

Algebraic Topology
Lecture Notes
Fall 2018

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Please send suggestions for corrections to gereon.quick@ntnu.no.

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LECTURE 1

Introduction

Organization:

Lectures: Tuesdays and Thursdays, both days at 12.15-14.00 in S21 in Sentralbygg 2.

Exercises: We will have some suggested exercises, hopefully on a weekly basis. You should try to solve as many exercises as possible, not just the ones I suggest, but also all that you find in other textbooks. We will not have an exercise class though. But you can discuss exercises with me at any time!

Course webpage: wiki.math.ntnu.no/ma3403/2018h/start where all news about the class will be announced. You will also find lecture notes a few hours after class on the webpage.

Office hours: Upon request.

Just send me an email: gereon.quick@ntnu.no

Textbooks: We will not follow just one book... but there are many good texts out there. For example, you can look at

[H] A. Hatcher, Algebraic Topology. It's available online for free. It contains much more than we have time for during one semester.

[Mu] J.R. Munkres, Elements of Algebraic Topology.

[V] J.W. Vick, Homology Theory - An Introduction to Algebraic Topology.

Two books that you can use as an outlook to future topics:

[Ma] J.P. May, A Concise Course in Algebraic Topology. It's also online somewhere.

[MS] J.W. Milnor, J. Stasheff, Characteristic Classes.

There are many other good books and lecture notes out there. Ask me if you need more.

What is required?

I will assume that you are starting your third year at NTNU (or more). You should have taken the equivalent of **Calculus 1-3** or **MA1101-1103, MA 1201-1202**. So you should be familiar with Euclidean space \mathbb{R}^n , multivariable calculus and linear algebra. Ideally, you have taken TMA4190 Introduction to Topology and/or General Topology.

You should also know a bit about algebra, like what is a group, an abelian group, a field, ideally also what is a ring and module over a ring.

Finally, it would be good if you knew what a topological space is and you would know what the words **open, closed, compact, etc** mean. But, in fact, you could also just have some few examples of topological spaces in mind, like n -spheres, torus etc. without knowing too many abstract stuff. For, the class is much more about the **ideas and methods** we develop than anything else. And these methods are useful almost everywhere.

Nevertheless, if you want to refresh your knowledge on Topology, you may want to have a look at the book

[J] K. Jänich, Topology.

What this class is about:

Note: If some of the following words do not yet make sense to you, no worries! For the moment we are just waving our hands and use fancy words. We will make sense of all this during the semester...

Very roughly speaking, **topology** studies spaces up to **continuous transformation** of one space into the other.

The correct place to do this is the **category of topological spaces** whose objects are topological spaces and whose morphisms are continuous maps. The isomorphisms in this category are called **homeomorphisms**, i.e., a continuous map with a continuous (left- and right-) inverse, or a continuous bijective map with a continuous inverse.

We can then describe topology as the science which studies properties of spaces which do not change under homeomorphisms. You have seen many such properties already, e.g. **compactness** (to be recalled in a bit).

This gives rise to a typical question in topology:

Typical question in topology

Given two topological spaces X and Y . Are X and Y homeomorphic, i.e., is there a homeomorphism $\phi: X \xrightarrow{\cong} Y$?

Let us look at a familiar example. Fix two natural numbers $n < m$.

- Is there a **linear isomorphism** (of vector spaces) $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$?
Answer: **No**, since linear algebra tells us that isomorphic vector spaces have equal dimension.
- Is there a **diffeomorphism** (bijective differential map with differentiable inverse) $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$?
Answer: **No**, since otherwise the derivative at 0 would be a linear isomorphism $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$ between tangent spaces.
- Is there a **bijective map** (of sets) $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$?
Answer: **Yes**. Surprisingly enough one can construct such maps, and it is actually not that difficult.
- Is there a **homeomorphism** $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$?

The answer to the last question is: **No**. But it is not so simple to show. In fact, one of the goals of algebraic topology is to develop tools that help us decide similar questions. For example, is there a homeomorphism between the 2-dimensional sphere S^2 and the torus? The answer is no. But how can we prove that? Both spaces are compact and (in some sense) two-dimensional and oriented...

Algebraic Topology in a nutshell

Translate problems in topology into problems in algebra which are (hopefully) easy to answer.

Key idea: develop **algebraic invariants** (numbers, groups, rings etc and homomorphisms between them) which decode the topological problem.

This should be done such that **homeomorphic** spaces should have the **same invariants** (that is where the name comes from).

In particular, this implies: if we find values of an invariant that differ for X and Y , then they cannot be homeomorphic.

Remark: We will later see that all the invariants we construct are preserved under **homotopy** equivalences, a **weaker** notion than homeomorphisms. This will finally lead to the idea of the stable homotopy category being the motive of topological spaces. We will not discuss this in class, but feel free to ask me about it. :)

For example, the first important tool that we are going to define soon is **singular homology**. It will allow us to use a simple algebraic argument to show that there cannot be a homeomorphism $\mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^n$.

Just to make you taste a little more of what algebraic topology can do:

Multiplicative Structures on \mathbb{R}^n

Let $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bilinear map with two-sided identity element $e \neq 0$ and no zero-divisors. Then $n = 1, 2, 4$, or 8 .

What we are looking for is a "multiplication map". You know the cases $n = 1$ and $n = 2$ very well. It's just \mathbb{R} and $\mathbb{C} \cong \mathbb{R}^2$. These are actually fields.

For $n = 4$, there are the Hamiltonians, or Quaternions, $\mathbb{H} \cong \mathbb{R}^4$ with a multiplication which is almost as good as the one in \mathbb{C} and \mathbb{R} , but it is not commutative. (You add elements i, j, k to \mathbb{R} with certain multiplication rules.)

For $n = 8$, there are the Octonions $\mathbb{O} \cong \mathbb{R}^8$. The multiplication is not associative and not commutative.

And that's it!

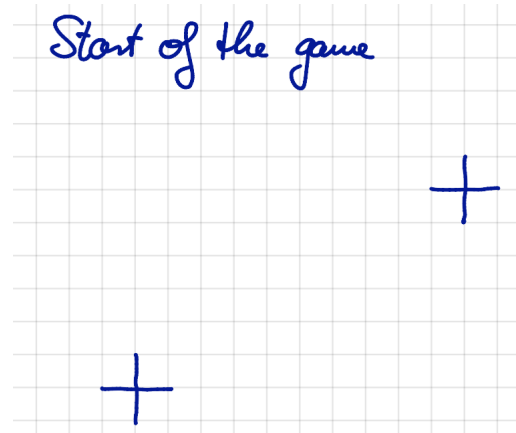
This is a really deep result!

The crucial and, at first glance maybe surprising, point to prove this fundamental result is that the statement has something to do with the behavior of tangent spaces on spheres. That's a topological problem. Frank Adams was the first to solve it.

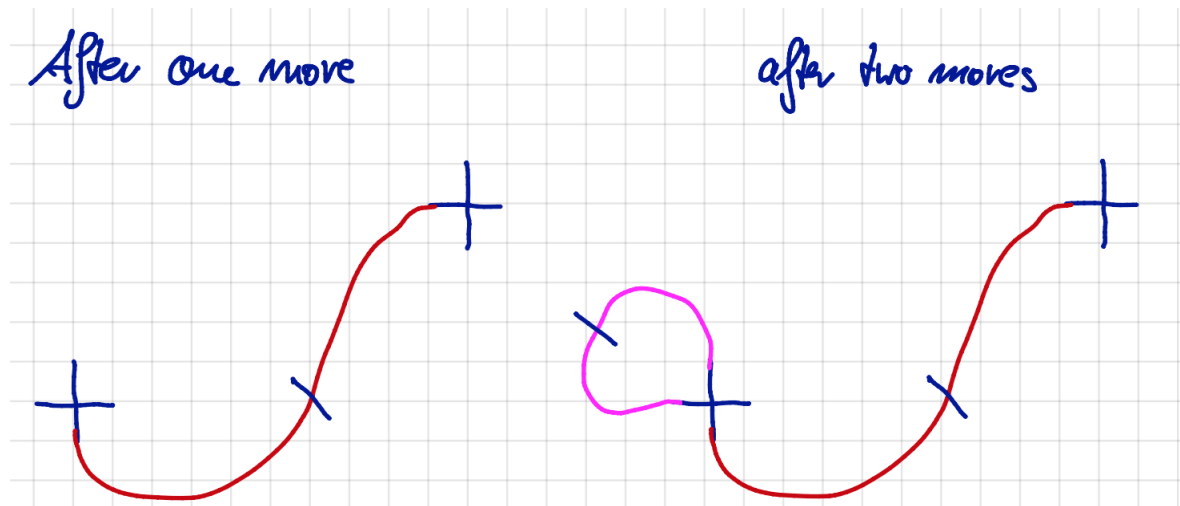
In this class we will start to walk on the path towards a proof of this problem. Unfortunately, we won't make it to the finish line within one semester. So, if you like, learn more about it in **Advanced Algebraic Topology...**

- Before we move on, let us play **a game** and see an invariant in action.

The rules: Take a piece of paper and draw two crosses, i.e. spots with four free ends.

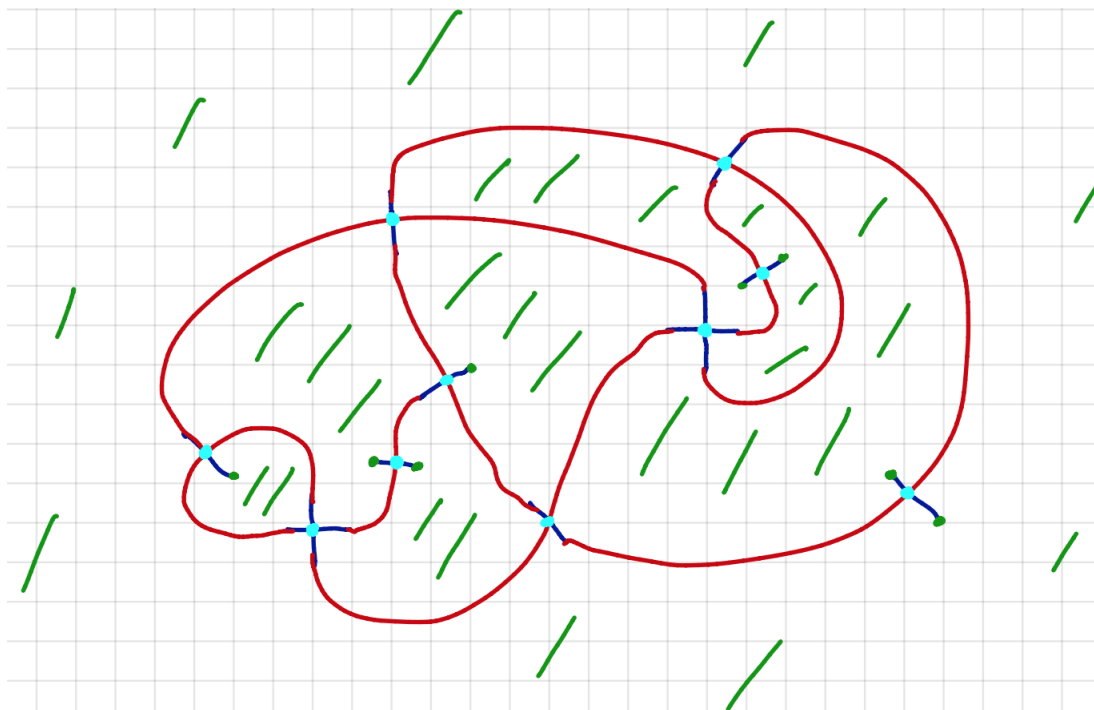


Each move involves joining two free ends with a curve which does not cross any existing line, and then putting a short stroke across the line to create two new free ends. The players play alternating moves.



If there are no legal moves left, **the player who made the last legal move wins.**

Let us assume that we know that the game ends after a finite number of moves, say m moves. At the end we will have created a connected, closed planar graph, in particular, a figure which has vertices, edges and faces.



We claim that no matter how you play, what strategy you use etc, there are always 8 moves before the game stops, it is always the second player who wins and there is a fixed number of vertices, edges and faces!

Why? Well, the number of moves and everything about the figure we create is determined by **Euler's formula** $v - e + f = 2$. The number 2 is an **example of an algebraic invariant**.

To understand how this works, we need to determine how the number v of vertices, the number e of edges, and the number f of faces depends on the number m of moves.

For the vertices, when we start the game we have two vertices. In each move, we create one new vertex. Thus we get

$$v = 2 + m.$$

For the edges, when we start the game we have no edges. In each move, we create one line, but we split it into two edges by adding a vertex in the middle. Hence in each move, we create 2 edges. Thus we get

$$e = 2m.$$

For the faces, we have to think backwards. At the end, there is exactly one free (or loose) end pointing into each face that we created. For, if there was a face with two free ends pointing into one face, then we could connect these two ends within that face and the game would not have stopped. Note that there is also exactly one loose end pointing out of the figure. (Again, if there were two we could connect them by going around the figure.)

Now we need to check how many free ends we produce. We start with 4 free ends per cross, that is 8 free ends. In each move, we connect two free ends, but we also create two new ones. Thus the number of free ends does not change during the whole game. Hence we get

$$f = 8.$$

In total we get

$$\begin{aligned} 2 &= v - e + f \\ 2 &= 2 + m - 2m + 8 \\ 0 &= -m + 8 \\ m &= 8. \end{aligned}$$

Hence no matter how we play, the game ends after 8 moves. Since this number is even, the second player always wins. Moreover, we always get $v = 10$, $e = 16$, and $f = 8$.

Alternative: Changing the starting setup changes the outcome of the game. For if we start with n crosses (or nodes), then we get with the same reasoning as above

$$\begin{aligned} v &= n + m \\ e &= 2m \\ f &= 4n. \end{aligned}$$

Euler's formula then yields

$$\begin{aligned} 2 &= v - e + f \\ 2 &= n + m - 2m + 4n \\ m &= 5n - 2. \end{aligned}$$

Thus the game ends after $m = 5n - 2$ moves. For example, if $n = 3$, this is an odd number and the first player always wins.

Here is the idea why Euler's formula holds: We can draw any connected planar graph (a graph we can draw in the plane such that its edges only intersect in the vertices and we can walk along the edges between any two vertices) as follows:

- 1) We start with a graph consisting of **just one vertex** and no edges, so $v = 1$ and $e = 0$. And we have one face, the outer face or the plane around the vertex, so $f = 1$. So in total the formula holds $v - e + f = 1 - 0 + 1 = 2$.
- 2) Now we can extend the graph by either
 - a) adding one vertex and connect it via an edge to the first one; that is we change $v \rightarrow v + 1$ and $e \rightarrow e + 1$ or
 - b) draw an edge from the existing vertex to itself; this way we create a new face as well, hence we change $e \rightarrow e + 1$ and $f \rightarrow f + 1$.

