MA3403 Algebraic Topology
Lecturer: Gereon Quick
Lecture 24

24. More on Poincaré duality

We continue our discussion of Poincaré duality. First we see two applications of the theorem with $\mathbb{F}_2$ coefficients. Then we will discuss what we need to do for other coefficients. This will lead to an important concept, orientations of manifolds, and an important algebraic structure, the cap product.

Dualities reflect fundamental properties

Poincaré duality is extremely interesting, since it reflects a **deep symmetry** in the homology and cohomology groups on manifolds. For, the cohomology in dimension p determines the cohomology in dimension $n-p$. This symmetry has many consequences which make the study of manifolds particularly interesting.
Duality theorems arise in many areas of mathematics and always reflect deep and interesting structures.

We start with an application of Poincaré duality modulo 2.

Applications of Poincaré duality with $\mathbb{F}_2$-coefficients

Recall the important theorem:

Theorem: Poincaré duality mod 2

Let M be a connected **compact manifold** of dimension n. Then there exists a unique class $[M] \in H_n(M;\mathbb{F}_2)$, called the **fundamental class of** M, such that, for every $p \geq 0$, the **pairing**

$$H^p(M;\mathbb{F}_2) \otimes H^{n-p}(M;\mathbb{F}_2) \xrightarrow{\cup} H^n(M;\mathbb{F}_2) \xrightarrow{\langle -,[M]\rangle} \mathbb{F}_2$$

is **perfect**.

Since real projective space is a compact connected n-dimensional manifold, Poincaré duality applies. And, in fact, we can use this result to deduce the algebra structure on the cohomology of real projective space:

Corollary: Cohomology of $\mathbb{R}\mathrm{P}^n$

Let x be the nonzero element in $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. Then x^k is the nonzero element of $H^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ for $k=2,\ldots,n$.
Thus $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is the **truncated polynomial algebra**

$$H^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)\cong \mathbb{F}_2[x]/(x^{n+1})$$

generated by x in degree 1 and truncated by setting $x^{n+1}=0$.
Moreover, $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2)$ is a polynomial algebra

$$H^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)\cong \mathbb{F}_2[x]$$

generated by x in degree 1.

Proof: The proof is **by induction** on n.

By the construction of the cell structure on $\mathbb{R}\mathrm{P}^n$, we know that the inclusion $j_k\colon \mathbb{R}\mathrm{P}^k \hookrightarrow \mathbb{R}\mathrm{P}^{k+1}$ is a **map of cell complexes** which induces an isomorphism

$$H^i(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)\xrightarrow{\cong} H^i(\mathbb{R}\mathrm{P}^{k+1};\mathbb{F}_2) \text{ for all } i=0,\ldots,k,$$

which sends the nonzero element $x\in H^1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ to the nonzero element in $H^1(\mathbb{R}\mathrm{P}^{k+1};\mathbb{F}_2)$ which we therefore also denote by x.

Hence, assuming x^k is the nonzero element in $H^k(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$, it suffices to show that $x\cup x^k$ is nonzero in $H^{k+1}(\mathbb{R}\mathrm{P}^{k+1};\mathbb{F}_2)$.

By **Poincaré duality**, the pairing

$$H^1(\mathbb{R}\mathrm{P}^{k+1};\mathbb{F}_2)\otimes H^k(\mathbb{R}\mathrm{P}^{k+1};\mathbb{F}_2)\xrightarrow{\cup} H^{k+1}(\mathbb{R}\mathrm{P}^{k+1};\mathbb{F}_2)\xrightarrow{\langle -,[\mathbb{R}\mathrm{P}^{k+1}]\rangle}\mathbb{F}_2$$

is perfect. Since x and x^k are nonzero by assumption, this implies $x\cup x^k=x^{k+1}$ is nonzero as well.

For $\mathbb{R}\mathrm{P}^n$, we know that $H^i(\mathbb{R}\mathrm{P}^n\ F_2)=0$ for $i>n$, since there are no cells in dimensions bigger than n. Thus $x^{n+1}=0$.

For $\mathbb{R}\mathrm{P}^\infty$ we just continue the induction process. **QED**

As an application of this calculation, we are going to prove another famous theorem, the **Borsuk-Ulam Theorem**.

Lemma

Let $f\colon \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ be a continuous map which induces a nontrival map

$$f_* \neq 0\colon H_1(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2) \to H_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$$

Then $m \leq n$.

