MA3403 Algebraic Topology
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Lecture 20

20. Cup products in cohomology

We are now going to define the **additional algebraic structure** on cohomology that we promised earlier: multiplication.

There are many different ways to define a product structure in cohomology. As always, each of these ways has its advantages and disadvanteges. We will take a direct path to the construction. This has the advantage to get a product right away. The price we are going to pay is that we will have to work harder for some results later. Note also that, even though we emphasized the importance of the diagonal map in a previous lecture, this will not become clear from our direct approach today. Though it matters nevertheless. :)

What we do take advantage of and which would not work for singular chains is that a cochain is by definition a map to a ring. So we can multiply images of cochains. Hence we could try to multiply cochains pointwise. We will just need to figure out the images of which points we need to multiply.

We need to assume that we work with coefficients in a ring R. We will always assume that R is commutative and that there is a neutral element 1 for multiplication (even though not all arguments require all these assumptions). Our main examples will be, of course, $\mathbb{Z}$, $\mathbb{Z}/n$, $\mathbb{Q}$.

Definition: Cup products

For cochains $\varphi \in S^p(X;R)$ and $\psi \in S^q(X;R)$, we define the **cup product** $\varphi \cup \psi \in S^{p+q}(X;R)$ to be the cochain whose value on the $p+q$-simplex $\sigma\colon \Delta^{p+q} \to X$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma_{|[e_0,\dots,e_p]})\psi(\sigma_{|[e_p,\dots,e_{p+q}]})$$

where the product is taken in R (here it comes already quite handy that we work with coefficients in a ring).
Note: The symbol $\sigma_{|[e_0,\dots,e_p]}$ refers to the restriction of σ to the **front face** of Δ^{p+q}

$$\sigma_{|[e_0,\dots,e_p]}\colon \Delta^p \hookrightarrow \Delta^{p+q} \xrightarrow{\sigma} X,\ \sigma_{|[e_0,\dots,e_p]}(t_0,\dots,t_p) = \sigma(t_0,\dots,t_p,0,\dots,0).$$

Similarly, the symbol $\sigma_{|[e_p,\ldots,e_{p+q}]}$ refers to the restriction of σ to the **back face** of Δ^{p+q}

$$\sigma_{|[e_p,\ldots,e_{p+q}]}\colon \Delta^q \hookrightarrow \Delta^{p+q} \xrightarrow{\sigma} X,\ \sigma_{|[e_p,\ldots,e_{p+q}]}(t_0,\ldots,t_q) = \sigma(0,\ldots,0,t_0,\ldots,t_q).$$

We would can think of this construction as evaluating φ on the front face of σ, evaluating ψ on the back face of σ, and then taking the product of the two results.

To make sure that this construction yields something meaningful on the level of cohomology we need to check a couple of things.

Lemma: Cup products and coboundaries

For cochains $\varphi \in S^p(X;R)$ and $\psi \in S^q(X;R)$, we have

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^p \varphi \cup \delta\psi.$$

For the next proof and the remaining lecture, recall that the notation $\hat{e}_i$ means that the vertex e_i is omitted.

Proof: By definition, for a simplex $\sigma \in \Delta^{p+q+1} \to X$, we have

$$\begin{aligned}
\delta(\varphi \cup \psi)(\sigma) &= (\varphi \cup \psi)(\partial\sigma) \\
&= (\varphi \cup \psi)\left(\sum_{i=0}^{p+q+1} (-1)^i \sigma_{|[e_0,\ldots,\hat{e}_i,\ldots,e_{p+q+1}]}\right) \\
&= \sum_{i=0}^{p+1} (-1)^i \varphi(\sigma_{|[e_0,\ldots,\hat{e}_i,\ldots,e_{p+1}]})\psi(\sigma_{|[e_{p+1},\ldots,e_{p+q+1}]}) \\
&+ \sum_{i=p}^{p+q+1} (-1)^i \varphi(\sigma_{|[e_0,\ldots,e_{p+1}]})\psi(\sigma_{|[e_p,\ldots,\hat{e}_i,\ldots,e_{p+q+1}]})
\end{aligned}$$

where the split into the two sums is justified by the fact that the last term of the first sum is exactly (-1)-times the first term of the second sum.

