E between $f_* \circ g_*$ and 1_{E_*} and a chain homotopy h_F between $g_* \circ f_*$ and 1_{F_*}.

Now we apply $\mathrm{Hom}(-,M)$. Then h_E induces a cochain homotopy $\mathrm{Hom}(h_E,M)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(E_0,M) & \longrightarrow & \mathrm{Hom}(E_1,M) & \longrightarrow & 0 \\ & & {\scriptstyle 1_{\mathrm{Hom}(E_0,M)}} \downarrow\downarrow {\scriptstyle g_0^* \circ f_0^*} & \overset{h^*}{\swarrow} & {\scriptstyle g_1^* \circ f_1^*} \downarrow\downarrow {\scriptstyle 1_{\mathrm{Hom}(E_1,M)}} & & \\ 0 & \longrightarrow & \mathrm{Hom}(E_0,M) & \longrightarrow & \mathrm{Hom}(E_1,M) & \longrightarrow & 0. \end{array}$$

between

$$\mathrm{Hom}(f_* \circ g_*,M) = \mathrm{Hom}(g_*,M) \circ \mathrm{Hom}(f_*,M) \text{ and } \mathrm{Hom}(1_{E_*},M) = 1_{\mathrm{Hom}(E_*,M)}.$$

Whereas h_F induces a cochain homotopy $\mathrm{Hom}(h_F,M)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(F_0,M) & \longrightarrow & \mathrm{Hom}(F_1,M) & \longrightarrow & 0 \\ & & {\scriptstyle 1_{\mathrm{Hom}(F_0,M)}} \downarrow\downarrow {\scriptstyle f_0^* \circ g_0^*} & \overset{h^*}{\swarrow} & {\scriptstyle f_1^* \circ g_1^*} \downarrow\downarrow {\scriptstyle 1_{\mathrm{Hom}(F_1,M)}} & & \\ 0 & \longrightarrow & \mathrm{Hom}(F_0,M) & \longrightarrow & \mathrm{Hom}(F_1,M) & \longrightarrow & 0. \end{array}$$

between

$$\mathrm{Hom}(g_* \circ f_*,M) = \mathrm{Hom}(f_*,M) \circ \mathrm{Hom}(g_*,M) \text{ and } \mathrm{Hom}(1_{F_*},M) = 1_{\mathrm{Hom}(F_*,M)}.$$

Thus, the maps induced by the compositions on cohomology are equal to the respective identity maps. In other words, the induced maps f^* and g^* on cohomology are mutual inverses to each other.

Moreover, since the chain homotopy type of f_* and g_* is unique by the lemma of the lecture on Tor, they induce in fact a unique isomorphism

$$\mathrm{Coker}(\mathrm{Hom}(i,M)) = H^1(\mathrm{Hom}(E_*,M) \xrightarrow{\cong} H^1(\mathrm{Hom}(F_*,M) = \mathrm{Coker}(\mathrm{Hom}(j,M)).$$

QED

Lemma: Induced exact sequence

Let M be an abelian group and assume we have a short exact sequence of abelian groups

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

Then there is an associated long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}(C,M) \longrightarrow \mathrm{Hom}(B,M) \longrightarrow \mathrm{Hom}(A,M) \\ \longrightarrow \mathrm{Ext}(C,M) \longrightarrow \mathrm{Ext}(B,M) \longrightarrow \mathrm{Ext}(A,M) \longrightarrow 0. \end{array}$$

Proof: Let $0 \to E_1 \to E_0 \to A \to 0$ be a free resolution of A, and $0 \to F_1 \to F_0 \to C \to 0$ be a free resolution of C. This data gives us a free resolution of B by forming direct sums:

$$0 \to E_1 \oplus F_1 \to E_0 \oplus F_0 \to B \to 0.$$

By the result of the previous lecture, we can lift the maps in the short exact sequence to maps of resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \dashrightarrow & E_1 \oplus F_1 & \dashrightarrow & F_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_0 & \dashrightarrow & E_0 \oplus F_0 & \dashrightarrow & F_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The horizontal sequences are short exact, since the middle term is a direct sum of the other terms. Hence we get a short exact sequence of chain complexes

$$0 \to E_* \to E_* \oplus F_* \to F_* \to 0.$$

Since all three complexes consist of free abelian groups, applying $\mathrm{Hom}(-,M)$ yields a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}(F_*,M) \to \mathrm{Hom}(E_* \oplus F_*,M) \to \mathrm{Hom}(E_*,M) \to 0.$$