Thus after both operations the formula $v - e + f = 2$ still holds. Now we continue this process until we have created the planar graph we had in mind.

Here is another example of the use of an algebraic invariant:

• **Football pattern:**

Question: How many pentagons and hexagons are there on a classical football?

We set $P := \text{\# of pentagons}$ and $H := \text{\# of hexagons}$. The collection of all vertices, edges and faces of all the pentagons and hexagons on the football forms a **graph on the surface of the football**. This graph and therefore the pattern on the football is governed by **Euler's formula** $v - e + f = 2$. Hence we need to calculate the number of vertices v , the number of edges e and the number of faces f .

The number of faces is obviously given by

$$f = P + H.$$

To calculate the number of edges e we observe that every pentagon has 5 edges and every hexagon has 6 edges. That yields $5P + 6H$ edges. **But** we have counted too many edges. For at each edge, there are **two faces which meet**. Thus we need to divide our number by 2 and get

$$e = \frac{5P + 6H}{2}.$$

To calculate the number of vertices v we observe again that every pentagon has 5 vertices and every hexagon has 6 vertices. That yields $5P + 6H$ edges. **But again** we have counted too many vertices. For at each vertex, there are **three faces which meet**. Thus we need to divide our number by 3 and get

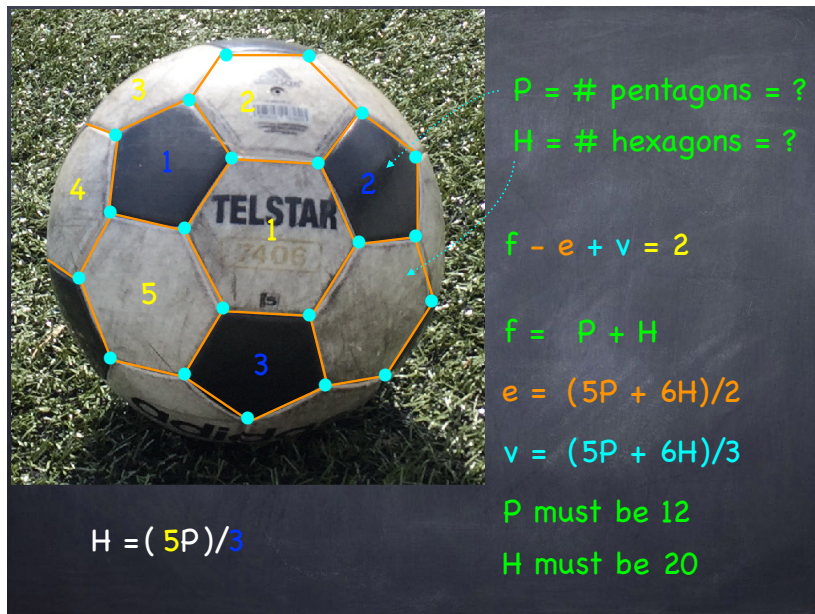
$$v = \frac{5P + 6H}{3}.$$

Now we apply Euler's formula:

$$\begin{aligned} v - e + f &= 2 \\ \frac{5P + 6H}{3} - \frac{5P + 6H}{2} + P + H &= 2 && \text{(multiply by 6)} \\ 10P + 12H - 15P - 18H + 6P + 6H &= 2 && \text{(simplify)} \\ P &= 12. \end{aligned}$$

To get H , we count how many hexagons there are per pentagon: Each pentagon is surrounded by 5 hexagons which would yield $H = 5P$. But each hexagon is attached to 3 pentagons at the same time. Hence we have counted three times as many hexagons as there really are. This yields

$$H = \frac{5P}{3} = \frac{5 \cdot 12}{3} = \frac{60}{3} = 20.$$



The Euler characteristic:

The number $v - e + f = 2$ is an example of the Euler characteristic of a surface, i.e., a two-dimensional manifold. In fact, 2 is the Euler characteristic of the sphere. The Euler characteristic can be defined for any topological space X . It is denoted by $\chi(X)$ and it is always an integer number. More generally, $\chi(X)$ is a topological invariant which means it does not change if we transform X continuously.

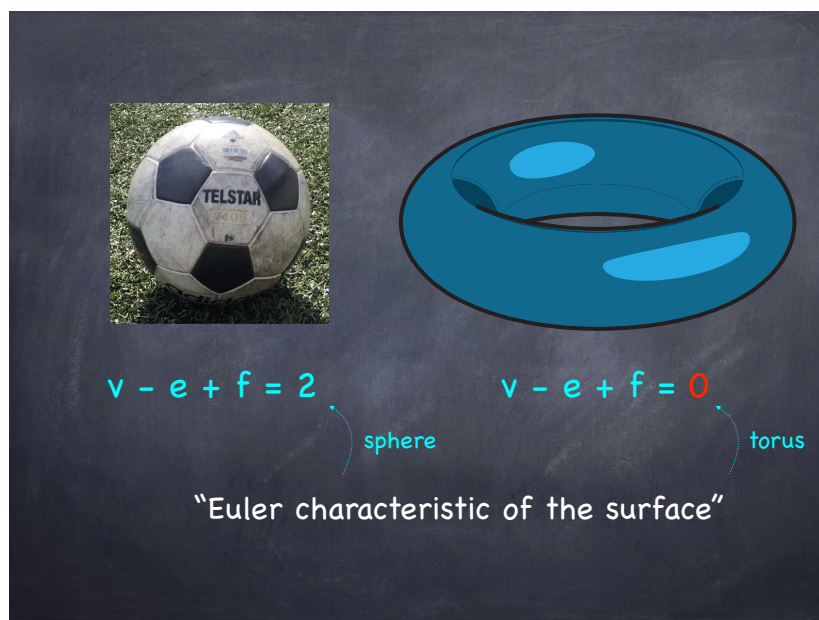
Here are some examples for surfaces:

- For a sphere $X = S^2$, it is 2: $\chi(S^2) = 2$.
- For a torus $X = T^2$, it is 0: $\chi(T^2) = 0$.
- For a surface with two holes, it is -2 .
- In general, for a surface with g holes, it is $2 - 2g$.

Now assume we have two spaces X and Y , defined in some complicated way which makes it difficult to understand how they look. But let us assume we can calculate their Euler characteristics by some method. Then, if $\chi(X) \neq \chi(Y)$, we know that we cannot transform X continuously into Y .

And there are also more positive examples. It often happens that an invariant defined one way turns out to encode a lot of other information as well.

You will learn more about these things soon...



The rough initiating idea for our algebraic invariants

Now back to the general situation. Let us try to get a first idea of how algebraic topologists think and come up with their fancy invariants. Let us say we have a space X and we want to characterise it, or at least be able to distinguish it from other spaces.

The initiating idea is to study X by taking **test spaces** we understand well and looking at the **space of all maps** from these test spaces into X . This may sound like an awkward detour, but it turns out to be pretty smart.

So what are those test spaces? The most simple space is a point. So let $T = \bullet$ the one-point space and let $C(\bullet, X)$ be the set/space of all (continuous) maps from \bullet into X . Since any map in $C(\bullet, X)$ is determined by the one-point image, we just get that $C(\bullet, X)$ is the set of points of X .

So what happens if we take a **one-dimensional test space** like the unit interval $[0,1]$? A continuous map $\gamma: [0,1] \rightarrow X$ is a path in X from $\gamma(0)$ to $\gamma(1)$. Each $\gamma(0)$ and $\gamma(1)$ also gives us also an element in $C(\bullet, X)$. Hence if we look at $C(\bullet, X)$ modulo those which can be connected by a path in $C([0,1], X)$, then we can read off how many “pieces” X has. Making this more precise gives us the set $\pi_0(X)$ of connected components of X . The set $\pi_0(X)$ is the first example of an **algebraic invariant**.

Let us keep going with this. A **two-dimensional test space** might be the square $[0,1] \times [0,1]$. Given a continuous map $\alpha: [0,1] \times [0,1] \rightarrow X$, the two restrictions $\alpha(0,t)$ and $\alpha(1,t)$ define two paths in X . If we assume that they have the same start and end points, then we get a relation on the set $C([0,1], X)$ of paths in X .

This leads, by looking only at paths which are loops, to the **fundamental group** $\pi_1(X)$ of X , the next algebraic invariant.

Continuing this way and to look at maps from an **n -dimensional test space modulo relations** that come from maps from a corresponding $n+1$ -dimensional test space we can produce a sequence of algebraic invariants $\pi_2(X), \pi_3(X), \dots$ which are called the **higher homotopy groups** of X . (Actually, for the n th homotopy group one uses the n -dimensional sphere, the n -dimensional space with the maximal symmetry.)

The collection of all homotopy groups encodes a lot of information about X . In fact, in many cases it contains all the information about X **up to homotopy**, i.e., continuous deformation of X .

However, **homotopy groups** are notoriously **difficult to compute**. That is why one also uses a **different type of test spaces**.

Starting again with the unit interval $[0,1]$ in dimension one, we could also proceed as follows. In dimension two we take an equilateral **triangle**, called a two-simplex. In dimension three we take a regular **tetrahedron**, called a three-simplex. In dimension four, we continue with a regular four-dimensional simplex and so forth.

This leads to the **singular homology groups** $H_n(X)$ of X . These groups will be the main object of our studies for a while. In general, they carry less information than homotopy groups. However, their big advantage is that they are **computable**! During the next couple of weeks we will develop the machinery to compute homology groups. Along the way we will witness many fundamental ideas that turned out to be extremely useful in many areas of mathematics...

LECTURE 2

Cell complexes and homotopy

Our main goal today is to introduce cell complexes as an important type of topological spaces and the concept of homotopy which is a fundamental idea to simplify problems.

We start with a super brief recollection of some basic notions in topology.

A crash course in topology

Roughly speaking, a topology on a set of points is a way to express that points are near to each other as a generalization of a space with a metric, i.e., a concrete distance function.

You know the fundamental example of a metric space. For, recall from Calculus 2 that the norm of a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is defined by

$$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \in \mathbb{R}.$$

For any n , the space \mathbb{R}^n with this norm is called **n -dimensional Euclidean space**. The norm induces a metric, i.e., a distance function by

$$d(x, y) := |x - y| \text{ for } x, y \in \mathbb{R}^n.$$

This turns \mathbb{R}^n into a metric space and therefore an example of a **topological space** in the following way:

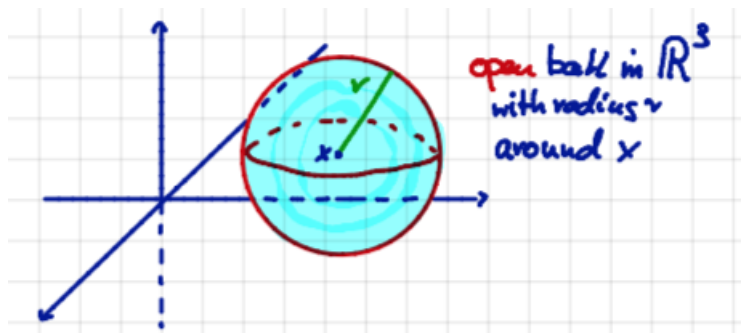
Open sets in \mathbb{R}^n

- Let x be a point in \mathbb{R}^n and $r > 0$ a real number. The ball

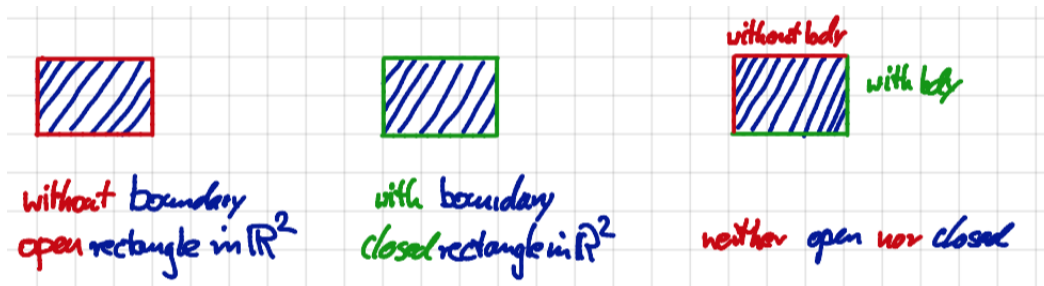
$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$$

with radius r around x is an **open** set in \mathbb{R}^n .

- The open balls $B_r(x)$ are the prototypes of open sets in \mathbb{R}^n .
- A subset $U \subseteq \mathbb{R}^n$ is called **open** if for every point $x \in U$ there exists a real number $r > 0$ such that $B_r(x)$ is contained in U .
- A subset $Z \subseteq \mathbb{R}^n$ is called **closed** if its complement $\mathbb{R}^n \setminus Z$ is open in \mathbb{R}^n .



- Familiar examples of open sets in \mathbb{R} are open intervals, e.g. $(0,1)$ etc.
- The cartesian product of n open intervals (an open rectangle) is open in \mathbb{R}^n .
- Similarly, closed intervals are examples of closed sets in \mathbb{R} .
- The cartesian product of n closed intervals (a closed rectangle) is closed in \mathbb{R}^n .
- The empty set \emptyset and \mathbb{R}^n itself are both open and closed sets.
- Not every subset of \mathbb{R}^n is open or closed. There are a lot of subsets which are neither open nor closed. For example, the interval $(0,1]$ in \mathbb{R} ; the product of an open and a closed interval in \mathbb{R}^2 .



The set of open sets in \mathbb{R}^n

$$\mathcal{T}_{\mathbb{R}^n} = \{U \subseteq \mathbb{R}^n \text{ open}\}$$

is a subset of all subsets of \mathbb{R}^n and has the following properties:

- $\emptyset, \mathbb{R}^n \in \mathcal{T}_{\mathbb{R}^n}$
- $U_j \in \mathcal{T}_{\mathbb{R}^n}$ for all $j \in J \Rightarrow \cup_{j \in J} U_j \in \mathcal{T}_{\mathbb{R}^n}$
- $U_1, U_2 \in \mathcal{T}_{\mathbb{R}^n} \Rightarrow U_1 \cap U_2 \in \mathcal{T}_{\mathbb{R}^n}$.

