MA3403 Algebraic Topology

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Lecture 21

21. Applications of cup products in cohomology

We are going to see some examples where we calculate or apply multiplicative structures on cohomology. But we start with a couple of facts we forgot to mention last time.

Relative cup products

Let (X,A) be a pair of spaces. The formula which specifies the cup product by its effect on a simplex

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma_{|[e_0,\ldots,e_p]})\psi(\sigma_{|[e_p,\ldots,e_{p+q}]})$$

extends to relative cohomology.

For, if $\sigma \colon \Delta^{p+q} \to X$ has image in A, then so does any restriction of σ. Thus, if either φ or ψ vanishes on chains with image in A, then so does $\varphi \cup \psi$.

Hence we get relative cup product maps

$$H^p(X;R) \times H^q(X,A;R) \to H^{p+q}(X,A;R)$$
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More generally, assume we have two open subsets A and B of X. Then the formula for $\varphi \cup \psi$ on cochains implies that cup product yields a map

$$S^p(X,A;R) \times S^q(X,B;R) \to S^{p+q}(X,A+B;R)$$

where $S^n(X,A+B;R)$ denotes the subgroup of $S^n(X;R)$ of cochains which vanish on sums of chains in A and chains in B.

The natural inclusion

$$S^n(X,A \cup B;R) \hookrightarrow S^n(X,A+B;R)$$

induces an isomorphism in cohomology. For we have a map of long exact cohomology sequences

$$\begin{array}{ccccccccc}
H^n(A\cup B) & \longrightarrow & H^n(X) & \longrightarrow & H^n(X,A\cup B) & \longrightarrow & H^{n+1}(A\cup B) & \longrightarrow & H^{n+1}(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^n(A+B) & \longrightarrow & H^n(X) & \longrightarrow & H^n(X,A+B) & \longrightarrow & H^{n+1}(A+B) & \longrightarrow & H^{n+1}(X)
\end{array}$$

where we omit the coefficients. The small chain theorem and our results on cohomology of free chain complexes imply that $H^n(A \cup B; R) \xrightarrow{\cong} H^n(A + B; R)$ is an isomorphism for every n. Thus, the Five-Lemma implies that

$$H^n(X,A \cup B; R) \xrightarrow{\cong} H^n(X,A + B; R)$$

is an isomorphism as well.

Thus composition with this isomorphism gives a cup product map

$$H^p(X,A; R) \times H^q(X,B; R) \to H^{p+q}(X,A \cup B; R).$$

Now one can check that all the formulae we proved for the cup product also hold for the relative cup products.

Cohomology ring

All we are going to say now also works for relative cohomology. But to keep things simple, we just describe the absolute case.

We will now often drop the symbol $\cup$ to denote the cup product and just write

$$\alpha\beta = \alpha \cup \beta.$$

The cohomology ring of a space X is the defined as

$$H^*(X; R) = \bigoplus_n H^n(X; R)$$

as the direct sum of all cohomology groups. Note that, while the symbol $*$ previously often indicated that something holds for an arbitrary degree, we now use it to denote the direct sum over all degrees.

The product of two sums is defined as

$$(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i \beta_j.$$

This turns $H^*(X; R)$ into a ring with unit, i.e., multiplication is associative, there is a multiplicatively neutral element 1, and addition and multiplication satisfy the distributive law.

We consider the cohomological degree n in $H^n(X; R)$ as a grading of $H^*(X; R)$. If an element α is in $H^p(X; R)$ we call p the degree of α and denote it also by $|\alpha|$.

Since multiplication respects this grading in the sense that it defines a map

$$H^p(X; R) \times H^q(X; R) \to H^{p+q}(X,A; R),$$

we call $H^*(X;R)$ a graded ring.

Moreover, as we have shown with a lot of effort last time, the multiplication is commutative up to a sign which depends on the grading:

$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha.$$

Hence $H^*(X;R)$ a **graded commutative ring**.

Moreover, there is an obvious scalar multiplication by elements in R which turns $H^*(X;R)$ into a graded R-algebra.

Finally, if $f\colon X \to Y$ is a continuous map, then the induced map on cohomology

$$f^*\colon H^*(Y;R) \to H^*(X;R)$$

is a **homomorphism of graded R-algebras**.