By taking cohomology of these cochain compelxes, we get an induced long exact sequence of the associated cohomology groups. This is the desired exact sequence together with the identifiction of H^1 with Ext and H^0 with Hom of the previous lemma. **QED**

This lemma also gives a hint to where the name Ext comes from:

Ext and extensions

• We can think of a short exact sequence of abelian groups

$$0 \to A \to B \to M \to 0$$

as an **extension of M by A**. We can then say that two extensions are **euqivalent** if they fit into an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow \cong & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

• Note that we can always construct a trivial extension by taking the direct sum of A and M:

$$0 \to A \xrightarrow{(1,0)} A \oplus M \to M \to 0.$$

Recall that we say that such a sequence splits.

• The group $\mathrm{Ext}(A,M)$ measures how far extensions of M by A can be from being from the trivial extension. For, we have

$$\mathrm{Ext}(A,M) = 0 \iff \text{every extension of } M \text{ by } A \text{ splits.}$$

Proof: Given an extension, applying $\mathrm{Hom}(-,M)$ yields an exact sequence

$$\mathrm{Hom}(B,M) \to \mathrm{Hom}(M,M) \to \mathrm{Ext}(A,M).$$

Thus the identity map $M \xrightarrow{1} M$ lifts to a map $B \to M$ if $\mathrm{Ext}(A,M) = 0$. But that is equivalent to that the initial short exact sequence splits. **QED**

• Now one can show in general that $\mathrm{Ext}(A,M)$ is in bijection with the **set of all equivalence classes** of extensions of M by A.
• For example, we computed $\mathrm{Ext}(\mathbb{Z}/2,\mathbb{Z}/2) = \mathbb{Z}/2$. The trivial element in Ext corresponds to the trivial extension

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

whereas the non-trivial element corresponds to the extension

$$0 \to \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

We summarize some further **properties of Ext**:

- Ext is **functorial**: For any homomorphisms of abelian groups $A \to A'$ and $M \to M'$, there are homomorphisms
$$\mathrm{Ext}(A',M) \to \mathrm{Ext}(A,M) \text{ and } \mathrm{Ext}(A,M) \to \mathrm{Ext}(A,M').$$
This follows from the lemma on liftings of resolutions.

- If A is **free**, then $\mathrm{Ext}(A,M) = 0$ for any abelian group A. This follows from the fact that $0 \to A \xrightarrow{1} A \to 0$ is a free resolution of A.

- Ext commutes with **finite** direct sums, i.e.,
$$\mathrm{Ext}(A_1 \oplus A_2,M) \cong \mathrm{Ext}(A_1,M) \oplus \mathrm{Ext}(A_2,M).$$
This follows from the fact that the direct sum of free resolutions of each A_1 and A_2 is a free resolution of $A_1 \oplus A_2$.

- Let A be a finitely generated abelian group and let $T(A)$ denote its torsion subgroup. Since $\mathrm{Ext}(\mathbb{Z}/m,\mathbb{Z}) = \mathbb{Z}/m$, the structure theorem for finitely generated abelian groups and the previous two points imply that
$$\mathrm{Ext}(A,\mathbb{Z}) \cong T(A).$$

Now we prove the main result which connects homology and cohomology and answers the question we raised last time about the Kronecker map κ:

Theorem: Universal Coefficient Theorem

Let C_* be a chain complex of **free** abelian groups and let M be an abelian group. We write $C^* = \mathrm{Hom}(C_*,M)$ for the induced cochain complex.
Then there are natural short exact sequences

$$0 \to \mathrm{Ext}(H_{n-1}(C_*),M) \to H^n(C^*) \xrightarrow{\kappa} \mathrm{Hom}(H_n(C_*),M) \to 0$$

for all n. These sequences split, but the splitting is not natural.

The proof builds on the same ideas as for the UCT in homology. But let us do it anyway to get more practice.

Proof: • We write Z_n for the **kernel** and B_{n-1} for the **image** of the differential $d\colon C_n \to C_{n-1}$. Since C_n and C_{n-1} are **free**, both Z_n and B_{n-1} are **free as well**.

By definition of Z_n and B_n, the restriction of the **differentials** to these groups **vanish**. This implies that (Z_*,d) and (B_*,d) are chain complexes (with trivial differentials).