We take these three properties as the model for a topology:

Definition: Topological spaces

Let X be a set together with a collection \mathcal{T}_X of subsets which satisfy

- (i) $\emptyset, X \in \mathcal{T}_X$
- (ii) $U_j \in \mathcal{T}_X$ for all $j \in J \Rightarrow \cup_{j \in J} U_j \in \mathcal{T}_X$
- (iii) $U_1, U_2 \in \mathcal{T}_X \Rightarrow U_1 \cap U_2 \in \mathcal{T}_X$.

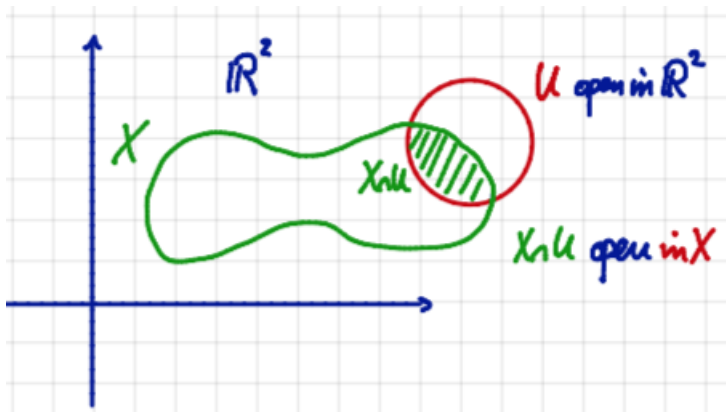
(Note that in (ii), J can be an arbitrary indexing set.)

Then we say that the pair (X, \mathcal{T}_X) is a topological space and the sets in \mathcal{T}_X are called open. We also say that \mathcal{T}_X defines a topology on X . We often drop mentioning \mathcal{T}_X and just say X is a topological space (when the topology \mathcal{T}_X is given otherwise). The complement of an open set is called a closed set.

Here are some examples of topological spaces which also demonstrate that some topologies are more interesting than others:

- \mathbb{R}^n with $\mathcal{T}_{\mathbb{R}^n}$ as described above.
- An arbitrary set X with the **discrete topology** $\mathcal{T}_X = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X , i.e., the set of all subsets of X . In the discrete topology, all subsets are open and hence all subsets are also closed.
- On an arbitrary set X , there is always the **coarse topology** $\mathcal{T}_X = \{\emptyset, X\}$.
- Let (X, d) be a **metric space**. Then we can imitate the construction of the standard topology on \mathbb{R}^n and define the **induced topology** as the set of all $U \subseteq X$ such that for each $x \in U$ there exists an $r > 0$ so that $B(x, r) \subseteq U$. Here $B(x, r) = \{y \in X : d(x, y) < r\}$ is the metric ball of radius r centred at x .
- Let (X, \mathcal{T}_X) be a topological space, let $Y \subset X$ be an arbitrary subset. The induced topology or **subspace topology of Y** is defined by

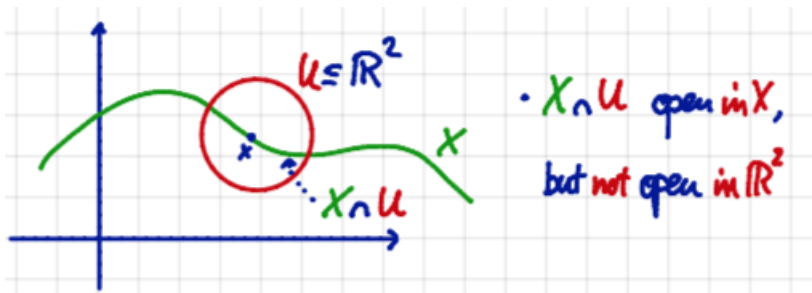
$$\mathcal{T}_Y := \{V \subset Y : \text{there is a } U \in \mathcal{T}_X \text{ such that } V = U \cap Y\}.$$



Warning

It is important to note that the property of being **an open subset** really depends on the bigger space we are looking at. Hence **open** always refers to being **open in** some given space.

For example, a set can be open in a space $X \subset \mathbb{R}^2$, but not be open in \mathbb{R}^2 , see the picture.



Open sets are nice for a lot of reasons. First of all, they provide us with a way to talk about things that happen **close to** a point.

Definition: Open neighborhoods

We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of x** if there is an open subset $U \subseteq V$ with $x \in U$. If V itself is open, we call V an **open neighborhood**.

The type of maps that preserve open sets are the continuous maps:

Definition: Continuous maps

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f: X \rightarrow Y$ is **continuous** if and only if, for every $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$, i.e., the **preimages of open sets are open**.

We denote the **set of continuous maps** $X \rightarrow Y$ by $C(X, Y)$.

Topological spaces form a **category** with morphisms given by continuous maps.

Examples of continuous maps include:

- Continuous maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$ that you are familiar with from Calculus 2.
- If X carries the **discrete topology** then **every** map $f: X \rightarrow Y$ is continuous.
- If Y carries the **coarse topology** then **every** map $f: X \rightarrow Y$ is continuous.

Definition: Homeomorphisms

A continuous map $f: X \rightarrow Y$ is a **homeomorphism** if it is one-to-one and onto, and its inverse f^{-1} is continuous as well. Homeomorphisms preserve the topology in the sense that $U \subset X$ is open in X if and only if $f(U) \subset Y$ is open in Y .

Homeomorphisms are the isomorphisms in the category of topological spaces.

Some examples are:

- $\tan: (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ is a homeomorphism.
- $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^3$ is a homeomorphism.

But not every continuous bijective map is a homeomorphism. Here is an example:

Example: A bijection which is not a homeomorphism

Let

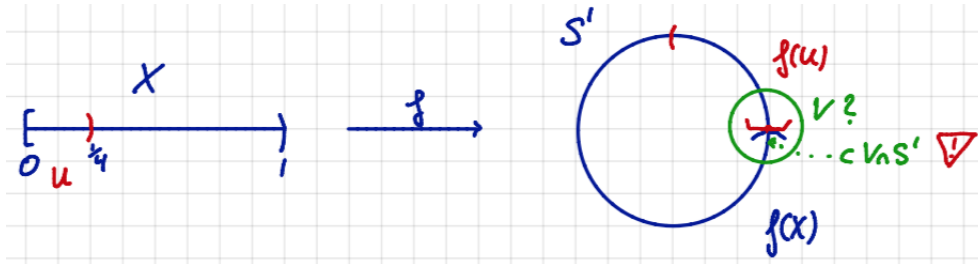
$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

be the unit circle considered as a subspace of \mathbb{R}^2 . Define a map

$$f: [0, 1) \rightarrow S^1, t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$

We know that f is bijective and continuous from Calculus and Trigonometry class. But the function f^{-1} **is not continuous**. For example, the image under f of the open subset $U = [0, \frac{1}{4})$ (open **in** $[0, 1]$!) is **not** open in S^1 . For the point $y = f(0)$ does not lie in any open subset V of \mathbb{R}^2 such that

$$V \cap S^1 = f(U).$$



Here is an extremely important property a subset in a topological space can have. We are going to use it quite often.

Definition: Compactness

Let X be a topological space. A subset $Z \subset X$ is called **compact** if for any collection $\{U_i\}_{i \in I}$, $U_i \subset X$ open, with $Z \subset \bigcup_{i \in I} U_i$ there exist **finitely many** $i_1, \dots, i_n \in I$ such that $Z \subset U_{i_1} \cup \dots \cup U_{i_n}$.

In other words, a subset Z in a topological space is compact iff every open cover $\{U_i\}_i$ of Z has a **finite** subcover.

- By the Theorem of Heine-Borel, a subset $Z \subset \mathbb{R}^n$ is **compact** if and only if it is **closed and bounded**. Being bounded means, that there is some (possibly huge) $r \gg 0$ such that $Z \subset B_r(0)$.
- In particular, neither \mathbb{R} nor any \mathbb{R}^n is compact.
- The n -dimensional disk $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and the n -sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ are compact.
- Finite sets, i.e., a subset which contains only finitely many elements, are always compact.
- If X carries the discrete topology, then a subset $Z \subset X$ is compact if and only if it is finite.
- If X carries the coarse topology, then every $Z \subset X$ is compact.

Definition: Connectedness

A topological space X is called **connected** if it is not possible to split it into the union of two non-empty, disjoint subsets which are both open and closed at the same time.

In other words, a space is connected if and only if the empty set and the whole space are the only subsets which are both open and closed.

Note that the image $f(X)$ of a connected space X under a continuous map $f: X \rightarrow Y$ is again connected.

Simple examples of connected spaces are given by intervals in \mathbb{R} .

Definition: Hausdorff spaces

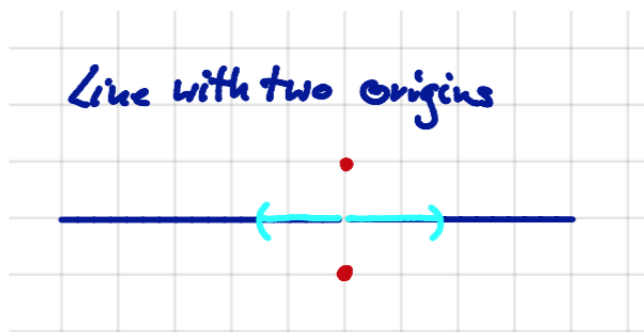
A topological space X is called **Hausdorff** if, for any two distinct points $x, y \in X$, there are two **disjoint** open subsets $U, V \subset X$ such that $x \in U$ and $y \in V$.

In other words, in a Hausdorff space we can separate points by open subsets.

Every subspace of \mathbb{R}^N (with the relative topology) is a Hausdorff space. Moreover, basically all the spaces we look at will be Hausdorff. However, there are spaces which are not Hausdorff.

For a **typical counter-example**, consider two copies of the real line $Y_1 := \mathbb{R} \times \{1\}$ and $Y_2 := \mathbb{R} \times \{2\}$ as subspaces of \mathbb{R}^2 . On $Y_1 \cup Y_2$, we define the equivalence relation $(x, 1) \sim (x, 2)$ for all $x \neq 0$.

Let X be the set of equivalence classes. The topology on X is the quotient topology defined as follows (see also below): a subset $W \subset X$ is open in X if and only if both its preimages in $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{2\}$ are open.



Then X looks like the real line except that the origin is replaced with two different copies of the origin. Away from the double origin, X looks perfectly nice and we can separate points by open subsets. But every neighborhood of one of the origins contains the other. Hence we cannot separate the two origins by open subsets, and X is **not Hausdorff**.

Here are some useful facts about compact spaces:

Lemma: Closed in compact implies compact

- 1) Let X be a **compact** topological space. Let $Z \subset X$ be a **closed** subset. Then Z is **compact**.
- 2) Let Y be a **Hausdorff** space. Then any compact subset of Y is **closed**.

Let us prove the first assertion. The other one is left as a little exercise.

Proof: Let $\{U_i\}_{i \in I}$ be an open cover of Z . We set $U := X \setminus Z$. Then $\{U, U_i\}_{i \in I}$ is an open cover of X . Since X is compact, there exist i_1, \dots, i_n such that $X \subset U \cup U_{i_1} \cup \dots \cup U_{i_n}$ and hence, by the definition of U , we have $Z \subset U_{i_1} \cup \dots \cup U_{i_n}$. **QED**

Another useful fact:

Lemma: Continuous images of compact sets are compact

Let $f: X \rightarrow Y$ be continuous. Let $K \subset X$ be compact. Then $f(K) \subset Y$ is compact.

But, in general, if $Z \subset Y$ is compact, then $f^{-1}(Z) \subset X$ does not have to be compact.

As a consequence we can deduce a useful criterion for when continuous bijections are homeomorphisms:

Lemma: Continuous bijection from compact to Hausdorff is a homeomorphism

Let X be a compact space and Y be Hausdorff. If $f: X \rightarrow Y$ is a continuous bijection, then f is a homeomorphism.

Proof: Since f is a bijection, there is a set-theoretic inverse map which we denote by $g := f^{-1}: Y \rightarrow X$. We need to show that g is continuous. So let $K \subset X$ be a closed subset. We are going to show that $g^{-1}(K) = f(K) \subset Y$ is closed in Y . Since X is compact, K is also compact as a closed subset. Hence its image $f(K) \subset Y$ is compact. Since Y is Hausdorff, this implies that $f(K)$ is closed in Y . **QED**

Compactness, being Hausdorff, and being connected are important examples of topological properties:

Homeomorphisms preserve topological properties

Slogan: **Topology** is the study of properties which are preserved under homeomorphisms. From this point of view, a **topological property** is by definition a property that is preserved under homeomorphisms.

Hence, roughly speaking, from the point of view of a topologist, two spaces which are homeomorphic are basically the same.

For example, if $f: X \rightarrow Y$ is a **homeomorphism**, then **X is compact if and only if Y is compact**. For, both f and its inverse f^{-1} are continuous and surjective maps. Hence if X is compact, so is $f(X) = Y$; and if Y is compact, so is $f^{-1}(Y) = X$.

We will remind ourselves of many other important topological properties along the way.

Constructing new spaces out of old

There are several ways to construct topological spaces. Here are two important constructions that we are going to use:

Definition: Product topology

Let X and Y be two topological spaces. The **product topology** on $X \times Y$ is the coarsest topology, i.e., the topology with fewest open sets, such that the projection maps $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$ are both continuous. More concretely, a subset $W \subset X \times Y$ is open in the product topology if for every point $w = (x, y) \in W$ there are open subsets $x \in U \subset X$ and $y \in V \subset Y$ with $U \times V \subset W$.

Definition: Disjoint unions or sums of spaces

Let X and Y be two topological spaces. We denote by $X \sqcup Y$ the **disjoint union** (or sum) of X and Y . Recall that as a set we can define $X \sqcup Y$ as

$$X \sqcup Y = X \times \{0\} \cup Y \times \{1\}.$$

(In other words, we take one copy of X and one copy of Y and by the indexing we make sure that we keep them apart.)

The disjoint union inherits a topology by defining

$$\mathcal{T}_{X \sqcup Y} = \{U \sqcup V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}.$$

Another important construction for producing new topological spaces is to take quotients.

• Quotient Spaces

Let X be a topological space. Let \sim be an equivalence relation on X . For any $x \in X$ let $[x]$ be the equivalence class of x . We denote as usually the set of equivalence classes by

$$X/\sim := \{\text{set of equivalence classes under } \sim\} = \{[x] : x \in X\}.$$

Let $\pi: X \rightarrow X/\sim$, $x \mapsto [x]$ be the natural projection. The **quotient topology** is defined by

$$U \subset X/\sim \text{ open} \iff \pi^{-1}(U) \subset X \text{ open}.$$

Note that the map $\pi: X \rightarrow X/\sim$ is continuous by definition.