Now we should determine some ring structures and see what they can tell us.

As a first, though disappointing, **example**, let us note that the product in the cohomology of a **sphere** S^n (with $n \geq 1$) is boring, since $H^0(S^n;R)$ is just R and the product on $H^n(S^n;R)$ is trivial for reasons of degrees:

$$H^n(S^n;R) \times H^n(S^n;R) \to H^{2n}(S^n;R) = 0.$$

So let us move on to more interesting cases.

Cohomology ring of the torus

Even though the cohomology ring of S^1 was boring, the cohomology ring of the product $T = S^1 \times S^1$, i.e., of the torus, is not. Let us assume $R = \mathbb{Z}$.

We computed the homology of T using its structure as a cell complex with one 0-cell, two 1-cells, and one 2-cell.

The cellular chain complex has the form

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

where $d_1(a,b) = a + b$ and $d_2(s) = (s, -s)$ (the attaching map of the 2-cell to the two 1-cells was $aba^{-1}b^{-1}$). This yields the homology of T.

We can then apply the UCT to deduce that the singular cohomology of T is given by

$$H^i(T;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \end{cases}$$

and $H^i(T;\mathbb{Z}) = 0$ for $i > 2$.

Let α and β be generators of $H^1(T;\mathbb{Z})$. We could obtain them for example as the dual of the basis $\{a{,}b\}$ of $H_1(T;\mathbb{Z})$ and the isomorphism of the UCT:

$$H^1(T;\mathbb{Z}) = \mathrm{Hom}(H_1(T;\mathbb{Z}){,}\mathbb{Z}).$$

Being a dual basis means, in particular,

$$\alpha(a) = \langle \alpha{,}a \rangle = 1,\ \alpha(b) = \langle \alpha{,}b \rangle = 0,\ \beta(a) = \langle \beta{,}b \rangle = 0,\ \beta(b) = \langle \beta{,}b \rangle = 1$$

where the funny brackets denote the Kronecker pairing we had defined earlier.

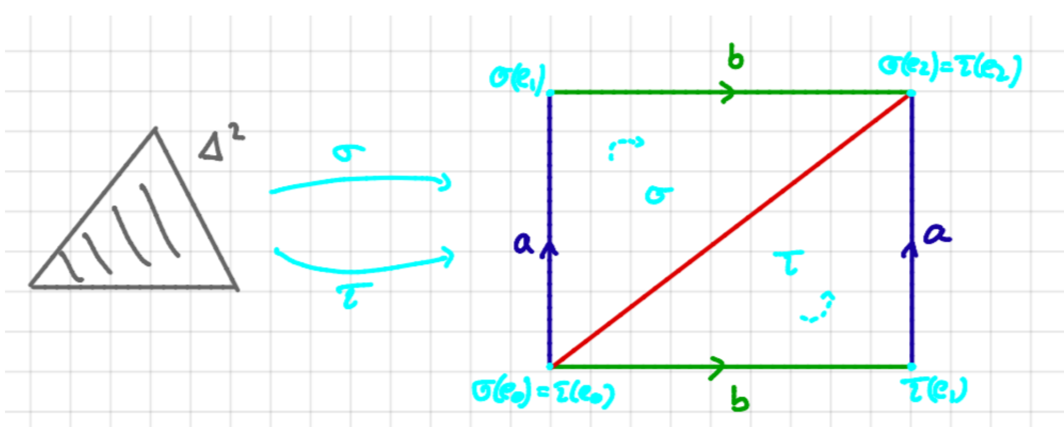
Since multiplication is graded commutative, we have

$$2\alpha^2 = 0 = 2\beta^2.$$

Since $\mathbb{Z}$ is torsion-free, this implies

$$\alpha^2 = 0 = \beta^2.$$

Now we would like to understand the product $\alpha\beta$. Therefore, we need to evaluate it on a generator of $H_2(T;\mathbb{Z})$. Such a generator is given by the 2-chain $\sigma - \tau$, where σ and τ are the 2-simplices indicated in the picture (that this is a generator needs to be checked; we just accept this for the moment):



It is a cycle, since

$$\partial(\sigma-\tau)=\partial(\sigma)-\partial(\tau)=b-d+a-(a-d+b)=0$$

where d denotes the diagonal.