Now it remains to observe that these two sums are exactly the definition of $(\delta\varphi \cup \psi)(\sigma)$ and $(-1)^p(\varphi \cup \delta\psi)(\sigma)$.

QED

We would like this construction to **descend to cohomology**. Therefore, we need to check:

- Assume that φ and ψ are **cocycles**, i.e., $\delta\varphi = 0$ and $\delta\psi = 0$. Then $\varphi \cup \psi$ is a **cocycle**, since
$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi \pm \varphi \cup \delta\psi = 0 \pm 0 = 0.$$
- Assume that φ is a **cocycle**, i.e., $\delta\varphi = 0$, and ψ is a **coboundary**, i.e., there is a cochain ψ' with $\psi = \delta\psi'$. Then $\varphi \cup \psi$ is a **coboundary**, since
$$\begin{aligned} \delta(\varphi \cup \psi') &= \delta\varphi \cup \psi \pm \varphi \cup \delta\psi' \\ &= 0 \pm \varphi \cup \psi. \end{aligned}$$
In other words, $\varphi \cup \psi$ is the image of $\pm\varphi \cup \psi'$ under δ.
- Similarly, we can show that $\varphi \cup \psi$ is a **coboundary** if φ is a **coboundary** and ψ is a **cocycle**.

Thus we have shown:

Cup product in cohomology

For any p and q, the cup product defines a map on cohomology groups
$$H^p(X; R) \times H^q(X; R) \xrightarrow{\cup} H^{p+q}(X; R).$$

As we can easily check by evaluating on a simplex:

- The product is **associative**, i.e.,
$$(\varphi \cup \psi) \cup \xi = \varphi \cup (\psi \cup \xi).$$
- The product is **distributive**, i.e.,
$$\varphi \cup (\psi + \xi) = \varphi \cup \psi + \varphi \cup \xi.$$
- The 0-cocycle $\epsilon \in H^0(X; R)$ defined by taking value 1 for every 0-simplex is a **neutral element**, i.e.,
$$\epsilon \cup \varphi = \varphi = \varphi \cup \epsilon \text{ for all } \varphi \in H^p(X; R).$$

Before we address commutativity, let us first check how the cup product behaves under induced homomorphisms:

Proposition: Cup products are natural

Let $f\colon X \to Y$ be a continuous map and let $f^*\colon H^{p+q}(Y; R) \to H^{p+q}(X; R)$ be the induced homomorphism. Then
$$f^*(\varphi \cup \psi) = f^*\varphi \cup f^*\psi$$

for all $\varphi \in H^p(Y;R)$ and $\psi \in H^q(Y;R)$.

Proof: We can check this formula already on the level of cochains. For, given a simplex $\sigma\colon \Delta^{p+q} \to X$, we get

$$\begin{aligned}(f^*\varphi \cup f^*\psi)(\sigma) &= f^*\varphi(\sigma_{|[e_0,\dots,e_p]})f^*\psi(\sigma_{|[e_p,\dots,e_{p+q}]}) \\ &= \varphi(f\circ\sigma_{|[e_0,\dots,e_p]})\psi(f\circ\sigma_{|[e_p,\dots,e_{p+q}]}) \\ &= (\varphi\cup\psi)(f\circ\sigma) \\ &= f^*(\varphi\cup\psi)(\sigma).\end{aligned}$$

QED

Now we are going to address the remaining natural property of multiplication: commutativity. It will turn out that the cup product is not exactly symmetric. This is annoying, but so is life sometimes. Howver, it is very close to being symmetric. For the next result, recall that we assume that R itself is commutative.

Theorem: Cup products are graded commutative

For any classes $\varphi \in H^p(X;R)$ and $\psi \in H^q(X;R)$, we have

$$\varphi\cup\psi = (-1)^{pq}(\psi\cup\varphi).$$

The proof of this result will require some efforts. Before we think about it, let us **collect some consequences** of this theorem and of the construction of the cup product in general.