The quotient topology is the coarsest topology, in the sense that it has fewest open sets, such that the quotient map π is continuous.

The quotient topology has the following **universal property**: For any topological space Y and for any maps $f: X \rightarrow Y$ which descends to a map $\bar{f}: X/\sim \rightarrow Y$, i.e., f is constant on equivalence classes, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \downarrow & \nearrow \bar{f} & \\ X/\sim & & \end{array}$$

commutes, the map f is continuous iff \bar{f} is continuous.

Many important examples of spaces that we will study arise as follows:

- Take a subset $X \subset \mathbb{R}^n$ and consider it with the induced topology as a subset.
- Consider an interesting equivalence relation \sim on X and take the quotient topological space X/\sim .

Let us look at some **examples** of this procedure:

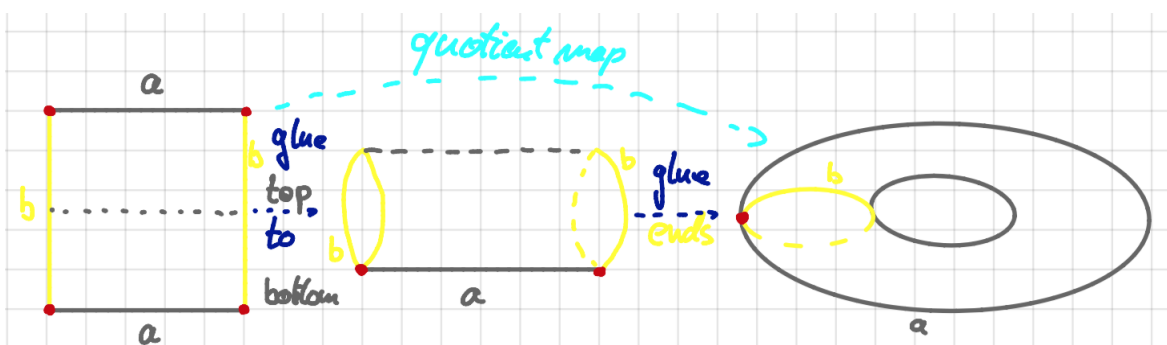
Torus

We start with the square

$$S := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\} \subset \mathbb{R}^2$$

with the subspace topology induced from the topology of \mathbb{R}^2 . Now we would like to glue opposite sides to each. This corresponds to taking the **quotient**

$$T := S / ((x,0) \sim (x,1) \text{ and } (0,y) \sim (1,y)).$$



Real projective space

Real projective space \mathbb{RP}^n is the space of lines in \mathbb{R}^{n+1} through the origin. As a topological space it can be constructed as follows:

We define the equivalence relation \sim on the n -sphere S^n by identifying antipodal points, i.e., $x \sim y \iff y = -x$. Then we have

$$\mathbb{RP}^n = S^n / \sim$$

and equip it with the quotient topology. Since S^n is compact and \mathbb{RP}^n is the continuous image of S^n (under the quotient map), we see that **\mathbb{RP}^n is compact**.

There is also a complex version:

Complex projective space

Again, complex projective space \mathbb{CP}^n is the space of one-dimensional \mathbb{C} -vector subspaces in \mathbb{C}^{n+1} . It can be topologized as follows:

We define the equivalence relation \sim on the sphere S^{2n+1} by $x \sim y$ if and only if there is a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $y = \lambda x$ where we think S^{2n+1} as the subspace of points x in \mathbb{C}^n with $|x| = 1$. Then we have

$$\mathbb{CP}^n = S^{2n+1} / \sim$$

and equip it with the quotient topology. Since S^{2n+1} is compact, \mathbb{CP}^n is compact.

Aside: Note that the “topological dimension” of \mathbb{CP}^n is $2n$ (we have not said what that means though). The n rather refers to the dimension as a complex manifold.

Projective spaces play an important role in geometry and topology. We will meet them quite frequently during this course (and future courses).

It happens also that it might be necessary to present a well-known space in a different form. For example, we can write spheres as quotients. We will see that this is just one example of a whole class of interesting spaces.

Sphere as a quotient

For every $n \geq 1$, there is a homeomorphism

$$\bar{\rho}: D^n / \partial D^n \xrightarrow{\sim} S^n.$$

There are in fact many different ways to construct such a homeomorphism. Let us write down one in concrete terms for the special case $n = 2$. The general case follows by throwing in more coordinates.

We define a continuous map $\rho: D^2 \rightarrow S^2$ such that

$$\begin{cases} \rho(0,0) = (0,0,-1) & \text{and} \\ \rho(x,y) = (0,0,+1) & \text{for all } (x,y) \in \partial D^2 = S^1. \end{cases}$$

Since ρ will be constant on ∂D^2 , it will induce a map $\bar{\rho}$ on the quotient $D^2 / \partial D^2$.

We define ρ as a rotation invariant map which sends the inner part of D^2 of points with radius less than $1/2$ mapping onto the lower hemisphere of S^2 and the outer part of D^2 of points with radius greater than $1/2$ mapping

onto the upper hemisphere $\rho: D^2 \rightarrow S^2$ by

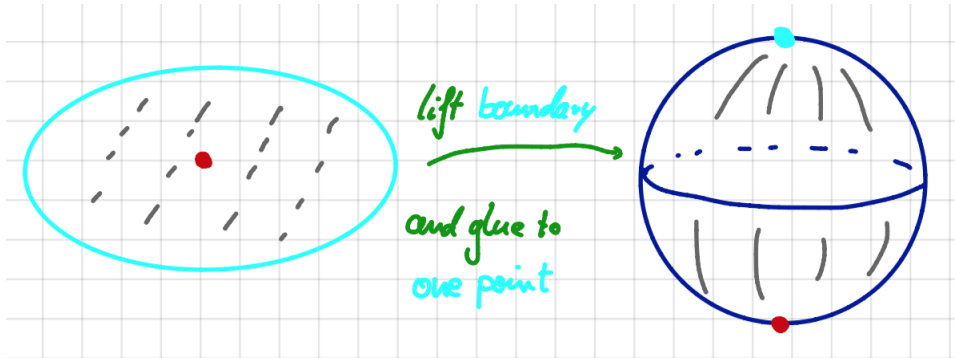
$$\rho(x,y) = \begin{cases} \left(2x, 2y, -\sqrt{1-4(x^2+y^2)} \right) & \text{if } x^2+y^2 \leq 1/4 \\ \left(f(x,y)x, f(x,y)y, \sqrt{1-f(x,y)^2(x^2+y^2)} \right) & \text{if } x^2+y^2 \geq 1/4 \end{cases}$$

where we denote $f(x,y) = 4 - 4\sqrt{x^2+y^2}$ (to make the formula fit in a frame). This map is well-defined also for points with $x^2+y^2 = 1/4$. Moreover, ρ is continuous, as a composite of continuous functions, and constant on ∂D^2 .

An inverse map can be defined by

$$S^2 \rightarrow D^2/\partial D^2, (x,y,z) \mapsto \begin{cases} (\frac{1}{2}x, \frac{1}{2}y) & \text{if } -1 \leq z \leq 0 \\ (g(x,y)x, g(x,y)y) & \text{if } 0 \leq z < 1 \\ \text{class of } \partial D^2 & \text{for } (0,0,1) \end{cases}$$

where we denote $g(x,y) = \frac{1-\sqrt{1-\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}$. Note that this map is well-defined also for $z = 0$, since then $x^2+y^2 = 1$ and $g(x,y) = 1/2$.



• Compactifications

The concrete maps we wrote down in the previous example are kind of ugly. But there is another way to show that there is such a homeomorphism $D^n/\partial D^n \approx S^n$.

For we can also consider S^n as the **one-point compactification** of \mathbb{R}^n . Let us first say what that means:

Definition: One-point compactification

1) If Y is a compact Hausdorff space and $X \subset Y$ is a proper subspace whose closure equals Y , then Y is called a **compactification** of X .

If $Y \setminus X$ consists of a single point, then Y is called the **one-point compactification** of X .

2) Let X be a topological space with topology \mathcal{T}_X . Let ∞ denote an abstract point which is not in X and let $\hat{X} := X \cup \{\infty\}$. We define a topology $\mathcal{T}_{\hat{X}}$ on \hat{X} as follows:

- each open set in X is an open set in \hat{X} , i.e., $\mathcal{T}_X \subset \mathcal{T}_{\hat{X}}$ and
- for each compact subset $K \subseteq X$, define an open subset $U_K \in \mathcal{T}_{\hat{X}}$ by $U_K := (X \setminus K) \cup \{\infty\}$.

Then \hat{X} is a **one-point compactification** of X .

To see that \hat{X} actually is compact, take any open cover of \hat{X} . Then at least one of the open sets contains ∞ . Hence that set covers $(X \setminus K) \cup \{\infty\}$ for some compact set K . Since K is compact, finitely many of the remaining open sets suffice to cover K and therefore all of \hat{X} .

Examples of one-point compactifications are **spheres**. For S^n is the one-point compactification of \mathbb{R}^n . For $n = 1$, one can think of S^1 as taking the real number line and connect the two ends at infinity in one point ∞ to close the circle. More generally, one can construct a homeomorphism via **stereographic projection**.

As an application, we give a new proof $D^n / \partial D^n \approx S^n$:

Sphere as a quotient revisited

For every $n \geq 1$, there is a homeomorphism

$$\bar{\rho}: D^n / \partial D^n \xrightarrow{\sim} S^n.$$

Since $S^n \approx \mathbb{R}^n \cup \{\infty\}$, it suffices to construct homeomorphism

$$\rho: D^n \xrightarrow{\sim} \mathbb{R}^n \cup \{\infty\}, x \mapsto \begin{cases} \frac{x}{1-|x|} & \text{if } |x| < 1 \\ \infty & \text{if } |x| = 1. \end{cases}$$

We claim that the map ρ is **continuous**. To show this, we use the sequential criterion of continuity. Let (a_n) be a sequence in D^n with $\lim_{n \rightarrow \infty} a_n = c$. If $c \in D^n \setminus \partial D^n$ is an interior point, then $\rho(c) \in \mathbb{R}^n$ and we know $\lim_{n \rightarrow \infty} \rho(a_n) = \rho(c)$, since the restriction of ρ to $D^n \setminus \partial D^n$ is a composite of continuous maps and the a_n will all be in $D^n \setminus \partial D^n$ for n sufficiently

large. If $c \in \partial D^n$ is a boundary point, then $\rho(c) = \infty$. Since $a_n \rightarrow c$, the sequence $(\rho(a_n))$ is **unbounded**, since the denominator of $\rho(a_n)$ tends to 0 while the norm of the nominator tends to 1.

Hence for any compact subset K in \mathbb{R}^n , i.e., for any closed and bounded $K \subset \mathbb{R}^n$, there is a natural number $N(K)$ such that $\rho(a_n) \notin K$ for all $n \geq N(K)$. That means that the sequence $(\rho(a_n))$ **converges in the topology** of $\mathbb{R}^n \cup \{\infty\}$ to $\rho(c) = \infty$. This shows that ρ is continuous.

We also know that $\bar{\rho}$ is **bijective**, since the restriction $\rho: D^n \setminus \partial D^n \rightarrow \mathbb{R}^n$ is bijective and ρ sends ∂D^n to ∞ . Hence $\bar{\rho}$ is a continuous bijection from a compact space to a Hausdorff space. As we have seen above, this implies that $\bar{\rho}$ is a homeomorphism.

• Cell complexes

Another way to think of the above procedure is the following. The sphere consists of two parts that we **glue together**:

- an open n -disk, i.e., the open interior $D^n \setminus \partial D^n$,
- and a single point, which corresponds to the class of the boundary ∂D^n ; on S^2 we can picture this point as the northpole (the light blue dot in the above picture).

Topologists think of such **building blocks** as the **cells** of a space. However, not all spaces can be built this way. So let us make precise what is needed:

Definition: Cell complexes

A **cell complex** or **CW-complex** is a space X which results from the following inductive procedure:

- (1) Start with a discrete set X^0 . The points of X^0 will be the **0-cells** of X .
- (2) If X^{n-1} is defined, we construct the **n -skeleton** X^n by attaching **n -cells** e_α^n to X^{n-1} via continuous maps $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$. This means that X^n is the **quotient space** of the disjoint union $X^{n-1} \sqcup_\alpha D_\alpha^n$ of X^{n-1} with a collection of n -disks D_α^n under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$ and $\varphi_\alpha: \partial D_\alpha^{n-1} = S_\alpha^{n-1} \rightarrow X^{n-1}$. Thus, as a set, X^n consists of X^{n-1} together with a union of n -cells e_α^n each of which is an open n -disk $D_\alpha^n \setminus \partial D_\alpha^n$.
- (3) If this process stops after finitely many steps, say N , then $X = X^N$. But it is also allowed to continue with the inductive process indefinitely. In this case, one defines $X = \bigcup_n X^n$ and equips X

with the **weak topology**, i.e., a set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n .