Now we can calculate

$$\begin{aligned}(\alpha\cup\beta)(\sigma-\tau)&=\alpha(\sigma_{|[e_0,e_1]})\beta(\sigma_{|[e_1,e_2]})-\alpha(\tau_{|[e_0,e_1]})\beta(\tau_{|[e_1,e_2]})\\&=\alpha(a)\beta(b)-\alpha(b)\beta(a)\\&=1-0=1.\end{aligned}$$

Thus, since $H^2(T;\mathbb{Z})=\mathrm{Hom}(H_2(T;\mathbb{Z}),\mathbb{Z})$ by the UCT, we see that $\alpha\beta$ is a generator of $H^2(T;\mathbb{Z})$.

Hence we can conclude that the cohomology ring of the torus is the ring with generators α and β and relations

$$H^*(T;\mathbb{Z})=\mathbb{Z}\{\alpha,\beta\}/\langle\alpha^2=0=\beta^2,\alpha\beta=-\beta\alpha\rangle.$$

Another way to formulate this is to say that $H^*(T;\mathbb{Z})$ is the exterior algebra over $\mathbb{Z}$ with generators α and β:

$$H^*(T;\mathbb{Z})=\Lambda_{\mathbb{Z}}[\alpha,\beta].$$

In gerenal, the exterior algebra $\Lambda_R[\alpha_1,\dots,\alpha_n]$ over a commuative ring R with unit is defined as the free R-module with generators $\alpha_{i_1}\cdots\alpha_{i_k}$ for $i_1<\cdots<i_k$ with associative and distributive multiplication defined by the rules

$$\alpha_i\alpha_j=-\alpha_j\alpha_i \text{ if } i\neq j, \text{ and } \alpha_i^2=0.$$

Setting $\Lambda^0=R$, $\Lambda_R[\alpha_1,\dots,\alpha_n]$ beceomes a graded commutative ring with odd degrees for the α_is and unit $1\in R$.

For the n-torus $T^n=S^1\times\cdots\times S^1$, defined as the n-fold product of S^1, we then get

$$H^*(T^n;\mathbb{Z})=\Lambda_{\mathbb{Z}}[\alpha_1,\dots,\alpha_n].$$

Cohomology of projective spaces

The cohomology rings of projective spaces are truncated polynomial algebras:

Cohomology rings of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$

- For every $n \geq 1$ and $\mathbb{F}_2$-coefficients, we have an isomorphism of graded rings

$$H^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1}), \text{ and } H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$$

with $|x| = 1$.

- For every $n \geq 1$ and integral coefficients, we have an isomorphism of graded rings

$$H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}), \text{ and } H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}[y]$$

with $|y| = 2$.

The proof of this result requires some efforts. We will postpone its proof and rather see some consequences of it.

Cup products detect more

Consider the wedge of spheres $S^2 \vee S^4$. We know that its homology is given by

$$\tilde{H}_*(S^2 \vee S^4; \mathbb{Z}) = \tilde{H}_*(S^2; \mathbb{Z}) \oplus \tilde{H}_*(S^4; \mathbb{Z}).$$

In other words,

$$H_i(S^2 \vee S^4; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, 4 \\ 0 & \text{else.} \end{cases}$$

Hence the homologies of $\mathbb{C}\mathrm{P}^2$ and $S^2 \vee S^4$ are the same. Since all the groups are free, this also implies that the cohomology groups of the two spaces are the same. Thus, neither homology nor cohomology groups can distinguish between these two spaces.

The cup product, however, can.

For, we know that the square of a generator in $H^n(S^n; \mathbb{Z})$ is zero, since $H^{2n}(S^n; \mathbb{Z}) = 0$. Thus

$$H^*(S^n; \mathbb{Z}) = \mathbb{Z}[t]/(t^2 = 0) \text{ with } |t| = n,$$

and hence we have a generator $s \in H^2(S^2 \vee S^4; \mathbb{Z})$ with $s^2 = 0$ and a generator $t \in H^2(S^2 \vee S^4; \mathbb{Z})$ with $t^2 = 0$.