- Many cup products are trivial just for degree reasons. For classes $\varphi \in H^p(X;R)$ and $\psi \in H^q(X;R)$ with $p+q$ such that $H^{p+q}(X;R) = 0$, then $\varphi\cup\psi = 0$ no matter what.
- This can happen for example if X is a finite cell complex.
- If $\varphi \in H^p(X;R)$ and p is **odd**, then

$$\varphi^2 = (-1)^{p^2}\varphi^2 = -\varphi^2.$$

Therefore, $2\varphi^2 = 0$ in $H^{2p}(X;R)$.

If R is **torsion-free** or if R is a field of **characteristic different from** 2, this implies

$$\varphi^2 = 0.$$

Proof for a special case: In order to find a strategy for the proof of the theorem, let us look at a special case. So let $[\varphi], [\psi] \in H^1(X;R)$, and let $\sigma \colon \Delta^2 \to X$ be a 2-simplex.

The respective cup products are then determined by their effect on a 2-simplex $\sigma \colon \Delta^2 \to X$:

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma_{|[e_0,e_1]})\psi(\sigma_{|[e_1,e_2]}).$$

and

$$\begin{aligned}(\psi \cup \varphi)(\sigma) &= \psi(\sigma_{|[e_0,e_1]})\varphi(\sigma_{|[e_1,e_2]}) \\ &= \varphi(\sigma_{|[e_1,e_2]})\psi(\sigma_{|[e_0,e_1]})\end{aligned}$$

where we use that R is commutative.

Hence in order to show that these two expressions are related, we would like to reshuffle the vertices. As a first attempt we are going to reverse the order of all vertices, i.e., we replace σ with $\bar{\sigma}$ defined by

$$\sigma(e_i) = \sigma(e_{2-i}).$$

We will also use the notation

$$\sigma_{|[e_2,e_1,e_0]} = \bar{\sigma}$$

which expresses

$$\bar{\sigma}(t_0,t_1,t_2) = \sigma(t_2,t_1,t_0).$$

Recall that, a long time ago, we showed that reversing the order of vertices on a 1-simplex corresponds, at least up to boundaries, multiplying the simplex with (-1).

Hence we should consider inserting a sign as well. So we define maps

$$\rho_1 \colon S_1(X) \to S_1(X) \text{ and } \rho_2 \colon S_2(X) \to S_2(X)$$

both defined by sending a simplex σ to $-\bar{\sigma}$.

Surprisingly, the comparison of the two cup products after taking pullbacks along the ρs becomes easier. For,

$$\begin{aligned}(\rho_1^*\varphi \cup \rho_1^*\psi)(\sigma) &= \varphi(-\sigma_{|[e_1,e_0]})\psi(-\sigma_{|[e_2,e_1]}) \\ &= \varphi(\sigma_{|[e_1,e_0]})\psi(\sigma_{|[e_2,e_1]})\end{aligned}$$

and

$$\begin{aligned}(\rho_2^*(\psi \cup \varphi))(\sigma) &= -\psi(\sigma_{|[e_2,e_1]})\varphi(\sigma_{|[e_1,e_0]}) \\ &= -\varphi(\sigma_{|[e_1,e_0]})\psi(\sigma_{|[e_2,e_1]})\end{aligned}$$

using that R is commutative.

Hence we get

$$\rho_1^*\varphi \cup \rho_1^*\psi = -\rho_2^*(\psi \cup \varphi).$$

In other words, up to ρ_1^* and ρ_2^* we have shown the desired equality.

Now we remember that we are still on the level of cochains. The theorem is about an equality of cohomology classes. Hence all we need to show is that ρ_1^* and ρ_2^* will vanish once we pass to cohomology.

This leads to the idea to show that ρ_1 and ρ_2 are part of a chain map which is chain homotopic to the identity. So let us try to do this.