We have already seen some **examples** of cell complexes:

- The **sphere** S^n is a cell complex with just two cells: **one 0-cell** e^0 (that is a point) and **one n -cell** e^n which is attached to e^0 via the constant map $S^{n-1} \rightarrow e^0$. Geometrically, this corresponds to expressing S^n as $D^n/\partial D^n$: we take the open n -disk $e^n = D^n \setminus \partial D^n$ and collapse the boundary ∂D^n to a single point which is e^0 .
- **Real projective space** \mathbb{RP}^n is a cell complex with **one cell in each dimension** up to n . To show this we proceed inductively. We know that \mathbb{RP}^0 consists of a single point, since it is S^0 whose two antipodal points are identified. Now we would like to understand how \mathbb{RP}^n can be constructed from \mathbb{RP}^{n-1} : We embed D^n as the **upper hemisphere** into S^n , i.e., we consider D^n as $\{(x_0, \dots, x_n) \in S^n : x_0 \geq 0\}$. Then

$$\mathbb{RP}^n = S^n / x \sim -x = D^n / (x \sim -x \text{ for } \textbf{boundary points } x \in \partial D^n).$$

But ∂D^n is just S^{n-1} . Thus the quotient map

$$S^{n-1} \rightarrow S^{n-1} / \sim = \mathbb{RP}^{n-1}$$

attaches an n -cell e^n , the open interior of D^n , at \mathbb{RP}^{n-1} . Thus we obtain \mathbb{RP}^n from \mathbb{RP}^{n-1} by attaching one n -cell via the quotient map $S^{n-1} \rightarrow \mathbb{RP}^{n-1}$. Summarizing, we have shown that \mathbb{RP}^n is a cell complex with one cell in each dimension from 0 to n :

$$\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n.$$

- We can continue this process and build the **infinite projective space** $\mathbb{RP}^\infty := \bigcup_n \mathbb{RP}^n$. It is a cell complex with one cell in each dimension. We can think of \mathbb{RP}^∞ as the space of lines in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

The **torus** is a cell complex with **one 0-cell**, **two 1-cells** and **one 2-cells**. This should be apparent from the construction of the torus as a quotient of a square that we have seen above. Starting with X^0 being a point **p** , the red dot in the picture below. Then we attach two open 1-cells $e_a^1, e_b^1 \subset D^1$ via the two constant maps

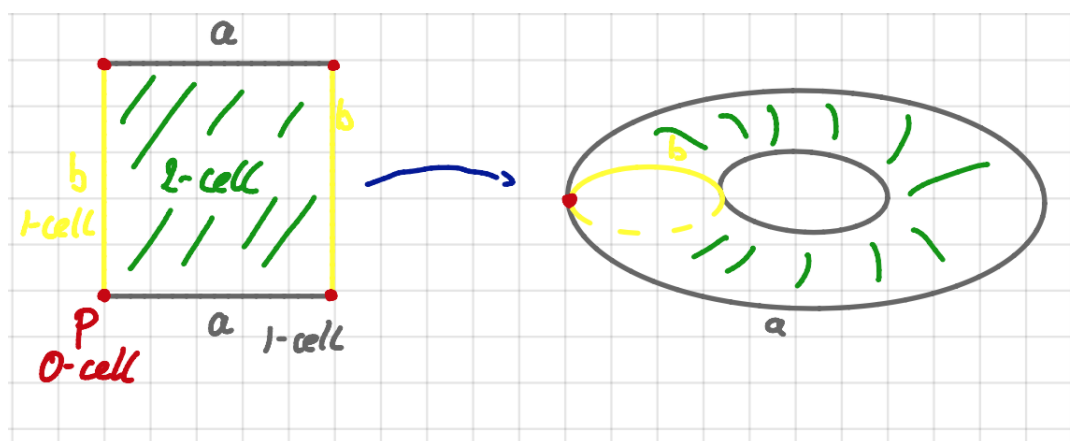
$$\varphi_a^1, \varphi_b^1: S^0 \rightarrow X^0$$

where we think of $e^1 = (0,1) \subset [0,1] = D^1$ as the open unit interval. (In the picture they look like two straight lines, but we should think of the end points being attached to **p** .)

Finally, we attach an open 2-cell $e^2 \subset D^2$ via the attaching map

$$\varphi^2: S^1 \rightarrow X^1, \varphi^2(x, y) = \begin{cases} x \in e_a^1 & \text{if } y > 0 \\ x \in e_b^1 & \text{if } y < 0 \\ p & \text{if } y = 0. \end{cases}$$

Note that this is a well-defined map, since we have identified the end-points in X^1 with p and hence $(1,0)$ and $(-1,0)$ are sent to the same point p .



- Actually, every **compact smooth manifold** can be turned into a finite cell complex. This illustrates the vast scope and importance of cell complexes in algebraic topology.

What makes topology unique

Note that the ability to **build spaces by gluing** together cells (or other specific spaces) makes life as a topologist particularly comfortable. For example, we will see that this procedure will often allow us to create spaces with given algebraic invariant. This flexibility together with the concept of **homotopy**, which we will explore next, puts **algebraic topologists** in a unique position and led to the **solution of a lot of problems**, not just in topology. **Geometry**, in its various forms, is usually much **more rigid** and does not allow us to perform such manoeuvres.

Note that there is a direct way to define the **Euler characteristic of cell complexes**. We will later see the reason why this is the correct definition using homology. Right now we can already check at the example of a tetrahedron that this definition agrees with Euler's formula we saw in the first lecture.

Definition: Euler characteristic for cell complexes

The Euler number of a cell complex X (with cells in dimension at most n) is defined to be the integer

$$\chi(X) = \sum_{k=0}^n (-1)^k \# \{k\text{-dimensional cells that are attached to } X^{k-1}\}.$$

For **example**, the Euler characteristic of S^n is

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

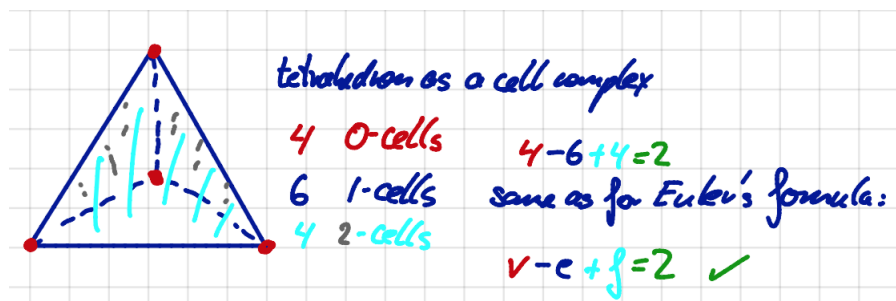
For real projective n -space we get

$$\chi(\mathbb{RP}^n) = 1 - 1 + 1 - \cdots + (-1)^n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

For the torus, we get

$$\chi(T) = 1 - 2 + 1 = 0.$$

To compare this definition with Euler's formula we used in the first lecture, let us look at the tetrahedron which is also a cell complex:



• Homotopy

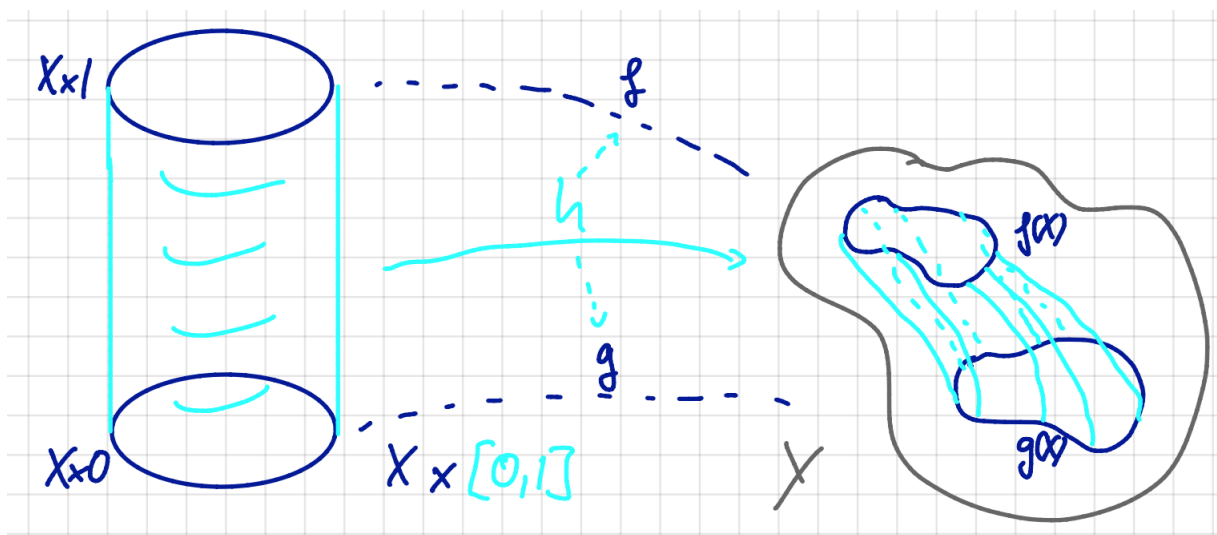
Homotopy is a fundamental notion in topology. Let us start with a definition and then try to make sense of this.

Definition: Homotopies

Let $f_0, f_1: X \rightarrow Y$ be two continuous maps. Then f_0 and f_1 are called **homotopic**, denoted $f_0 \simeq f_1$, if there is a continuous map $h: X \times [0,1] \rightarrow Y$ such that, for all $x \in X$,

$$h(x,0) = f_0(x), \text{ and } h(x,1) = f_1(x).$$

Homotopy defines an equivalence relation (exercise!) on the set of continuous maps from X to Y . The set of equivalence classes of continuous maps from X to Y modulo homotopy is denoted by $[X,Y]$.



Definition: Homotopy equivalences and contractible spaces

- A continuous map $f: X \rightarrow Y$ is called a **homotopy equivalence** if there is a continuous map $g: Y \rightarrow X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.
- Two spaces X and Y are called **homotopy equivalent** if there exists a homotopy equivalence f between X and Y . This is often denoted by $X \simeq Y$.
- A space which is homotopy equivalent to a one-point space is called **contractible**.

For **example**, \mathbb{R}^n is contractible, since

$$h: \mathbb{R}^n \times [0,1] \rightarrow \mathbb{R}^n, (t,x) \mapsto (1-t)x$$

defines a homotopy between the identity map on \mathbb{R}^n and the constant map $\mathbb{R}^n \rightarrow \{0\} \subset \mathbb{R}^n$ to the one-point space consisting of the origin. For the same reason, the n -disk D^n is contractible.

However, it is not always obvious which spaces are homotopy equivalent to each other. So it will be useful to develop some intuition for homotopy equivalences. There is a particular type that is easier to spot:

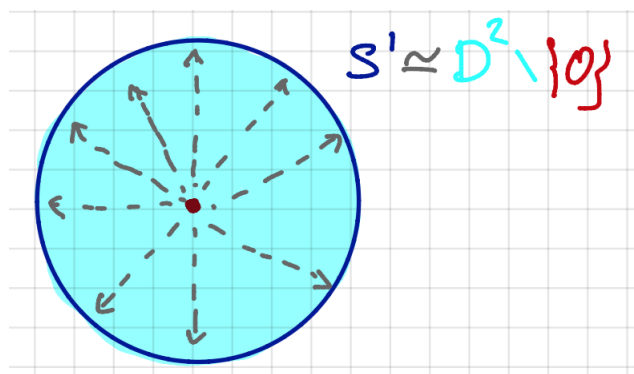
Definition: Deformation retracts

Let X be a topological space and $A \subset X$ a subspace.

- Then A is called a **retract** of X if there is a **retraction** $\rho: X \rightarrow A$, i.e., there is a continuous map $\rho: X \rightarrow A$ with $\rho|_A = \text{id}_A$.
- Note that we can **consider** ρ also as a **map** $X \rightarrow X$ via the inclusion $X \xrightarrow{\rho} A \subset X$. If ρ is then in addition homotopic to the identity of X , then A is called a **deformation retract** of X . In this case, ρ is called a **deformation retraction**. Note that in this case, ρ and the inclusion $A \subset X$ are **mutual homotopy inverses**.
- If this homotopy between ρ and id_X can be chosen such that all points of A **remain fixed**, i.e., the homotopy $h(t,a) = a$ for all $a \in A$ and all $t \in [0,1]$, then ρ is called a **strong deformation retraction** and A is called a **strong deformation retract** of X .

For a deformation retraction, one can think of the homotopy h as a map which during the time from 0 to 1 pulls back all the points of X into the subspace A , and leaves the whole time the points in A fixed. Here are some examples:

- The origin $\{0\}$ is a strong deformation retract of \mathbb{R}^n and of the n -disk D^n .
- For any topological space Y , the product $Y \times \{0\}$ is a strong deformation retract of $Y \times \mathbb{R}^n$ and $Y \times D^n$. For example, the circle $S^1 \times \{0\}$ is a strong deformation retract of the solid torus $S^1 \times D^2$.
- The n -sphere S^n is a strong deformation retract of the punctured disk $D^{n+1} \setminus \{0\}$ and also of $\mathbb{R}^{n+1} \setminus \{0\}$.



Why homotopy?

The simplest reason why we consider the homotopy relation is that **it works**. It is **fine enough** such that all the tools that we are going to define are invariant under homotopy, i.e., they are constant on equivalence classes. But it is also **coarse enough** that it identifies enough things such that many problems become simpler and in fact solvable.

With respect to first point, one can consider the **homotopy category** **hoTop** of spaces, i.e., the category whose objects are topological spaces and whose sets of morphisms from X to Y are the sets of homotopy classes of maps $[X, Y]$, satisfies a universal property for invariants.

With respect to the second point, we just indicate that life in **hoTop** is much easier because there are much **fewer morphisms**. For example, there are many and complicated continuous maps $S^1 \rightarrow \mathbb{C} \setminus \{0\}$. But there are very few homotopy classes of such maps, since $[S^1, \mathbb{C} \setminus \{0\}] = \mathbb{Z}$, **up to homotopy** a map $S^1 \rightarrow \mathbb{C} \setminus \{0\}$ is determined by the winding number, i.e., the number of times it goes around the origin.

To convince ourselves that homotopy actually works, we remark that homotopy is even fine enough to detect diffeomorphism classes between smooth manifolds and helped for example to classify manifolds up to bordism. But this is a story we save for a future lecture/class.

If you are still not convinced, then let us remark that to study things up-to-homotopy is **so useful** that mathematicians work hard to find analogs of the homotopy relation and the **homotopy category** in many different areas. If you want to learn more about this, have a look at **Quillen's** highly influential book on **Homotopical Algebra**. You will also see an example in **homological algebra** where one talks about homotopies between chain complexes.

LECTURE 3

Singular chains and homology

We would like to make the idea to study a topological space X by considering all continuous maps from test spaces into X precise. We start with defining an important class of **test spaces**:

Definiton: The standard n -simplex

For $n \geq 0$, the **standard n -simplex** Δ^n is the set $\Delta^n \subset \mathbb{R}^{n+1}$ defined by

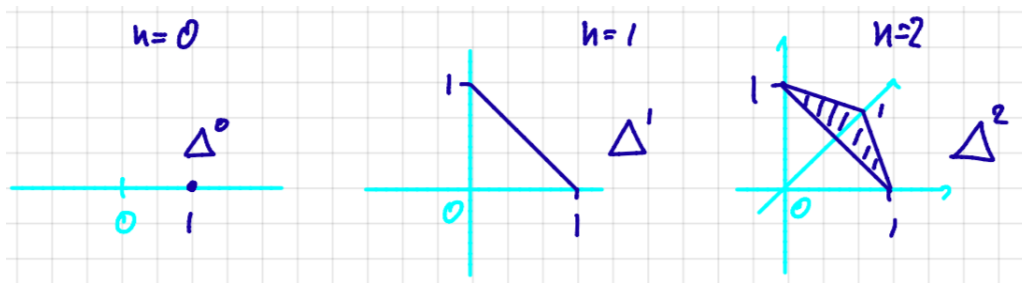
$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1, t_i \geq 0 \text{ for all } i\}.$$

Another way to describe Δ^n is to say that it is the **convex hull** of the standard basis $\{e_0, \dots, e_n\}$ in \mathbb{R}^{n+1} :

$$\Delta^n = \left\{ \sum_i t_i e_i : \sum_i t_i = 1, t_i \geq 0 \right\}.$$

The t_i are called **barycentric coordinates**.

