MA3403 Algebraic Topology
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Lecture 22

22. Poincaré duality and intersection form

We are going to meet an important class of topological spaces and study one of their fundamental cohomological properties. This lecture will be short of proofs, but rather aims to see an important theorem and structures at work.

Manifolds

We start with defining an important class of spaces.

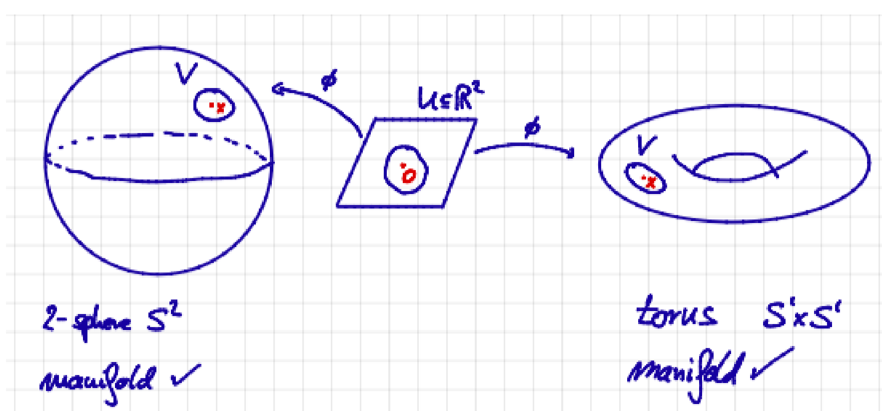
Definition: Topological manifolds

A n-dimensional topological manifold is a Hausdorff space in which each point has an open neighborhood which is homeomorphic to $\mathbb{R}^n$.

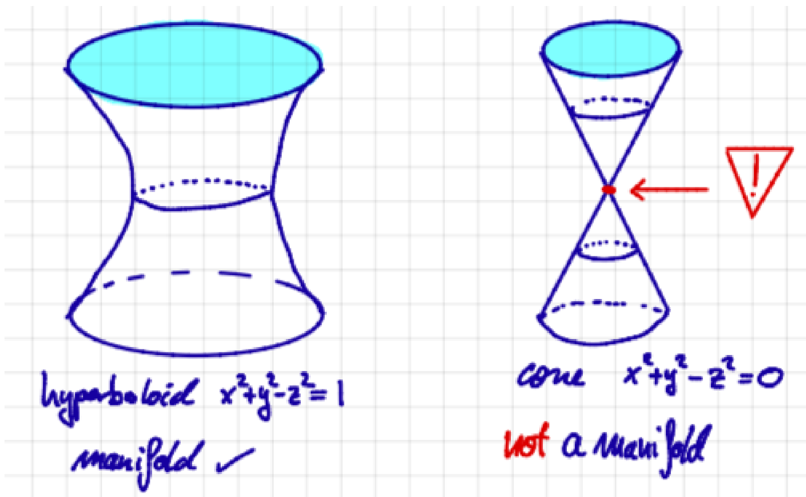
In this lecture, the word **manifold** will always mean a topological manifold.

You know many examples of manifolds, most notably $\mathbb{R}^n$ itself, any open subset of $\mathbb{R}^n$, n-spheres S^n, tori, Klein bottle, projective spaces. Even though the definition does not refer to this information, any manifold M can be embedded in some $\mathbb{R}^N$ for some large N (which depends on M).

Though it is a crucial point that N and n can and usually are different. For example, S^2 is a subset of $\mathbb{R}^3$, but each point on S^2 has a neighborhood which looks like a plane, i.e., is homeomorphic to $\mathbb{R}^2$.



There are many reasons why manifolds are important. One of them is that we understand and can study them locally, while they can be very complicated globally.



Poincaré duality

In this lecture, all homology and cohomology groups will be with $\mathbb{F}_2$-coefficients. Recall that there is a pairing

$$H^k(X;\mathbb{F}_2)\otimes H_k(X;\mathbb{F}_2)\xrightarrow{\langle -,-\rangle}\mathbb{F}_2$$

defined by evaluating a cocycle φ on a cycle σ which is an element in $\mathbb{F}_2$.

We are going to study the consequences of the following famous fact:

Theorem: Poincaré duality mod 2

Let M be a **compact topological manifold** of dimension n. Then there exists a unique class $[M]\in H_n(M;\mathbb{F}_2)$, called the **fundamental class of M**, such that, for every $p\geq 0$, the **pairing**

$$H^p(M;\mathbb{F}_2)\otimes H^{n-p}(M;\mathbb{F}_2)\xrightarrow{\cup}H^n(M;\mathbb{F}_2)\xrightarrow{\langle -,[M]\rangle}\mathbb{F}_2$$

is **perfect**.