First, we want that ρ_1 and ρ_2 commute with the boundary operator:

$$\begin{aligned}(\rho_1 \circ \partial)(\sigma) &= \rho(\sigma_{|[e_1,e_2]} - \sigma_{|[e_0,e_2]} + \sigma_{|[e_0,e_1]})\\ &= -\sigma_{|[e_2,e_1]} + \sigma_{|[e_2,e_0]} - \sigma_{|[e_1,e_0]})\\ &= \partial(-\sigma_{|[e_2,e_1,e_0]})\\ &= (\partial \circ \rho_2)(\sigma).\end{aligned}$$

Now we would like to construct a chain homotopy between ρ and the identity chain map.

The **idea** is to interpolate between the identity and ρ by permuting the vertices one after the other until the order is completely reversed. Then we sum up all these maps. Along the way we need to introduce some signs.

Before we can define maps, we need to recall the prism operator we used to construct a chain homotopy which showed that singular homology is homotopy invariant.

These were maps

$$p_i^n \colon \Delta^{n+1} \to \Delta^n \times [0,1]$$

determined by

$$p_i^n(e_k) = \begin{cases} (e_k,0) & \text{if } 0 \le k \le i \\ (e_{k-1},1) & \text{if } k > i. \end{cases}$$

Let us write $e_k^0 := (e_k,0)$ and $e_k^1 := (e_k,1)$. Given an n-simplex σ, we would like to compose it with p_i^n and also permute vertices.

Consider the **permutation of simplices**

$$\Delta^{n+1} \xrightarrow{s_i} \Delta^{n+1},\ (e_0, \dots, e_{n+1}) \mapsto (e_0, \dots, e_i, e_{n+1}, \dots, e_{i+1}).$$

To simplify the notation, we are going to write

$$\sigma_{|[e_0,\dots,e_i,e_n,\dots,e_i]} \colon \Delta^{n+1} \to X$$

for the $n+1$-simplex defined by the composition of s_i with

$$\Delta^{n+1} \xrightarrow{p_i^n} \Delta^n \times [0,1] \xrightarrow{\mathrm{pr}} \Delta^n \xrightarrow{\sigma} X.$$

Now we **define three maps**

$$h_0 \colon S_0(X) \to S_1(X),\ \sigma \mapsto \sigma_{|[e_0,e_0]},$$

for $n = 0$,

$$h_1 \colon S_1(X) \to S_2(X),\ \sigma \mapsto -\sigma_{|[e_0,e_1,e_0]} - \sigma_{|[e_0,e_1,e_1]},$$

for $n = 1$, and

$$h_2 \colon S_2(X) \to S_3(X),\ \sigma \mapsto -\sigma_{|[e_0,e_2,e_1,e_0]} + \sigma_{|[e_0,e_1,e_2,e_1]} + \sigma_{|[e_0,e_1,e_2,e_2]}$$

for $n = 2$.

(You will see that it does not matter so much how these maps are defined. It is just important that we have some consistent way of moving from $S_n(X)$ to $S_{n+1}(X)$.)

For a 1-simplex $\sigma \colon \Delta^1 \to X$, we compute

$$\begin{aligned}(\partial \circ h_1)(\sigma) &= \partial(-\sigma_{|[e_0,e_1,e_0]} - \sigma_{|[e_0,e_1,e_1]}) \\ &= -(\sigma_{|[e_1,e_0]} - \sigma_{|[e_0,e_0]} + \sigma_{|[e_0,e_1]}) \\ &\quad - (\sigma_{|[e_1,e_1]} - \sigma_{|[e_0,e_1]} + \sigma_{|[e_0,e_1]})\end{aligned}$$

and

$$\begin{aligned}(h_0 \circ \partial)(\sigma) &= h_0(\sigma_{|[e_1]} - \sigma_{|[e_0]}) \\ &= \sigma_{|[e_1,e_1]} - \sigma_{|[e_0,e_0]}.\end{aligned}$$