It will be convenient to keep both these descriptions in mind.



The standard simplices are related by **face maps** for $0 \leq i \leq n$ which can be described as

$$\phi_i^n(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

with the 0 inserted at the i th coordinate (t_0 is the 0th coordinate).

Using the **standard basis**, ϕ_i^n can be described as the affine linear map (a translation plus a linear map)

$$\phi_i^n: \Delta^{n-1} \hookrightarrow \Delta^n \text{ determined by } \phi_i^n(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i. \end{cases}$$

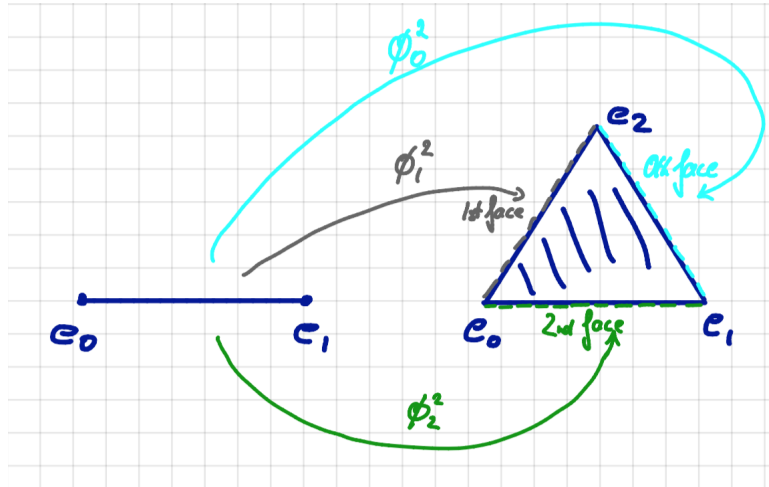
A short way of expressing the above formula for ϕ_i^n is that it embeds Δ^{n-1} into Δ^n by omitting the i th vertex (that is what the hat in the following formula means):

$$\phi_i^n = [e_0, \dots, e_{i-1}, \widehat{e_i}, e_{i+1}, \dots, e_n]: \Delta^{n-1} \rightarrow \Delta^n.$$

Definiton: Faces

Note that ϕ_i^n maps Δ^{n-1} onto the subsimplex opposite to the i th corner, or in the standard basis, opposite to e_i . We call the image of ϕ_i^n the **i th face** of Δ^n (which is opposite to e_i).

Note that the union of the images of all the face inclusions is the boundary of Δ^n .



The face maps satisfy a useful identity, sometimes called simplicial identity:

Lemma: A useful identity

For all $0 \leq j < i \leq n+1$ we have

$$(1) \quad \phi_i^n \circ \phi_j^{n-1} = \phi_j^n \circ \phi_{i-1}^{n-1}.$$

The first composition, $\phi_i^n \circ \phi_j^{n-1}$, results in a 0 at the j th and $i + 1$ st place. The second composition, $\phi_j^n \circ \phi_{i-1}^{n-1}$, has the effect to insert a 0 at the $(i - 1)$ st place and then one at the j th place. But since $j < i$, this means that, in both cases, we have an extra 0 at the j th and at the $(i + 1)$ st spot.

Thus both compositions yield

$$\begin{aligned} & \phi_i^n \circ \phi_j^{n-1}(t_0, \dots, t_{n-2}) \\ &= (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{i-2}, 0, t_{i-1}, \dots, t_{n-2}) \\ &= \phi_j^n \circ \phi_{i-1}^{n-1}(t_0, \dots, t_{n-2}). \end{aligned}$$

We are going to study a topological space X by looking at all the continuous maps from simplices into X . We give those sets of maps a name:

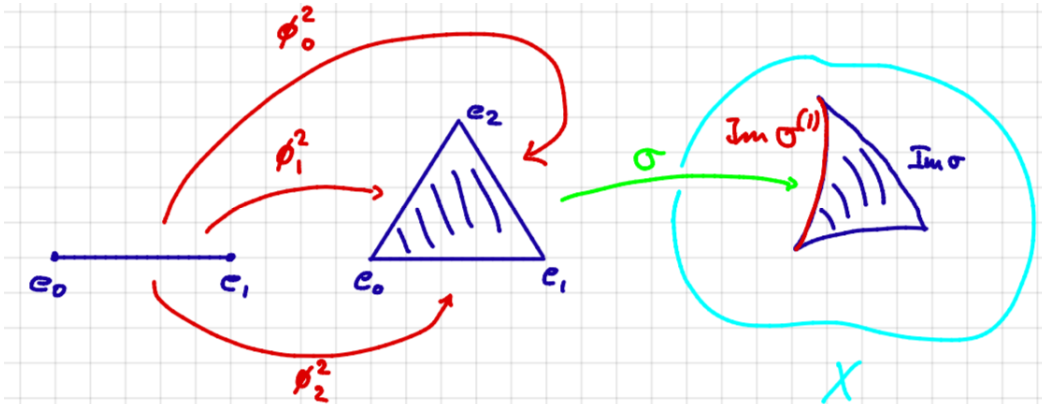
Definition: Singular n -simplices

Let X be any topological space. A **singular n -simplex in X** is a continuous map $\sigma: \Delta^n \rightarrow X$. We denote by $\text{Sing}_n(X)$ the set of all n -simplices in X . For example, $\text{Sing}_0(X)$ is just the set of points of X . But, in general, $\text{Sing}_n(X)$ carries more interesting information for $n \geq 1$.

For $0 \leq i \leq n$, we can use the face maps ϕ_i^n to define maps

$$d_i^n: \text{Sing}_n(X) \rightarrow \text{Sing}_{n-1}(X), \sigma \mapsto \sigma \circ \phi_i^n$$

by sending an n -simplex σ to the $n - 1$ -simplex defined by precomposition with the i th face inclusion. The image $d_i^n(\sigma) = \sigma \circ \phi_i^n$ is called the **i th face of σ** . We will sometimes use the notation $\sigma^{(i)} := \sigma \circ \phi_i^n$ for the i th face.



Since the collection of all face inclusions ϕ_i^n forms the boundary of Δ^n , we can use the maps d_i^n to talk about the **boundary** of an n -simplex. The boundaries of simplices will actually play a crucial role in the story.

We need to make this precise. First let us look at a simple example. Let X be some space and $\sigma: \Delta^1 \rightarrow X$ be a 1-simplex in X . Assume $\sigma(e_0) = x_0 \neq x_1 = \sigma(e_1)$. Then we would like to say that the boundary of σ is given by x_0 and x_1 .

Now let us assume that $\sigma: \Delta^1 \rightarrow X$ is another 1-simplex in X which forms a **closed loop**, i.e., $\sigma(e_0) = \sigma(e_1) = x \in X$. Now we would like to say that σ has no boundary (since it is a loop). Our face maps express $\sigma(e_0) = \sigma(e_1)$ as

$$d_0^1(\sigma) = d_1^1(\sigma).$$

It would be nice if we had a short way to formulate that the boundary of σ vanishes. For example, it would be nice if we were allowed to rewrite this equation as

$$\partial(\sigma) = d_0^1(\sigma) - d_1^1(\sigma) = 0.$$

But, so far, $\text{Sing}_0(X)$ is just a set and we are not allowed to add or subtract elements. We are now going to remedy this defect, since algebraic operations make life much easier. Therefore, we **formally extend** $\text{Sing}_n(X)$ into an abelian group.

The general way to turn a set B into an abelian group, is to form the associated free abelian group. **The idea is to add the minimal amount of structure and relations** to turn B into an abelian group. Since this is an important construction, we recall how this works:

Good to know about free abelian groups

- Any abelian group A can be seen as a \mathbb{Z} -module with $n \cdot a := a + \dots + a$ (n summands), for $n \in \mathbb{N}$ and $a \in A$, and $(-n) \cdot a := -n \cdot a$. Thus, abelian groups are in bijection with \mathbb{Z} -modules. An abelian group A is called **free** over a subset $B \subset A$ if B is a **\mathbb{Z} -basis**, i.e., if any element $a \in A$ can be **written uniquely as a \mathbb{Z} -linear combination** of elements in B . The cardinality of a basis is the same for any choice of basis and is called the **rank of A** .
- The group \mathbb{Z}^r is free abelian with **basis** $\{e_1, \dots, e_r\}$ with $e_i = (0, \dots, i, 0, \dots, 0)$ (the 1 in the i th position).
- Note that, for example, the group $\mathbb{Z}/2\mathbb{Z}$ is **not free**, since it does not admit a basis: the vector $1 \in \mathbb{Z}/2\mathbb{Z}$ cannot be in a basis since $2 \cdot 1 = 0$.

- Given a set B , there is an **associated free abelian group** $\mathbb{Z}B$ with basis B which is characterized by the following **universal property**: any map $f: B \rightarrow A$ of sets into an arbitrary abelian group A can be **extended uniquely** to a group homomorphism $\phi: \mathbb{Z}B \rightarrow A$ with $\phi(b) = f(b)$ for all $b \in B$.

In terms of category theory, this means that the functor **AbGroups** \rightarrow **Sets** which forgets the group structure, is right adjoint to the functor

$$\mathbf{Sets} \rightarrow \mathbf{AbGroups}, B \mapsto \mathbb{Z}B.$$

In other words,

$$\mathrm{Hom}_{\mathbf{Sets}}(B, A) = \mathrm{Hom}_{\mathbf{AbGroups}}(\mathbb{Z}B, A).$$

- Any **subgroup** of a free abelian group F is a free abelian group.

We apply this construction to the set $B = \mathrm{Sing}_n(X)$:

Definition: Singular n -chains

The group $S_n(X)$ of **singular n -chains in X** is the **free abelian group** generated by n -simplices

$$S_n(X) := \mathbb{Z}\mathrm{Sing}_n(X).$$

Thus an **n -chain** is a finite \mathbb{Z} -linear combination of simplices

$$\sum_{i=1}^k a_i \sigma_i, \quad a_i \in \mathbb{Z}, \quad \sigma_i \in \mathrm{Sing}_n(X).$$

Note: If $n < 0$, $\mathrm{Sing}_n(X)$ is defined to be empty and $S_n(X)$ is the trivial abelian group $\{0\}$. So whenever we talk about n -chains, n will be assumed to be nonnegative.

Definition: Boundary operators

We define the **boundary operator** by

$$\partial_n: \mathrm{Sing}_n(X) \rightarrow S_{n-1}(X), \quad \partial(\sigma) = \sum_{i=0}^n (-1)^i d_i^n \sigma = \sum_{i=0}^n (-1)^i \sigma^{(i)}.$$

We can then extend this to a homomorphism, which we also call boundary operator, by additivity, i.e.,

$$\partial_n: S_n(X) \rightarrow S_{n-1}(X), \partial \left(\sum_{j=1}^m a_j \sigma_j \right) := \sum_{j=1}^m a_j \partial(\sigma_j).$$

Note that we will often just write ∂ instead of ∂_n .

In particular, for the loop σ we considered above we are allowed to write in $S_0(X)$

$$\partial_1(\sigma) = d_0^1(\sigma) + (-1)d_1^1(\sigma) = d_0^1(\sigma) - d_1^1(\sigma) = 0.$$

A loop is an example of a particularly **important class of chains**. For, the equation $\partial(\sigma) = 0$ **expresses algebraically** that σ has **no boundary**. We give such chains a special name:

Definition: Cycles

An **n -cycle in X** is an n -chain $c \in S_n(X)$ with $\partial_n c = 0$. We denote the **group of n -cycles** by

$$\begin{aligned} Z_n(X) &:= \text{Ker}(\partial_n: S_n(X) \rightarrow S_{n-1}(X)) \\ &= \{c \in S_n(X) : \partial_n(c) = 0\} \subseteq S_n(X). \end{aligned}$$

Note that the group of **0-cycles** is all of $S_0(X)$, since every 0-chain is mapped to 0:

$$Z_0(X) = S_0(X).$$

To find another **example** of a **1-cycle** we could consider a 1-chain $c = \alpha + \beta + \gamma$ where $\alpha, \beta, \gamma: \Delta^1 \rightarrow X$ are singular 1-simplices such that

$$\alpha(e_1) = \beta(e_0), \beta(e_1) = \gamma(e_0), \gamma(e_1) = \alpha(e_0).$$

For then we get

$$\begin{aligned} \partial(c) &= d_0(\alpha) - d_1(\alpha) + d_0(\beta) - d_1(\beta) + d_0(\gamma) - d_1(\gamma) \\ &= \alpha(e_1) - \alpha(e_0) + \beta(e_1) - \beta(e_0) + \gamma(e_1) - \gamma(e_0) \\ &= 0. \end{aligned}$$

As the notation suggests, we are going to think of a chain of the form $\partial(c)$ as the boundary of c :

Definition: Boundaries

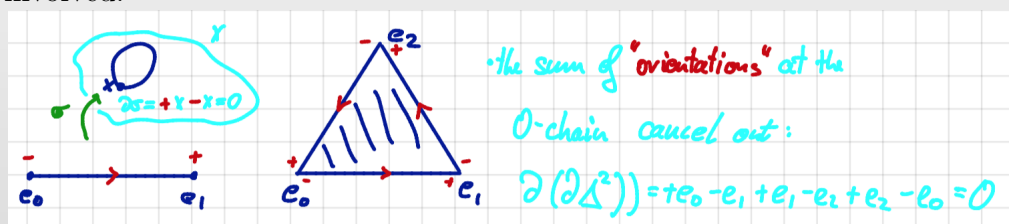
An **n -dimensional boundary in X** is an n -chain $c \in S_n(X)$ such that there exists an $(n+1)$ -chain b with $\partial_{n+1}b = c$. We denote the group of n -boundaries by

$$\begin{aligned} B_n(X) &:= \text{Im}(\partial_{n+1}: S_{n+1}(X) \rightarrow S_n(X)) \\ &= \{c \in S_n(X) : \text{there is a } b \in S_{n+1}(X) \text{ with } \partial_{n+1}(b) = c\}. \end{aligned}$$

As an aside, here is another way of thinking of the algebraic process.