Proof: Since $H^1(X;\mathbb{F}_2) \cong \mathrm{Hom}(H_1(X;\mathbb{F}_2))$, the assumption implies that the induced map in cohomology

$$f^*\colon H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to H^1(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$$

is nontrivial as well.

Let $x \neq 0$ be the nonzero element in $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. Then $f^*(x) \neq 0$ is nonzero in $H^1(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$. By the calculation of the $H^*(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$, we have

$$0 \neq (f^*(x))^m = f^*(x^m).$$

Thus, $x^m \neq 0$ in $H^m(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ which implies $m \leq n$. **QED**

Lemma: Paths between antipodal points

Let $p \in S^n$ and let $\sigma\colon \Delta^1 \to S^n$ be a 1-simplex on S^n which connects p and its antipodal point $-p$ in S^n, i.e., $\sigma(e_0) = p$ and $\sigma(e_1) = -p$. Let

$$\pi\colon S^n \to \mathbb{R}\mathrm{P}^n$$

be the quotient map.
Then $\pi_*(\sigma) = \pi \circ \sigma$ is a **cycle** on $\mathbb{R}\mathrm{P}^n$ which represents a **nonzero element** in $H_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$.

Proof: First, that $\pi_*(\sigma)$ is a **cycle on** $\mathbb{R}\mathrm{P}^n$ just follows from the fact

$$[\pi(\sigma(e_0))] = [-\pi(\sigma(e_0))] = [\pi(\sigma(e_1))] \text{ in } \mathbb{R}\mathrm{P}^n.$$

It remains to show that it is **not a boundary**.

Recall that there is a cell structure on S^n with skeleta

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n.$$

By symmetry, we can assume that p and $-p$ are the points of S^0.

• For $n = 1$, we have a **homeomorphism** $\mathbb{R}\mathrm{P}^1 \approx S^1$ (for example, one could use the stereographic projection). Since σ connects p and $-p$ on S^1, there is an integer k such that σ walks around S^1 $(k+1/2)$-many times. Thus $\pi_*(\sigma)$ walks around $\mathbb{RP}^1$ $(2k+1)$-many times, i.e., an **odd** number of times.

Now recall that we showed

$$\mathbb{Z} \xrightarrow{\cong} H_1(S^1;\mathbb{Z}),\ m \mapsto (z \mapsto z^m)$$

where we use the identification $\pi(S^1) = H_1(S^1;\mathbb{Z})$ that we showed in the exercises. This implies that with $\mathbb{F}_2$-coefficients, even numbers correspond to 0 in $H_1(\mathbb{RP}^1;\mathbb{F}_2)$ and odd numbers correspond to the nonzero element in $H_1(\mathbb{RP}^1;\mathbb{F}_2)$.

Thus, the image of $\pi_*(\sigma)$ in $H_1(\mathbb{RP}^1;\mathbb{F}_2)$ is nonzero.

- For $n > 1$, we first choose a path τ on $S^1 \subset S^n$ which connects p and $-p$ on S^1. By the previous case, we know $[\pi_*(\tau)] \neq 0$ in $H_1(\mathbb{RP}^1;\mathbb{F}_2)$. The **inlcusion** map $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ induces an **isomorphism**

$$H_1(\mathbb{RP}^1;\mathbb{F}_2) \xrightarrow{\cong} H_1(\mathbb{RP}^n;\mathbb{F}_2).$$

Hence $[\pi_*(\tau)] \neq 0$ in $H_1(\mathbb{RP}^n;\mathbb{F}_2)$ as well.

But for $n > 1$, the difference $\sigma - \tau$ is a boundary, since it is homotopic to a constant map. This implies

$$[\pi_*(\sigma)] = [\pi_*(\tau)] \neq 0 \text{ in } H_1(\mathbb{RP}^n;\mathbb{F}_2).$$

QED

Lemma: No antipodal maps

For any n, there is no continuous map $f\colon S^{n+1} \to S^n$ with

$$f(-p) = -f(p) \text{ for all } p \in S^{n+1}.$$

Proof: Assume there was such a map f. Since $f(-p) = f(-p)$ for all p, f induces a map

$$\bar{f}\colon \mathbb{RP}^{n+1} \to \mathbb{RP}^n$$

which fits into a commutative diagram

$$\begin{array}{ccc} S^{n+1} & \xrightarrow{f} & S^n \\ {\scriptstyle \pi^{n+1}}\downarrow & & \downarrow{\scriptstyle \pi^n} \\ \mathbb{RP}^{n+1} & \xrightarrow[\bar{f}]{} & \mathbb{RP}^n. \end{array}$$

Now we take take a 1-simplex σ which connects two antipodal points on S^{n+1}. Its image $f_*(\sigma) = f \circ \sigma$ is then a 1-simplex which connects two antipodal points on S^n, since $f(-p) = f(-p)$.