Taking these terms together we get

$$\begin{aligned}(\partial \circ h_1)(\sigma) + (h_0 \circ \partial)(\sigma) &= -\sigma_{|[e_1,e_0]} + \sigma_{|[e_0,e_0]} - \sigma_{|[e_0,e_1]} \\ &\quad - \sigma_{|[e_1,e_1]} + \sigma_{|[e_0,e_1]} - \sigma_{|[e_0,e_1]} \\ &\quad + \sigma_{|[e_1,e_1]} - \sigma_{|[e_0,e_0]} \\ &= -\sigma_{|[e_1,e_0]} - \sigma_{|[e_0,e_1]} \\ &= \rho(\sigma) - \sigma.\end{aligned}$$

Thus, we have shown the homotopy relation

$$\partial \circ h_1 + h_0 \circ \partial = \rho_1 - \mathrm{id}.$$

Similarly, for a 2-simplex $\sigma\colon \Delta^2 \to X$, we calculate

$$\begin{aligned}
(\partial \circ h_2)(\sigma) &= \partial(-\sigma_{|[e_0,e_2,e_1,e_0]} - \sigma_{|[e_0,e_1,e_2,e_1]} + \sigma_{|[e_0,e_1,e_2,e_2]}) \\
&= -(\sigma_{|[e_2,e_1,e_0]} - \sigma_{|[e_0,e_1,e_0]} + \sigma_{|[e_0,e_2,e_0]} - \sigma_{|[e_0,e_2,e_1]}) \\
&+ (\sigma_{|[e_1,e_2,e_1]} - \sigma_{|[e_0,e_2,e_1]} + \sigma_{|[e_0,e_1,e_1]} - \sigma_{|[e_0,e_1,e_2]}) \\
&+ (\sigma_{|[e_1,e_2,e_2]} - \sigma_{|[e_0,e_2,e_2]} + \sigma_{|[e_0,e_1,e_2]} - \sigma_{|[e_0,e_1,e_2]}) \\
&= -\sigma_{|[e_2,e_1,e_0]} + \sigma_{|[e_0,e_1,e_0]} - \sigma_{|[e_0,e_2,e_0]} + \sigma_{|[e_1,e_2,e_1]} + \sigma_{|[e_0,e_1,e_1]} \\
&+ \sigma_{|[e_1,e_2,e_2]} - \sigma_{|[e_0,e_2,e_2]} - \sigma_{|[e_0,e_1,e_2]}
\end{aligned}$$

and

$$\begin{aligned}
(h_1 \circ \partial)(\sigma) &= h_1(\sigma_{|[e_1,e_2]} - \sigma_{|[e_0,e_2]} + \sigma_{|[e_0,e_1]}) \\
&= -\sigma_{|[e_1,e_2,e_1]} - \sigma_{|[e_1,e_2,e_2]} + \sigma_{|[e_0,e_2,e_0]} + \sigma_{|[e_0,e_2,e_2]} - \sigma_{|[e_0,e_1,e_0]} - \sigma_{|[e_0,e_1,e_1]}.
\end{aligned}$$

This gives

$$(\partial \circ h_2 + h_1 \circ \partial)(\sigma) = \rho_2(\sigma) - \sigma.$$

Thus, we have again shown the homotopy relation

$$\partial \circ h_2 + h_1 \circ \partial = \rho_2 - \mathrm{id}.$$

This indicates that ρ_1 and ρ_2 are part of a chain map which is chain homotopic to the identity.

To prove the general case, we adapt this strategy we developed for $n = 1$.

Proof of the theorem: When we evaluate the two cup products on a simplex $\sigma\colon \Delta^{p+q} \to X$, they differ only by a permutation of the vertices of σ. The idea of the proof consists of

- chosing a nice permuation which simplifies notation and computations,
- and then to construct a chain homotopy between the resulting cup product and the identity.

Now let us get to work:

• For an n-simplex σ, let $\bar{\sigma}$ be the n-simplex obtained by composing it first with the linear transformation which **reverses the order** of the vertices.