If there was an isomorphism of graded $\mathbb{Z}$-algebras

$$\tilde{H}^*(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \cong \tilde{H}^*(S^2 \vee S^4; \mathbb{Z})$$

it would have to send the generator $y \in H^2(\mathbb{CP}^2;\mathbb{Z})$ to the generator $s \in H^2(S^2 \vee S^4;\mathbb{Z})$. But $y^2 \neq 0$ in $H^4(\mathbb{CP}^2;\mathbb{Z})$, whereas $s^2 = 0$ in $H^4(S^2 \vee S^4;\mathbb{Z})$.

Thus, such an isomorphism of graded rings cannot exist.

Thus, the cup product structures show that there does not exist a homotopy equivalence between $\mathbb{CP}^2$ and $S^2 \vee S^4$, something our previous invariants could not prove.

Hopf maps

As an important application of what we just learned, we consider the following situation.

Many problems can be reduced to checking whether a map is **null-homotopic**, i.e., **homotopic to a constant map**, or not.

Given a map $f\colon X \to Y$, we can form the mapping cone $C_f = CX \cup_f Y$ (which we introduced in the exercises). It is the pushout of the diagram

$$\begin{array}{ccc} X \times \{1\} & \longrightarrow & CX \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \xrightarrow{\ i\ } & C_f. \end{array}$$

If f is homotopic to a constant map, then the diagram is equivalent to the diagram

$$\begin{array}{ccc} \mathrm{pt} & \longrightarrow & CX/(X \times \{1\}) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\ i\ } & SX \vee Y \end{array}$$

where we use that $CX/(X \times \{1\})$ is the suspension SX of X.

Thus, if f is **null-homotopic**, then there is a homotopy equivalence

$$C_f \xrightarrow{\simeq} SX \vee Y.$$

Let us look at an **example**. Let

$$\eta\colon S^3 \to \mathbb{CP}^1 \approx S^2,\ x \mapsto [\mathbb{C}x] = \{\lambda x \in \mathbb{C}^2 : \lambda \in \mathbb{C}\}$$

be the **complex Hopf map** which sends a point $x \in S^3 \subset \mathbb{C}^2$ to the complex line in $\mathbb{C}^2$ which passes through x.

This is exactly the map which attaches the 4-cell to $\mathbb{CP}^1 \approx S^2$ in the cell structure of $\mathbb{CP}^2$. The mapping cone C_η of η is $\mathbb{CP}^2$, since the cone of S^3 is just

D^4:

$$CS^3 = (S^3 \times [0,1])/(X \times \{0\}) \approx D^4$$

and hence

$$C_\eta = CS^3 \cup_\eta S^2 \approx D^4 \cup_\eta S^2 \approx D^4 \cup_\eta \mathbb{C}\mathrm{P}^1 \approx \mathbb{C}\mathrm{P}^2.$$

Now we use that we showed in the exercises that the suspension of S^3 is homeomorphic to S^4. Thus, if η was null-homotopic, then the argument above would imply

$$\mathbb{C}\mathrm{P}^2 \approx C_\eta \xrightarrow{\simeq} S^2 \vee S^4.$$

But we just showed that such a homotopy equivalence cannot exits. Thus, **η is not null-homotopic**.

More Hopf maps

Note that there is also a **quaternionic Hopf map**

$$\nu\colon S^7 \to S^4,$$

and an **octonionic Hopf map**

$$\sigma\colon S^{15} \to S^8.$$

They are constructed in the same way as η by replacing $\mathbb{C}$ with the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$, respectively. There are corresponding projective spaces $\mathbb{H}\mathrm{P}^n$ and $\mathbb{O}\mathrm{P}^n$ with $\mathbb{H}\mathrm{P}^1 \approx S^4$ and $\mathbb{O}\mathrm{P}^1 \approx S^8$, and polynomial rings as cohomology rings:

$$H^*(\mathbb{H}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[z]/(z^3),\ |z| = 4,\ \text{and}\ H^*(\mathbb{O}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[w]/(w^3),\ |w| = 8.$$

The homotopy classes of η, ν and σ

$$[\eta] \in \pi_3(S^2),\ [\nu] \in \pi_7(S^4),\ [\sigma] \in \pi_{15}(S^8)$$

play a crucial role in the stable homotopy category.

Is there a multiplication on $\mathbb{R}^n$?