That the pairing is perfect means that the **adjoint map**

$$H^p(X;\mathbb{F}_2)\xrightarrow{\langle a\cup -,[M]\rangle}\mathrm{Hom}(H^{n-p}(X;\mathbb{F}_2),\mathbb{F}_2),\ a\mapsto\langle a\cup -,[M]\rangle$$

is an **isomorphism**.

Here are some **first consequences** of this theorem:

- Since cohomology vanishes in negative dimensions, we must have $H^p(X;\mathbb{F}_2) = 0$ for $p > n$ as well.
- Since M is assumed to be compact, we know that $\pi_0(M)$, the set of connected components of M, is **finite**. Moreover, we once showed that $H^0(M;\mathbb{F}_2)$ equals $\mathrm{Map}(\pi_0(M),\mathbb{F}_2)$. Hence we get

$$H^n(M;\mathbb{F}_2) = \mathrm{Hom}(H^0(M;\mathbb{F}_2),\mathbb{F}_2) = \mathrm{Hom}(\mathrm{Map}(\pi_0(M),\mathbb{F}_2),\mathbb{F}_2) = \mathbb{F}_2[\pi_0(M)].$$

- A vector space admitting a perfect pairing is finite-dimensional. Hence $H^p(M;\mathbb{F}_2)$ is **finite-dimensional** for all p.

There is a version of the **Universal Coefficient Theorem** with $\mathbb{F}_2$-coefficients. Since $\mathbb{F}_2$ is a field, it implies that there is an isomorphism

$$\mathrm{Hom}(H^{n-p}(M;\mathbb{F}_2),\mathbb{F}_2) \cong H_{n-p}(M;\mathbb{F}_2).$$

(Note that we formulated the UCT with the roles of homology and cohomology reversed. But, since the map arose from the Kronecker pairing, we can also produce the claimed version of the UCT. As mentioned in the intro to this lecture, we rush through some points for the sake of telling a good story.)

Composition with the above pairing yields an isomorphism

$$\begin{tikzcd} H^p(X;\mathbb{F}_2) \arrow[r, "\cong"] \arrow[rr, bend right, dotted, "\cong"'] & \mathrm{Hom}(H^{n-p}(M;\mathbb{F}_2),\mathbb{F}_2) & H_{n-p}(M;\mathbb{F}_2). \arrow[l, "\cong"'] \end{tikzcd}$$

Definition: Poincaré duals

Homology and cohomology corresponding to each other under the dotted isomorphism are said to be **Poincaré dual** to each other.

Intersection pairing

Combining this isomorphism for different dimensions, we can write the cup product pairing in cohomology as a pairing in homology (where we drop the coefficients which are still $\mathbb{F}_2$)

$$\begin{tikzcd} H_p(M) \otimes H_q(M) \arrow[r, "\pitchfork"] \arrow[d, "\cong"'] & H_{p+q-n}(M) \arrow[d, "\cong"] \\ H^{n-p}(M) \otimes H^{n-q}(M) \arrow[r, "\cup"] & H^{2n-p-q}(M). \end{tikzcd}$$

The top map is called the **intersection pairing** in homology.

Here is how we should think about it:

- Let $\alpha \in H_p(M)$ and $\beta \in H_q(M)$ be homology classes.
- Represent them, if possible, as the image of fundamental classes of submanifolds of M. That means that there are submanifolds Y and Z in M of dimensions p and q, respectively, such that

  $$\alpha = i_*[Y] \text{ and } \beta = j_*[Z]$$

  where $i_*\colon H_p(Y) \to H_p(M)$ and $j_*\colon H_q(Z) \to H_q(M)$ are the homomorphisms induced by the inclusions $i\colon Y \hookrightarrow M$ and $j\colon Z \hookrightarrow M$.
- Move them a bit if necessary to make them **intersect transversally**.
- Then their intersection is a submanifold of dimension $p + q - n$ and its image will represent the homology class $\alpha \pitchfork \beta$.