In other words,

$$\bar{\sigma}(e_i) = \sigma(e_{n-i}) \text{ for all } i = 0, \ldots, n.$$

or

$$\bar{\sigma}(t_0, \ldots, t_n) = \sigma(t_n, \ldots, t_0).$$

We will also use the notation

$$\sigma_{|[e_n,\ldots,e_0]} = \bar{\sigma}.$$

For this will make it easier to combine it with the restriction to the $n-1$-dimensional faces of Δ^n.

• Since the reversal of the vertices is the product of $n + (n-1) + \cdots + 1 = n(n+1)/2$ many transpositions, our test case motivates the definition of the homomorphism

$$\rho_n \colon S_n(X) \to S_n(X),\ \rho_n(\sigma) = (-1)^{\frac{n(n+1)}{2}}\bar{\sigma}.$$

To simplify the notation we will $\epsilon_n := (-1)^{\frac{n(n+1)}{2}}$.

• **We claim** that ρ is a **map of chain complexes** which is chain **homotopic to the identity** map. Assuming that the claim is true we can finish the proof of the theorem as follows.

For $\sigma \colon \Delta^{p+q} \to X$, we can then calculate

$$\begin{aligned}(\rho^*\varphi \cup \rho^*\psi)(\sigma) &= \varphi(\epsilon_p \sigma_{|[e_p,\ldots,e_0]})\psi(\epsilon_q \sigma_{|[e_{p+q},\ldots,e_p]}) \\ &= \epsilon_p\epsilon_q \varphi(\sigma_{|[e_p,\ldots,e_0]})\psi(\sigma_{|[e_{p+q},\ldots,e_p]})\end{aligned}$$

and

$$(\rho^*(\psi \cup \varphi))(\sigma) = \epsilon_{p+q}\psi(\sigma_{|[e_{p+q},\ldots,e_p]})\varphi(\sigma_{|[e_p,\ldots,e_0]}).$$

Now we observe

$$\begin{aligned}\frac{(p+q)(p+q+1)}{2} &= \frac{p^2 + 2pq + q^2 + p + q}{2} \\ &= \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + \frac{2pq}{2} \\ &= \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + pq.\end{aligned}$$

Thus

$$\epsilon_{p+q} = (-1)^{pq}\epsilon_p\epsilon_q.$$

We conclude from these two computations

$$\rho^*\varphi \cup \rho^*\psi = (-1)^{pq}\rho^*(\psi \cup \varphi).$$

Now we use that ρ is chain homotopic to the identity. That implies that when we pass to cohomology classes, ρ^* is the identity and we obtain the desired identity

$$\varphi \cup \psi = (-1)^{pq}(\psi \cup \varphi).$$

Now we are going to prove the claims we made:

- ρ is a chain map.

We need to show that $\partial \circ \rho = \rho \circ \partial$. For an n-simplex σ we calculate the effects of the two maps:

$$\begin{aligned}
(\rho \circ \partial)(\sigma) &= \rho(\sum_{i=0}^{n}(-1)^i \sigma_{|[e_0,\dots,\hat{e}_i,\dots,e_n]}) \\
&= \epsilon_{n-1}\sum_{i=0}^{n}(-1)^i \sigma_{|[e_n,\dots,\hat{e}_i,\dots,e_0]} \\
&= \epsilon_{n-1}\sum_{i=0}^{n}(-1)^{n-i} \sigma_{|[e_n,\dots,\hat{e}_{n-i},\dots,e_0]} \text{ by changing the order of summation} \\
&= \epsilon_{n-1}\sum_{i=0}^{n}(-1)^{i-n} \sigma_{|[e_n,\dots,\hat{e}_{n-i},\dots,e_0]} \text{ using } (-1)^j = (-1)^{-j} \\
&= \epsilon_{n-1}(-1)^n\sum_{i=0}^{n}(-1)^{i} \sigma_{|[e_n,\dots,\hat{e}_{n-i},\dots,e_0]} \text{ again using } (-1)^n = (-1)^{-n} \\
&= \epsilon_{n}\sum_{i=0}^{n}(-1)^{i} \sigma_{|[e_n,\dots,\hat{e}_{n-i},\dots,e_0]} \\
&= \partial(\epsilon_n \sigma_{|[e_n,\dots,e_0]}) \\
&= (\partial \circ \rho)(\sigma)
\end{aligned}$$

where we used the identity $\epsilon_n = (-1)^n \epsilon_{n-1}$.