Hence we get a short exact sequence of chain complexes

$$0 \to Z_* \to C_* \xrightarrow{d} B_{*-1} \to 0. \tag{1}$$

• Since all groups in these chain complexes are **free**, applying the functor $\mathrm{Hom}(-,M)$ yields again a **short exact sequence of cochain complexes**

$$0 \to \mathrm{Hom}(B_{*-1},M) \to \mathrm{Hom}(C_*,M) \to \mathrm{Hom}(Z_*,M) \to 0.$$

This follows from the lemma we proved in the previous lecture.

• Since the **differentials** in Z_* and B_* are **trivial**, the nth cohomology of $\mathrm{Hom}(B_{*-1},M)$ is just $\mathrm{Hom}(B_{n-1},M)$, and the nth cohomology of $\mathrm{Hom}(Z_*,M)$ is just $\mathrm{Hom}(Z_n,M)$.

Hence the long exact sequence in cohomology associated to the short exact sequence (1) looks like

$$\cdots \to \mathrm{Hom}(Z_{n-1},M) \xrightarrow{\partial} \mathrm{Hom}(B_{n-1},M) \xrightarrow{d^*} H^n(\mathrm{Hom}(C_*,M)) \xrightarrow{i^*} \mathrm{Hom}(Z_n,M) \xrightarrow{\partial} \mathrm{Hom}(B_n,M) \to \cdots$$

• The **connecting homomorphism** $\mathrm{Hom}(Z_n,M) \xrightarrow{\partial} \mathrm{Hom}(B_n,M)$ in this sequence is $i_n^* = \mathrm{Hom}(i_n,M)$, where $i_n\colon B_n \hookrightarrow Z_n$ denotes the inclusion. For, the connecting homomorphism is defined as follows. Consider the maps

$$\begin{array}{ccccc} & & \mathrm{Hom}(C_n,M) & \longrightarrow & \mathrm{Hom}(Z_n,M) \\ & & \big\downarrow{\scriptstyle\delta} & & \\ \mathrm{Hom}(B_n,M) & \xrightarrow[\delta]{} & \mathrm{Hom}(C_{n+1},M). & & \end{array}$$

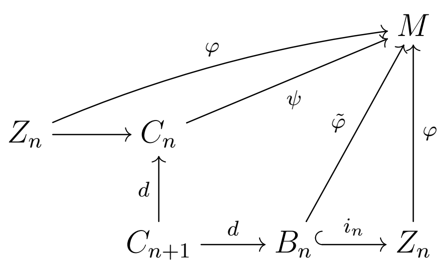
A preimage of $\varphi \in \mathrm{Hom}(Z_n,M)$ is any map $\psi\colon C_n \to M$ which restricts to Z_n. Such a preimage exists since the upper horizontal map is surjective. Then ψ is mapped to $\psi \circ d \in \mathrm{Hom}(C_{n+1}, M)$ by δ. Since every boundary is a cycle, we have

$$\psi \circ d = \varphi \circ d.$$

Now it remains to find a map $\tilde{\varphi}\colon B_n \to M$ such that

$$\psi \circ d = \varphi \circ d = \tilde{\varphi} \circ d.$$

There is a canonical candidate for $\tilde{\varphi}$, namely the restriciton of φ to B_n. This is exactly $i_n^*(\varphi)$.



• A long exact sequence can always be **cut into short exact sequences** of the from

$$0 \to \mathrm{Coker}(\mathrm{Hom}(i_{n-1}, M)) \to H_n(C^*) \to \mathrm{Ker}\,(\mathrm{Hom}(i_n, M)) \to 0.$$

Since the functor $\mathrm{Hom}(-, M)$ sends **cokernels to kernels**, the kernel on the right-hand side is just

$$\mathrm{Ker}\,(\mathrm{Hom}(i_n, M)) = \mathrm{Hom}(\mathrm{Coker}(i_n), M) = \mathrm{Hom}(Z_n/B_n, M) = \mathrm{Hom}(H_n(C_*), M).$$

For the cokernel on the left-hand side, we use that

$$B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C_*) \to 0$$

is a **free resolution** of $H_{n-1}(C_*)$.