By the previous lemma, $\pi^n_*(f_*(\sigma)) \neq 0$ in $H_1(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$. Thus

$$\bar{f}_*(\pi^{n+1}_*(\sigma)) = \pi^n_*(f_*(\sigma)) \neq 0.$$

In other words,

$$\bar{f}_* \neq 0\colon H_1(\mathbb{R}\mathrm{P}^{n+1}; \mathbb{F}_2) \to H_1(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$$

is nontrivial. By the other lemma, this is not possible. Hence f cannot exist. **QED**

The Borsuk-Ulam Theorem

Let $g\colon S^n \to \mathbb{R}^n$ be a continuous map. Then there is a point $p \in S^n$ with

$$g(p) = g(-p).$$

Proof: If there is no such point, we can define a continuous map

$$f\colon S^n \to S^{n-1},\ p \mapsto \frac{g(p) - g(-p)}{|g(p) - g(-p)|}.$$

But this map satisfies

$$f(-p) = -f(p).$$

This contradicts the previous lemma. **QED**

Orientation and fundamental classes

We now leave the world of $\mathbb{F}_2$-coefficients and contemplate on what we need for a Poincaré duality theorem with other coefficients. Since we will only sketch the main ideas anyway, we will just look at $\mathbb{Z}$-coefficients.

We start with the following observation on the homology groups of a manifold at a point:

Lemma: Local homology on manifolds

Let M be an n-dimensional topological manifold. For any point $x \in M$, there is an isomorphism

$$H_n(M,M-\{x\};\mathbb{Z}) \cong \mathbb{Z}.$$

and $H_i(M,M-\{x\};R) = 0$ for all $i \neq n$.

Proof: Since M is a manifold, there is an open neighborhood U around x in M such that $U \cong \mathbb{R}^n$. We set $Z = M - U$ and apply excision to get

$$\begin{aligned} H_i(M,M-\{x\};\mathbb{Z}) &\cong H_i(M-Z,(M-\{x\})-Z);\mathbb{Z}) \text{ (by excision)} \\ &= H_i(U,U-\{x\};\mathbb{Z}) \\ &\cong H_i(\mathbb{R}^n,\mathbb{R}^n-\{0\};\mathbb{Z}) \\ &\cong H_{i-1}(\mathbb{R}^n-\{0\};\mathbb{Z}) \text{ (by homotopy invariance and long ex. seq.)} \\ &\cong H_{i-1}(S^{n-1};\mathbb{Z}) \text{ (by homotopy invariance).} \end{aligned}$$

This implies

$$H_i(M,M-\{x\};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

QED

Local orientation

The group $H_n(M,M-\{x\};\mathbb{Z})$ is often called the **local homology of M at x**. It is an infinite cyclic group and therefore has **two generators**.
A **choice** of a generator $\mu_x \in H_n(M,M-\{x\};\mathbb{Z})$ is a **local orientation** of M at x.

For every point $x \in M$, we can choose such a generator. Note that such a choice was not necessary in $\mathbb{F}_2$, since there is only one generator. That makes $\mathbb{F}_2$-coefficients quite special.

The **natural question** is how all these choices are related. In other words, is it possible to **choose** these generators **in a compatible way?**

More precisely, let x and y be two points in M which both lie in some subset $U \subset M$. The inclusions $i_x \colon \{x\} \hookrightarrow M$ and $i_y \colon \{y\} \hookrightarrow M$ induce maps

$$H_n(M,M-\{x\};\mathbb{Z}) \xleftarrow{i_{x*}} H_n(M,M-U;\mathbb{Z}) \xrightarrow{i_{y*}} H_n(M,M-\{y\};\mathbb{Z}).$$

A class $\mu_U \in H_n(M,M-U;\mathbb{Z})$ which maps to generators in $H_n(M,M-\{x\};\mathbb{Z})$ and $H_n(M,M-\{y\};\mathbb{Z})$. Such an μ_U would define local orientations $\mu_x := i_{x*}(\mu_U)$ and $\mu_y := i_{y*}(\mu_U)$ at x and y, respectively. We call such an element μ_U a **fundamental class at U**.

Around every point in M there is a little neighborhood U with a fundamental class at U. The crucial question is: how large can we choose such a U? Ideally, we would like to be able to choose $U = M$ such that $H_n(M,M-U) = H_n(M)$.