- There is a chain homotopy between ρ and the identity.

We are going to use again the notation we introduced for the special case above. The idea for the chain homotopy is to interpolate between ρ which reverses the order of all vertices and the identity by, step by step, reversing the order up to some vertex while the others remain fixed. Then we throw in some signs to make things work.

We define homomorphisms h_n for each n by

$$\begin{aligned} h_n\colon S_n(X) &\to S_{n+1}(X) \\ \sigma &\mapsto \sum_{i=0} (-1)^i \epsilon_{n-i} \sigma_{|[e_0,\dots,e_i,e_n,\dots,e_i]}. \end{aligned}$$

Now we can show by calculating $\partial \circ h_n$ and $h_{n-1} \circ \partial$ that h is a chain homotopy, i.e., we have

$$\partial \circ h_n + h_{n-1} \circ \partial = \rho - \mathrm{id}.$$

We have

$$\begin{aligned} (\partial \circ h_n)(\sigma) &= \partial\Big(\sum_{i=0} (-1)^i \epsilon_{n-i} \sigma_{|[e_0,\dots,e_i,e_n,\dots,e_i]}\Big) \\ &= \sum_{j \le i} (-1)^i (-1)^j \epsilon_{n-i} \sigma_{|[e_0,\dots,\hat{e}_j,\dots,e_i,e_n,\dots,e_i]} \\ &+ \sum_{j \ge i} (-1)^i (-1)^{i+1+n-j} \epsilon_{n-i} \sigma_{|[e_0,\dots,e_i,e_n,\dots,\hat{e}_j,\dots,e_i]}. \end{aligned}$$

The overlap of the summation indices is necessary. For, at $j = i$, only the two sums together yield all the summands we need:

$$\epsilon_n \sigma_{|[e_n,\dots,e_0]} + \sum_{i>0} \epsilon_{n-i} \sigma_{|[e_0,\dots,e_{i-1},e_n,\dots,e_i]}$$
$$+ \sum_{i<n} (-1)^{n+i+1} \epsilon_{n-i} \sigma_{|[e_0,\dots,e_i,e_n,\dots,e_{i+1}]} - \sigma_{|[e_0,\dots,e_n]}.$$

Now we observe that the two sums in the lest epxression cancel out, since if we replace i by $i-1$ in the second sum turns the sign into

$$(-1)^{n+i} \epsilon_{n-i+1} = -\epsilon_{n-i}.$$

Hence, for $j = i$, what remains is exactly

$$\epsilon_n \sigma_{|[e_n,\dots,e_0]} - \sigma_{|[e_0,\dots,e_n]} = \rho(\sigma) - \sigma.$$

Hence it suffices to show that the terms with $j \neq i$ in $(\partial \circ h_n)(\sigma)$ cancel out with $(h_{n-1} \circ \partial)(\sigma)$. So we calculate

$$
\begin{aligned}
(h_{n-1} \circ \partial)(\sigma) &= h_{n-1}(\sum_{j=0} (-1)^j \sigma_{|[e_0,\ldots,\hat{e}_j,\ldots,e_n]}) \\
&= \sum_{j<i} (-1)^{i-1}(-1)^j \epsilon_{n-i} \sigma_{|[e_0,\ldots,\hat{e}_j,\ldots,e_i,e_n,\ldots,e_i]} \\
&+ \sum_{j>i} (-1)^{i}(-1)^j \epsilon_{n-i-1} \sigma_{|[e_0,\ldots,e_i,e_n,\ldots,\hat{e}_j,\ldots,e_i]}.
\end{aligned}
$$

Since $\epsilon_{n-i} = (-1)^{n-i}\epsilon_{n-i-1}$, the two sums cancel with the two corresponding sums in $(\partial \circ h_n)(\sigma)$. Hence h is a chain homotopy between ρ and the identity. **QED**