Signs are like orientations... just not exactly

We want to express the fact that a loop has **no boundary** by saying that the signs of the **boundary points cancel out**. The following picture illustrates that the something similar happens when several vertices are involved:



In general, we can **think of the signs** as giving the faces of the simplices an **orientation**. And if an n -simplex is a **face** of an $(n+1)$ -simplex, then it inherits an **induced orientation** which is determined by how it fits into the bigger simplex. **Going down two steps** of inherited signs means things cancel out.

However, thinking of signs as orientations is formally not correct as we will notice in an example below. But, as we will see soon, we can **algebraically remedy this defect**.

As the above picture suggests, **every boundary is a cycle**:

Theorem: Boundaries of boundaries vanish

For every topological space X , the boundary operator satisfies $\partial \circ \partial = 0$, or more precisely

$$\partial_n \circ \partial_{n+1} = 0: S_{n+1}(X) \rightarrow S_{n-1}(X).$$

Proof: It suffices to check this for an $(n+1)$ -simplex σ . The general case follows, since each ∂ is a homomorphism. For σ , we just calculate:

$$\begin{aligned}
 \partial_n \circ \partial_{n+1}(\sigma) &= \partial_n \left(\sum_{i=0}^{n+1} (-1)^i d_i^{n+1} \sigma \right) = \sum_{i=0}^{n+1} \partial_n(\sigma \circ \phi_i^{n+1}) \\
 &= \sum_{i=0}^{n+1} (-1)^i \sum_{j=0}^n (-1)^j \sigma \circ \phi_i^{n+1} \circ \phi_j^n \\
 &= \sum_{0 \leq j < i \leq n+1} (-1)^{i+j} \sigma \circ \phi_i^{n+1} \circ \phi_j^n + \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} \sigma \circ \phi_i^{n+1} \circ \phi_j^n \\
 &\stackrel{(*)}{=} \sum_{0 \leq j < i \leq n+1} (-1)^{i+j} \sigma \circ \phi_j^{n+1} \circ \phi_{i-1}^n + \sum_{0 \leq j' < i' \leq n+1} (-1)^{j'+i'-1} \sigma \circ \phi_{j'}^{n+1} \circ \phi_{i'-1}^n \\
 &= 0.
 \end{aligned}$$

Note that at $(*)$ we applied identity (1) to the left hand sum and just changed the labels of the indices as $i \rightarrow j'$ and $j \rightarrow i' - 1$. Since both sums run over the same indices (it does not matter how we label them) and the right hand sum is the left hand sum multiplied by (-1) , both sums cancel out. **QED**

As an immediate consequence we get:

Corollary: Every boundary is a cycle

For every $n \geq 0$, we have

$$B_n(X) \subseteq Z_n(X).$$

This basic result shows that the sequence $\{S_n(X), \partial_n\}_n$ has an important property:

Definition: Chain complexes

A graded abelian group is a sequence of abelian groups, indexed by the integers. A **chain complex** is a graded abelian group $\{A_n\}_n$ together with homomorphisms $\partial_n: A_n \rightarrow A_{n-1}$ with the property that $\partial_{n-1} \circ \partial_n = 0$.

Hence we have shown that we obtain for any topological space X a complex of (free) abelian groups

$$\cdots \xrightarrow{\partial} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0.$$

It is called the **singular chain complex** of X . We will see next lecture what such chain complexes are good for.

LECTURE 4

Singular homology, functoriality and H_0

Recall that we constructed, for any topological space X , the **singular chain complex** of X

$$\cdots \xrightarrow{\partial} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0.$$

The homomorphisms ∂_n satisfy the fundamental rule: $\partial \circ \partial = 0$.

The following definition of homology groups applies to any chain complex. However we formulate it only for the singular chain complex:

Definition: Singular homology

The **n th singular homology** group of X is defined to be the quotient group of n -cycles modulo n -boundaries:

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\text{Ker}(\partial: S_n(X) \rightarrow S_{n-1}(X))}{\text{Im}(\partial: S_{n+1}(X) \rightarrow S_n(X))}.$$

We are going to say that two cycles whose difference is a boundary are **homologous**.

Let us make a first attempt to understand what is going on here:

Singular what?

In algebra, homology is a way to **measure the difference between cycles and boundaries**. Singular homology is an application of homology in order to understand the structure of a space.

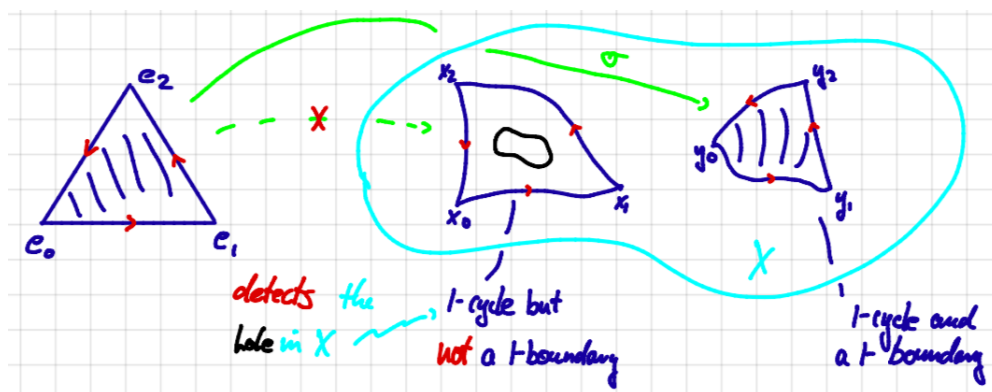
Given a space X , the group of n -cycles measures how often we can map an n -dimensional simplex into X without collapsing it to any of its $n - 1$ -dimensional faces.

Let $\sigma(\Delta^n)$ be the image in X of such a cycle. If we can even map an $(n + 1)$ -dimensional simplex $\sigma'(\Delta^{n+1})$ into X whose boundary is $\sigma(\Delta^n)$, then we can continuously collapse all of $\sigma(\Delta^n)$ to a point. In this case, we would like to

forget about this σ . For, from an n -dimensional point of view, this $\sigma(\Delta^n)$ is not interesting. That is what it means geometrically/topologically to take the quotient by $B_n(X)$.

But if we **cannot find** an $n + 1$ -dimensional simplex such that $\sigma(\Delta^n)$ is its boundary, then $\sigma(\Delta^n)$ potentially **carries interesting n -dimensional information** about X .

The slogan is: $H_n(X)$ measures n -dimensional wholes in X .



Before we see some examples of homology groups we go back to the idea of “**orientations of simplices**” and see why taking the quotient by boundaries is a good thing. We said that we think of the signs as orientations, but this is not completely correct. But **modulo boundaries** we are good:

Orientations revisited

Let X be some space, and suppose we have a one-simplex $\sigma: \Delta^1 \rightarrow X$. Define

$$\phi: \Delta^1 \rightarrow \Delta^1, (t, 1-t) \mapsto (1-t, t).$$

Precomposing with ϕ gives another singular simplex $\bar{\sigma} = \sigma \circ \phi$ which reverses the orientation of σ . It is **not true** that $\bar{\sigma} = -\sigma$ in $S_1(X)$.

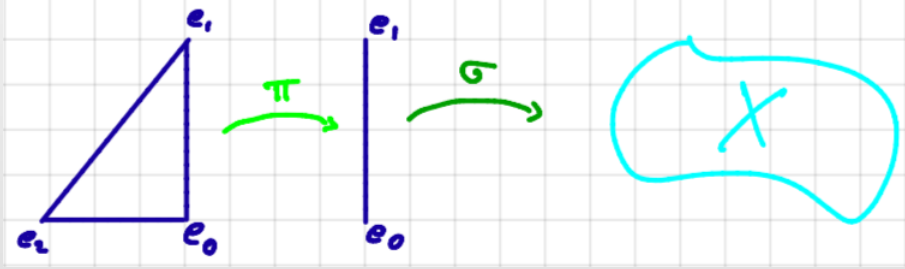
However, we claim that

$$\bar{\sigma} \equiv -\sigma \pmod{B_1(X)}.$$

This means that there is a 2-chain in X whose boundary is $\bar{\sigma} + \sigma$. If $d_0(\sigma) = d_1(\sigma)$ such that $\sigma \in Z_1(X)$ is a 1-cycle, then $\bar{\sigma}$ and σ are homologous and $[\bar{\sigma}] = [\sigma]$ in $H_1(X)$.

To prove the claim we need to construct an appropriate 2-chain. Let $\pi: \Delta^2 \rightarrow \Delta^1$ be the affine map determined by sending e_0 and e_2 to e_0

and e_1 to e_1 . For $x \in X$ and $n \geq 0$, we write $\kappa_x^n: \Delta^n \rightarrow X$ for the constant map with value x .



Now we calculate

$$\partial(\sigma \circ \pi) = \sigma \circ \pi \circ \phi_0^2 - \sigma \circ \pi \circ \phi_1^2 + \sigma \circ \pi \circ \phi_2^2 = \bar{\sigma} - \kappa_{\sigma(0)}^1 + \sigma.$$

Hence up to the term $-\kappa_{\sigma(0)}^1$ we get what we want. So we would like to eliminate this term. To do that we define the constant 2-simplex $\kappa_{\sigma(0)}^2: \Delta^2 \rightarrow X$ at $\sigma(0)$. Its boundary is

$$\partial(\kappa_{\sigma(0)}^2) = \kappa_{\sigma(0)}^1 - \kappa_{\sigma(0)}^1 + \kappa_{\sigma(0)}^1 = \kappa_{\sigma(0)}^1.$$

Thus

$$\bar{\sigma} + \sigma = \partial(\pi \circ \sigma + \kappa_{\sigma(0)}^2)$$

which proves the claim.

Actually, we will have to get back to orientations almost regularly, in particular when we talk about simplicial complexes, and step by step improve our understanding and control.

Aside: The sequence of homology groups $\{H_n(X)\}_n$ also forms a graded abelian group. Note that even though $Z_n(X)$ and $B_n(X)$ are free abelian groups because they are subgroups of the free abelian group $S_n(X)$, the quotient $H_n(X)$ is **not necessarily free**. Moreover, while $Z_n(X)$ and $B_n(X)$ may be uncountably generated, $H_n(X)$ turns out to be finitely generated for the spaces we are interested in.

Let us look at two **simple examples**:

- (1) **Let $X = \emptyset$.** Then $\text{Sing}_*(\emptyset) = \emptyset$ and $S_*(\emptyset) = 0$ is just the trivial abelian group by convention. Hence $\cdots \rightarrow S_2 \rightarrow S_1 \rightarrow S_0$ is the zero chain complex and $Z_*(\emptyset) = B_*(\emptyset) = 0$. The homology in all dimensions is therefore 0.
- (2) **Let $X = \text{pt}$** be a one-point space. Then, for each n , there is only one singular n -simplex, namely the constant map $\sigma_n: \Delta^n \rightarrow \text{pt}$. In

other words, $S_n(X) = \mathbb{Z} \cdot \sigma_n$ is generated by a single element. Hence $\sigma_n^{(i)} = \sigma_n \circ \phi_i^n = \sigma_{n-1}$ and

$$\partial \sigma_n = \sum_{i=0}^n (-1)^i \sigma_n^{(i)} = \sum_{i=0}^n (-1) \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \\ 0 & n = 0. \end{cases}$$

For cycles and boundaries this means

$$Z_n(X) = \begin{cases} \mathbb{Z} \cdot \sigma_n & n \text{ odd or } n = 0 \\ 0 & n \text{ even and } n \neq 0, \end{cases}$$

and

$$B_n(X) = \begin{cases} \mathbb{Z} \cdot \sigma_n & n \text{ odd or} \\ 0 & n \text{ even.} \end{cases}$$

For the homology groups we get

$$H_n(\text{pt}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

To complete the picture, the singular chain complex looks like

$$\dots \xrightarrow{\partial=\text{id}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\text{id}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z} \rightarrow 0.$$

• Functoriality

Now that we have defined homology we can ask how it behaves under continuous maps. So let X and Y be topological spaces and $f: X \rightarrow Y$ be a continuous map. Since singular simplices are just maps, we can define an **induced map**

$$f_*: \text{Sing}_n(X) \rightarrow \text{Sing}_n(Y), \sigma \mapsto f \circ \sigma$$

just by composition with f .

The same construction yields an induced map on chains:

$$f_* = S_n(f): S_n(X) \rightarrow S_n(Y), \sum_{j=1}^m a_j \sigma_j \mapsto \sum_{j=1}^m a_j (f \circ \sigma_j).$$

The induced map is compatible with the boundary operator in the following way:

Lemma: The singular chain complex is natural

For every $n \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{S_n(f)} & S_n(Y) \\ \partial_X \downarrow & & \downarrow \partial_Y \\ S_{n-1}(X) & \xrightarrow{S_{n-1}(f)} & S_{n-1}(Y). \end{array}$$

Proof: We just calculate and check that both ways have the same outcome for any singular n -simplex σ on X :

$$\begin{aligned} \partial_Y(S_n(f))(\sigma) &= \sum_{i=0}^n (-1)^i (f \circ \sigma) \circ \phi_i^n \\ &= \sum_{i=0}^n (-1)^i f \circ (\sigma \circ \phi_i^n) \\ &= S_{n-1}(f) \left(\sum_{i=0}^n (-1)^i \sigma \circ \phi_i^n \right) \\ &= S_{n-1}(f)(\partial_X \sigma). \end{aligned}$$

□

The lemma has the important consequence that

$$S_n(f)(Z_n(X)) \subset Z_n(Y) \text{ and } S_n(f)(B_n(X)) \subset B_n(Y).$$

For, if $c \in Z_n(X)$, then

$$\partial_Y(S_n(f)(c)) = S_{n-1}(f)(\partial_X(c)) = S_{n-1}(f)(0) = 0;$$

and, if $c \in B_n(X)$, then there is a $b \in B_{n+1}(X)$ with $\partial_X(b) = c$ and

$$\partial_Y(S_{n+1}(f)(b)) = S_n(f)(\partial_X(b)) = S_n(f)(c),$$

i.e., there is an element, $b' = S_{n+1}(f)(b)$, with $\partial_Y(b') = S_n(f)(c)$.

Proposition: Homology is functorial

Thus we get a well defined induced homomorphism on homology groups

$$H_n(f): H_n(X) \rightarrow H_n(Y), [c] \mapsto [S_n(f)(c)].$$

The homomorphisms $S_n(f)$ and $H_n(f)$ have the following **properties**:

- $S_n(\text{id}_X) = \text{id}_{S_n(X)}$ and $H_n(\text{id}_X) = \text{id}_{H_n(X)}$
- $S_n(f \circ g) = S_n(f) \circ S_n(g)$ and $H_n(f \circ g) = H_n(f) \circ H_n(g)$.