Let us look at an example:

Example: Intersection on a torus

Let $M = T^2 = S^1 \times S^1$ be the two-dimensional torus. We know

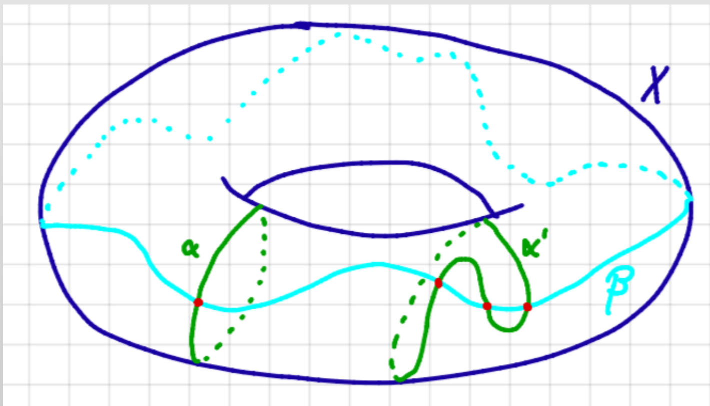
$$H^1(M) = \mathbb{F}_2\langle a,b\rangle$$

with $a^2 = 0 = b^2$, and $H^2(M)$ is generated by $ab = ba$.
The **Poincaré duals** α and β of a and b are represented by cycles which wrap around one or the other factor circle of M.

The cycles α and β can be made to intersect in a single point. This reflects the equation

$$\langle a \cup b, [M] \rangle = 1.$$

But this equation also tells us that α and β can only be moved in such a way that they intersect in an **odd** number of points.
The fact that $a^2 = 0$ reflects that the fact that its Poincaré dual α can be moved so as not to intersect itself.



Intersection form

Let us look at a particular case of Poincaré duality. Let us assume that M is **even-dimensional**, say of dimension $n = 2p$. Then Poincaré duality implies that we have a **symmetric bilinear form** on the $\mathbb{F}_2$-vector space $H^p(M)$:

$$H^p(M) \otimes_{\mathbb{F}_2} H^p(M) \to H^{2p}(M) \cong \mathbb{F}_2.$$

As we just observed, this can be interpetreted as a bilinear form on homology $H_p(M)$. Evaluating this form can be wiewed as describing (modulo 2) the number of points where two p-cycles intersect, after they have put moved in general position, i,.e., a position where they intersect transversally.

Definition: Intersection form

This form $H_p(M) \otimes H_p(M) \to \mathbb{F}_2$ is called **intersection form** and will be denoted

$$\alpha \cdot \beta := \langle a \cup b, [M] \rangle$$

where a and b are Poincaré dual to α and β, respectively.

Let us consider two examples:

- For the **sphere** S^2, the first homology is **trivial**, and so is the intersection form on S^2.
- In the **example of the torus**, the intersection form can be described in terms of the basis α and β by the matrix (since any such form looks like $(v,w) \mapsto v^T H w$)

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  Such a form is called **hyperbolic**.

Apparently, it would good to know a bit more about such forms. We are going to review what we need to know about them now and then get back to the application in topology in the next lecture.

A digression on symmetric bilinear forms

We need to have a brief look at such forms.

So let V be a finite-dimensional vector space over $\mathbb{F}_2$ together with a nondegenerate symmetric bilinear form. Such a form restricts to any subspace W of V, but the restricted form may be degenerate. But any subspace has an orthogonal complement

$$W^\perp = \{v \in V : v \cdot w = 0 \text{ for all } w \in W\}.$$

Then we have the following lemma:

Lemma

The restriction of a nondegenerate symmetric bilinear form on V to a subspace W is nondegenerate if and only if $W \cap W^\perp = 0$.
In this case, the restriction to $W^\perp$ is also nondegenerate and the splitting

$$V \cong W \oplus W^\perp$$

respects the forms.

We can use this lemma to inductively decompose all finite-dimensional symmetric bilinear forms:

- If there is a vector $v \in V$ with $v \cdot v = 1$, then it generates a nondegenerate subspace, i.e., a subspace on which the restriction of the form is nondegenerate, and

$$V = \langle v \rangle \oplus \langle v \rangle^{\perp}$$

where $\langle v \rangle$ denotes the subspace generated by v.

- Continue to split off one-dimensional subspaces until we reach a nondegenerate symmetric bilinear form such that $v \cdot v = 0$ for all vectors.
- Unless we ended up with zero space, we can pick a nonzero vector v. Since the form is nondegenerate, there must be a vector w such that $v \cdot w = 1$.
- The two vectors v and w generate a hyperbolic subspace, i.e., one on which the form is represented by the matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- Split this space off, and continue the process.

This procedure shows:

Proposition: Classification of nondegenerate forms

Any finite-dimensional nondegenerate symmetric bilinear form over $\mathbb{F}_2$ splits as an orthogonal sum of forms with matrices

$$I = (1) \text{ and } H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We are going to continue the study of forms and get bakc to the topology in the next lecture.