Unfortunately, this is not always possible. This leads to an important concept:

Orientations

Let M be a compact connected n-dimensional manifold.

- An **orientation** of M is a function $x \mapsto \mu_x$, where $\mu_x \in H_n(M,M-\{x\};\mathbb{Z})$ is a generator, which satisfies the following condition:
At any point $x \in M$, there is a neighborhood U around x and an element $\mu_U \in H_n(M,M-U;\mathbb{Z})$ such that $i_{y*}(\mu_U) = \mu_y$ for all $y \in U$.
- If such an orientation exists, we say that M is **orientable**.

If M is orientable, then there are **exactly two orientations**. If M is orientable, and we have **chosen an orientation**, then we say that M is **oriented**.

We can reformulate this in terms of a particular class in homology, the fundamental class. The following statement is both a definition and proposition. We skip the proof, since we only have time for a rough sketch of the story.

Fundamental classes and orientability

Let M be a compact connected n-dimensional manifold.

- A fundamental class of M is an element $\mu \in H_n(M;\mathbb{Z})$ such that, for every point $x \in M$, the image of μ under the map

$$H_n(M;\mathbb{Z}) \to H_n(M,M-\{x\};\mathbb{Z})$$

induced by the inclusion $(M,\emptyset) \hookrightarrow M,M-\{x\})$ is a generator.

- M is orientable if and only if M has a fundamental class.
- M is orientable if and only if $H_n(M;\mathbb{Z}) = \mathbb{Z}$.

For **example**, $\mathbb{R}\mathrm{P}^{2n}$ is not orientable, since $H_{2n}(\mathbb{R}\mathrm{P}^{2n};\mathbb{Z}) = 0$. Whereas $\mathbb{R}\mathrm{P}^{2n+1}$ is orientable with $H_{2n+1}(\mathbb{R}\mathrm{P}^{2n+1};\mathbb{Z}) = \mathbb{Z}$.

Spheres and tori are orientable. The Klein bottle is not orientable.

The cap product

There is another type of product that has elements in both cohomology and homology and has a homology class as output. Actually, there are several other such products. But that is a story for another day.

Cap products are defined for arbitrary spaces. So we leave the world of manifolds for a moment and get back to it afterwards. Again we only discuss $\mathbb{Z}$-coefficients, but everything works for any ring R as coefficients.

Definition: Cap product

Let X be any space. The **cap product** is defined to be the $\mathbb{Z}$-bilinear map

$$S^q(X) \times S_p(X) \xrightarrow{\cap} S_{p-q}(X)$$

defined by sending a q-cochain $\varphi \in S^q(X)$ and a p-simplex $\sigma \colon \Delta^p \to X$ to the $p-q$-chain

$$\varphi \cap \sigma := \varphi(\sigma_{|[e_0,\dots,e_q]})\sigma_{|[e_q,\dots,e_p]}.$$

If $p < q$, then the cap product is defined to be 0.

After checking the relation

$$\partial(\varphi \cap \sigma) = \varphi \cap (\partial\sigma)$$

we see that the cap product descends to a $\mathbb{Z}$-linear map on cohomology and homology

$$H^q(X) \otimes H_p(X) \xrightarrow{\cap} H_{p-q}(X).$$

Given a continuous map $f\colon X \to Y$, there is the following formula which expresses the naturality of the cap product:

$$\varphi \cap f_*(\sigma) = f_*(f^*\varphi \cap \sigma).$$

The cap product is important for us, since (one form of) Poincaré duality can be formulated by saying that the cap product with the fundamental class is an isomorphism:

Poincaré duality

Let M be a compact n-dimensional oriented manifold. Let $[M] \in H_n(M;\mathbb{Z})$ be its fundamental class. Then taking the cap product with $[M]$ yields an isomorphism

$$D\colon H^p(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-p}(M;\mathbb{Z}),\ \varphi \mapsto \varphi \cap [M].$$

Note that there are many different ways to formulate Poincaré duality. In particular, there is also the stronger statement in terms of perfect pairings on cohomolgy groups that we have seen in the mod 2-case.

The **idea of the proof** of this theorem is to study the case of open subsets of $\mathbb{R}^n$ first. Then we use that every point in M has an open neighborhood which is homeomorphic to an open subset in $\mathbb{R}^n$. Since M is compact, we only need to take finitely many such open neighborhoods to cover M. The Mayer-Vietoris sequence then allows to patch the overlapping open subsets together. Unfortunately, there are some technical difficulties to take care of along the way, e.g., that certain diagrams actually commute.