Hence, after applying $\mathrm{Hom}(-, M)$, we get an **exact sequence**

$$0 \to \mathrm{Hom}(H_{n-1}(C_*), M) \to \mathrm{Hom}(B_{n-1}, M) \xrightarrow{i_{n-1}^*} \mathrm{Hom}(Z_{n-1}, M) \to \mathrm{Coker}(\mathrm{Hom}(i_{n-1}, M)) \to 0.$$

Thus, since $\mathrm{Ext}(-, M)$ is **independent of the chosen free resolution**,

$$\mathrm{Coker}(\mathrm{Hom}(i_{n-1}, M)) = \mathrm{Ext}(H_{n-1}(C_*), M).$$

Finally, to obtain the asserted **splitting** we use that subgroups of free abelian groups are free. That implies that sequence (1) splits and we have

$$C_n \cong Z_n \oplus B_{n-1}.$$

Applying $\mathrm{Hom}(-,M)$ yields

$$\mathrm{Hom}(C_n,M) \cong \mathrm{Hom}(Z_n,M) \oplus \mathrm{Hom}(B_{n-1},M).$$

Now one has to work a little bit more to get that this induces a direct sum decomposition in homology.

It remains to check that the right-hand map in the theorem is in fact the previously defined map κ. We leave this as an exercise. **QED**

Now we can prove the result we claimed in the previous lecture:

Corollary: From isos in homology to isos in cohomology

Let C_* and D_* be two chain complexes of **free** abelian groups. Let M be an abelian group.
Assume that there is a map $C_* \xrightarrow{\varphi} D_*$ which induces an **isomorphism in homology**

$$\varphi_* \colon H_*(C_*) \xrightarrow{\cong} H_*(D_*).$$

Then this map also induces an **isomorphism in cohomology** with coefficients in M

$$\varphi^* \colon H^*(C^*) \xrightarrow{\cong} H^*(D^*).$$

Proof: Since the construction of the long exact sequence we used in the proof of the theorem is functorial, we see that φ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C_*),M) & \longrightarrow & H^n(C^*) & \longrightarrow & \mathrm{Hom}(H_n(C_*),M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\varphi_*)^*} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle (\varphi_*)^*} & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(D_*),M) & \longrightarrow & H^n(D^*) & \longrightarrow & \mathrm{Hom}(H_n(D_*),M) & \longrightarrow & 0 \end{array}$$

The assumption that φ_* induces an isomorphism implies that the two outer vertical maps are isomomorphisms. The Five-Lemma implies that the middle vertical map φ^* is an isomorphism as well. **QED**

Our previous oberservations about Ext and torsion subgroups together with the theorem imply:

Corollary: Computing cohomology from homology

Assume that the homology groups $H_n(C_*)$ and $H_{n-1}(C_*)$ of the chain complex are finitely generated. Let $T_n \subseteq H_n(C_*)$ and $T_{n-1} \subseteq H_{n-1}(C_*)$ denote the torsion subgroups. Then we can calculate the integral cohomology of $C^* = \mathrm{Hom}(C_*,\mathbb{Z})$ by

$$H^n(C^*;\mathbb{Z}) \cong (H_n(C_*)/T_n) \oplus T_{n-1}.$$

Since the **singular chain complex** $S_*(X,A)$ is an example of a chain complex of **free** abelian groups, the theorem implies:

Corollary: UCT for singular cohomology

For each pair of spaces (X,A) there are split short exact sequences

$$0 \to \mathrm{Ext}(H_{n-1}(X,A),M) \to H^n(X,A;M) \to \mathrm{Hom}(H_n(X,A),M) \to 0$$

for all n, and these sequences are natural with respect to maps of pairs $(X,A) \to (Y,B)$.

As a final remark, we mention that there are versions of Ext for the category of R-modules for any ring. The corresponding Ext-groups $\mathrm{Ext}_R(M,N)$ will depend on the ring R as well as on the modules M and N. Moreover, there might be non-trivial higher Ext-groups $\mathrm{Ext}^i_R(M,N)$ for $i \geq 2$, in general.

But the theory is very similar to the case of abelian groups, i.e., $\mathbb{Z}$-modules, as long as R is a principal ideal domain (PID). For, then submodules of free R-modules are still free over R (which is not true in general). Hence free resolutions of length two exist, and higher Ext groups vanish also in this case.

For example, fields are examples of PIDs. However, note that, for example, $\mathrm{Ext}(\mathbb{Z}/2,\mathbb{Z}/2) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2) = \mathbb{Z}/2$ whereas $\mathrm{Ext}^1_{\mathbb{Z}/2}(\mathbb{Z}/2,\mathbb{Z}/2) = 0$. Hence the base rings matter.