For the next application, we are going to assume one more result, nemaly that the cohomology ring of the product of $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ is given

$$H^*(\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n;\mathbb{Z}/2) \cong \mathbb{F}_2[\alpha_1,\alpha_2](\alpha_1^{n+1},\alpha_2^{n+1}).$$

This implies the following algebraic fact:

Theorem: Multiplication on $\mathbb{R}^n$

Assume there is a $\mathbb{R}$-bilinear map

$$\mu \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

such that $\mu(x,y) = 0$ implies $x = 0$ or $y = 0$.
Then n must be a power of 2.

In fact, n must be 1, 2, 4 or 8. In all these dimensions we have such multplications by identifying

$$\mathbb{R}^2 \cong \mathbb{C},\ \mathbb{R}^4 \cong \mathbb{H},\ \mathbb{R}^8 \cong \mathbb{O}.$$

But to show that there are no other such algebra structures on $\mathbb{R}^n$ is a much harder task. The only known proofs of this fact are using algebraic topology! In fact, for showing this we need to study the famous **Hopf Invariant One-Problem**. This is beyond the scope of this lecture. So let us be modest and just prove the result stated above.

Proof: • Since μ is linear in both variables, it induces a continuous map

$$\bar{\mu} \colon \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}.$$

Then $\bar{\mu}$ induces a homomorphism of cohomology rings which has the form

$$\bar{\mu}^* \colon \mathbb{F}_2[\alpha]/(\alpha^n) \to \mathbb{F}_2[\alpha_1,\alpha_2](\alpha_1^n,\alpha_2^n).$$

• Since μ does not have a zero-divisor, the restriction of μ to $\mathbb{R}^n \times \{a\}$ for any $a \in \mathbb{R}^n$ is an isomorphism. Hence the restriction of $\bar{\mu}$ to $\mathbb{R}\mathrm{P}^{n-1} \times \{y\}$ for any point $y \in \mathbb{R}\mathrm{P}^{n-1}$ is a homeomorphism.

This implies that the composite

$$\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1} \times \{y\} \hookrightarrow \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \xrightarrow{\bar{\mu}} \mathbb{R}\mathrm{P}^{n-1}$$

is a homeomorphism as well. Hence the induced homomorphism of cohomology rings must send α to α.

Repeating this argument for $\{y\} \times \mathbb{R}\mathrm{P}^{n-1}$, we see that the image of α under $\bar{\mu}^*$ must be

$$\bar{\mu}^*(\alpha) = \alpha_1 + \alpha_2.$$

Since both rings are polynomial algebras, $\bar{\mu}^*$ is completely determined by this identity.

- Since $\alpha^n = 0$, we must have $\bar{\mu}^*(\alpha)^n = 0$, i.e.,

$$(\alpha_1 + \alpha_2)^n = \sum_k \binom{n}{k} \alpha_1^k \alpha_2^{n-k} = 0.$$

The sum on the right-hand side can only be zero if all the coefficients of the monomials $\alpha_1^k \alpha_2^{n-k}$ vanish for $0 < k < n$. Since we are working over $\mathbb{F}_2$, this means that all the numbers $\binom{n}{k}$ for $0 < k < n$ must be even.

To prove this fact is equivalent to proving the following claim about the polynomial ring $\mathbb{F}_2[x]$:

- **Claim:** In $\mathbb{F}_2[x]$, we have

$$(1 + x)^n = 1 + x^n \iff n \text{ is a power of } 2.$$

First, if n is a power of 2, then the eqation $(a + b)^2 = a^2 + b^2$ modulo 2 shows the if part:

$$(1 + x)^{2^r} = (1 + x^2)^{2^{r-1}} = (1 + x^{2^2})^{2^{r-2}} = \cdots = 1 + x^{2^r} \text{ in } \mathbb{F}_2[x].$$

For the other direction, write n as

$$n = 2^r m \text{ with } m \text{ odd and } m > 1.$$

Then

$$(1 + x)^n = (1 + x)^{2^r m} = (1 + x^{2^r})^m = 1 + m x^{2^r} + \ldots + x^n \neq 1 + x^n \text{ in } \mathbb{F}_2[x]$$

since m is **odd**. **QED**