To **summarize** our observations: $S_n(-)$ and $H_n(-)$ are **functors** from the category of topological spaces to the category of abelian groups. For the sequence of all $S_n(-)$ even more is true: $S_*(-)$ is a functor from the category of topological spaces to the category of chain complexes of abelian groups (with chain maps as morphisms).

Invariance

As a consequence, if $f: X \rightarrow Y$ is a **homeomorphism**, then $H_n(f)$ is an **isomorphism** of abelian groups. In other words, homology groups only depend on the topology of a space.

In fact, we will soon see that homology is a coarser invariant in the sense that homotopic maps induce the same map in homology.

• The homology group H_0

Let us try to understand the simplest of the homology groups.

Lemma: Augmentation

For any topological space X , there is a homomorphism

$$\epsilon: H_0(X) \rightarrow \mathbb{Z}$$

which is nontrivial whenever $X \neq \emptyset$.

Proof: If $X = \emptyset$, then $H_*(\emptyset) = 0$ by definition. In this case, we define ϵ to be the zero homomorphism.

Now let $X \neq \emptyset$. Then there is a unique map $X \rightarrow \text{pt}$ from X to the one-point space. By functoriality, it induces a homomorphism

$$\epsilon: H_0(X) \rightarrow H_0(\text{pt}) = \mathbb{Z}.$$

□

Let us try to understand this ϵ a bit better. The map $X \rightarrow \text{pt}$ induces a homomorphism of chain complexes $S_*(X) \rightarrow S_*(\text{pt})$ which sends any 0-simplex

$\sigma: \Delta^0 \rightarrow X$ to the **constant map**

$$\kappa^0: \Delta^0 \xrightarrow{\sigma} X \rightarrow \text{pt}$$

which is the generator of $S_0(\text{pt}) = \mathbb{Z}$. Hence we get a map $\sigma \mapsto 1$ which extends to a homomorphism $\tilde{\epsilon}: S_0(X) \rightarrow \mathbb{Z}$ by additivity, i.e.

$$\tilde{\epsilon}\left(\sum_j a_j \sigma_j\right) = \sum_j a_j \in \mathbb{Z}.$$

To double check that this map **descends to a homomorphism** ϵ on $H_0(X)$ we need to show that it maps boundaries to 0. (We know this already, but let us do it anyway.)

So let b be a 0-chain which is the boundary of a 1-chain c , i.e., $b = \partial c$, and let $c = \sum_j a_j \gamma_j$ with **finitely many** 1-simplices $\gamma_j: \Delta^1 \rightarrow X$. Then each $\gamma_j \circ \phi_0^1$ and $\gamma_j \circ \phi_1^1$ are 0-simplices and are sent to 1 by $\tilde{\epsilon}$. Thus we get

$$\epsilon(b) = \epsilon(\partial c) = \epsilon\left(\sum_j a_j (\gamma_j \circ \phi_0^1 - \gamma_j \circ \phi_1^1)\right) = \sum_j a_j - \sum_j a_j = 0.$$

We learn from this discussion that, since a 0-simplex $\Delta^0 \rightarrow X$ can be identified with its image point, **ϵ counts the points on X , with multiplicities**. And if two points can be connected by a 1-simplex, i.e., by a path in X , then they add up to 0. This leads us to:

Theorem: H_0 for path-connected spaces

If X is path-connected and non-empty, then ϵ is an isomorphism

$$\epsilon: H_0(X) \xrightarrow{\cong} \mathbb{Z}.$$

Proof: Since X is non-empty, there is a point $x \in X$. The 0-simplex $\sigma = \kappa_x^0$ with value x is an element in $S_0(X)$ which is sent to $1 \in \mathbb{Z}$. Additivity implies that ϵ is **surjective**. To show that ϵ is also injective, we need to show that the classes of the 0-simplices given by constant maps at any two points are homologous.

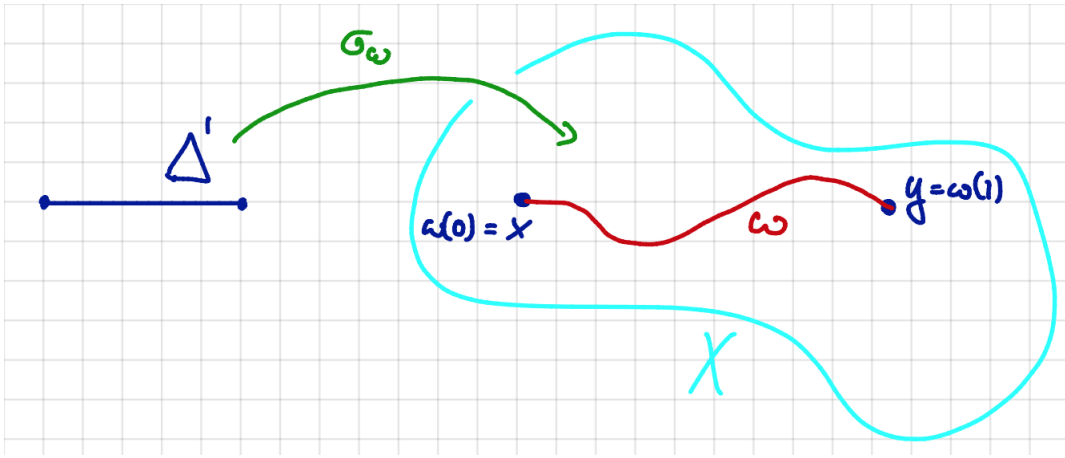
So let $y \in X$ be another point. Since X is path-connected, there is a path $\omega: [0,1] \rightarrow X$ with $\omega(0) = x$ and $\omega(1) = y$. We define a 1-simplex σ_ω by

$$\sigma_\omega(t_0, t_1) := \omega(1 - t_0) = \omega(t_1) \text{ for } t_0 + t_1 = 1, 0 \leq t_0, t_1 \leq 1.$$

The boundary of σ_ω is

$$\begin{aligned}\partial(\sigma_\omega) &= d_0(\sigma_\omega) - d_1(\sigma_\omega) = \sigma_\omega(e_1) - \sigma_\omega(e_0) \\ &= \sigma_\omega(0,1) - \sigma_\omega(1,0) (= \omega(0) - \omega(1)) \\ &= \kappa_x^0 - \kappa_y^0\end{aligned}$$

(where we identify 0-simplices and their image points). Hence the 0-simplices κ_x^0 and κ_y^0 are **homologous**. Since 0-simplices generate $H_0(X)$ and ϵ is a homomorphism, this implies that ϵ is **injective**. **QED**



Corollary: H_0 is generated by path components

If X is a disjoint union $X = \bigsqcup_{i \in I} X_i$ where each X_i is path-connected and non-empty, then, for all $n \geq 0$,

$$H_n(X) \cong \bigoplus_{i \in I} H_n(X_i).$$

In particular, for $n = 0$ we get

$$H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}.$$

In other words, $H_0(X)$ is the free abelian group generated by the set of path-components of X .

Proof: If $\sigma: \Delta^n \rightarrow X$ is an n -simplex, then its image lies in exactly one connected component X_i . Otherwise, we could write Δ^n as the disjoint union of two open and closed subsets contradicting the fact that Δ^n is connected. Hence σ factors into $\Delta^n \rightarrow X_i \hookrightarrow X$.

Since singular n -chains are freely generated by n -simplices, this shows that the singular chain complex of X splits into a direct sum

$$S_*(X) = \bigoplus_{i \in I} S_*(X_i).$$

For the same reason the boundary operators

$$\partial: S_n(X) \cong \bigoplus_{i \in I} S_n(X_i) \rightarrow \bigoplus_{i \in I} S_{n-1}(X_i) \cong S_{n-1}(X)$$

split into components $\partial_{X_i}: S_n(X_i) \rightarrow S_{n-1}(X_i)$. Hence we get an isomorphism. The statement for $n = 0$ then follows from the previous result on path-connected spaces. \square

LECTURE 5

Relative homology and long exact sequences

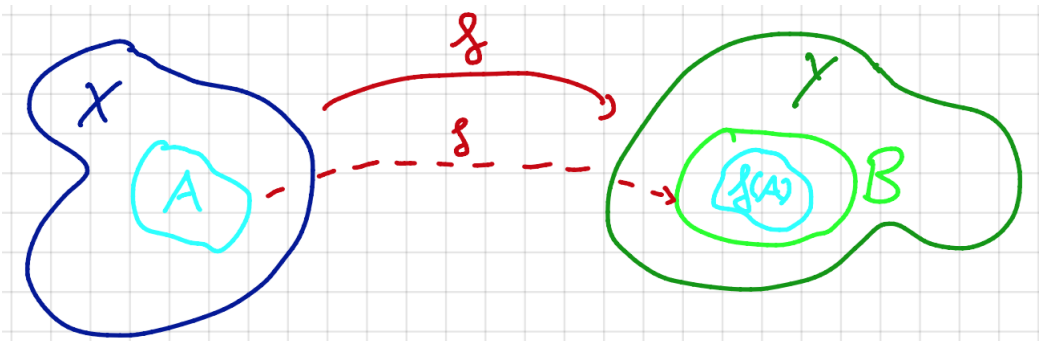
If we want to show that singular homology groups are useful, we need to be able to compute them. For H_0 that was not so difficult. But for $n \geq 1$, we need to develop some techniques.

In general, if you would like to compute something for spaces, it is always a **good idea to think about the relation to subspaces**. Maybe the information on smaller subspaces provides insights on the whole space. That is the idea we are going to exploit now for homology groups.

Let X be a topological space and let $A \subset X$ be a subset. We can consider (X, A) as a **pair of spaces**. If (Y, B) is another such pair, then we denote by

$$C((X, A), (Y, B)) := \{f \in C(X, Y) : f(A) \subset B\}$$

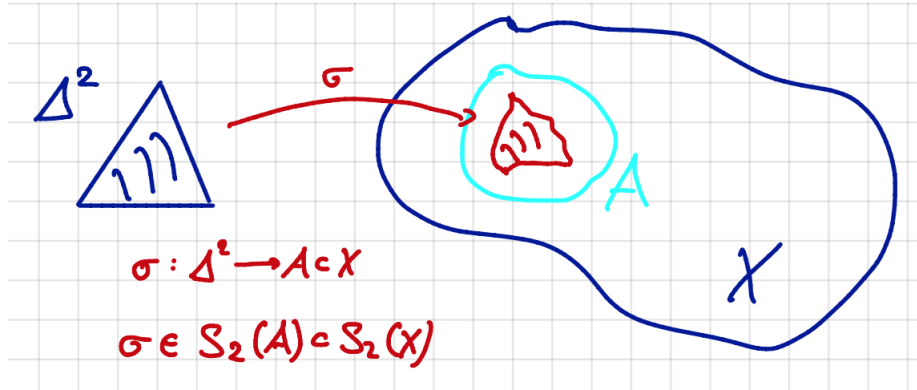
the set of continuous maps which respect the subspaces. In fact, we get a category **Top₂** of pairs of topological spaces.



Given a pair of spaces $A \subset X$, any n -simplex of A defines an n -simplex on X :

$$(\Delta^n \xrightarrow{\sigma} A) \mapsto (\Delta^n \xrightarrow{\sigma} A \subset X).$$

The induced map $S_n(A) \rightarrow S_n(X)$ is **injective**. Hence we are going to identify $S_n(A)$ with its image in $S_n(X)$ and consider $S_n(A)$ as a **subgroup** of $S_n(X)$.



Definiton: Relative chains

We define the group of **relative n -chains** by

$$S_n(X, A) := S_n(X) / S_n(A).$$

The group $S_n(X, A)$ is free, since the quotient map sends basis elements to basis elements, and is generated by the classes of n -simplices of X whose image is not entirely contained in A .

Since the boundary operator is defined via composition with the face maps, it satisfies

$$\partial(S_n(A)) \subset S_{n-1}(A) \subset S_{n-1}(X).$$

For, if the image of $\sigma: \Delta^n \rightarrow X$ lies in A , then so does the image of the composite $\Delta^{n-1} \hookrightarrow \Delta^n \rightarrow X$.

Thus ∂ **induces a homomorphism $\bar{\partial}$** on $S_n(X, A)$ and we have a commutative diagram

$$\begin{array}{ccc} S_n(X) & \longrightarrow & S_n(X, A) \\ \partial \downarrow & & \downarrow \bar{\partial} \\ S_{n-1}(X) & \longrightarrow & S_{n-1}(X, A). \end{array}$$

Since $\partial \circ \partial = 0$ and since $S_n(X) \rightarrow S_n(X, A)$ is surjective, we also have

$$\bar{\partial} \circ \bar{\partial} = 0.$$

We define **relative n -cycles** and **relative n -boundaries** by

$$Z_n(X, A) := \text{Ker}(\bar{\partial}: S_n(X, A) \rightarrow S_{n-1}(X, A)) \text{ and}$$

$$B_n(X, A) := \text{Im}(\bar{\partial}: S_{n+1}(X, A) \rightarrow S_n(X, A)).$$

Definition: Relative homology

The **n th relative homology group** of the pair (X, A) is defined as

$$H_n(X, A) := Z_n(X, A) / B_n(X, A).$$

Roughly speaking, relative homology groups measure the difference between the homology of X and the homology of A . Let us try to make this more precise. That an n -chain c in $S_n(X)$ represents a **relative** n -cycle means that $\bar{\partial}(c) = 0$ in $S_{n-1}(X)/S_{n-1}(A)$, i.e., $\partial(c) \in S_{n-1}(A)$. Hence it just means that the image of the boundary of c lies in A .

So let us consider the **preimage** of $Z_n(X, A)$ under the quotient map $S_n(X) \rightarrow S_n(X, A)$ and define

$$Z'_n(X, A) := \{c \in S_n(X) : \partial(c) \in S_{n-1}(A)\}.$$

Similarly, that an n -chain c in $S_n(X)$ represents a **relative** n -boundary means that there is an $n+1$ -chain b such that

$$c \equiv \partial(b) \pmod{S_n(A)}, \text{ i.e., } c - \partial(b) \in S_n(A).$$

Hence the **preimage** of $B_n(X, A)$ under the quotient map is

$$B'_n(X, A) := \{c \in S_n(X) : \exists b \in S_{n+1}(X) \text{ such that } c - \partial(b) \in S_n(A)\}.$$

Now we observe that $Z_n(X, A) = Z'_n(X, A) / S_n(A)$ (since $S_n(X, A)$ is $S_n(X) / S_n(A)$) and $B_n(X, A) = B'_n(X, A) / S_n(A)$. Hence we get

$$H_n(X, A) = \frac{Z_n(X, A)}{B_n(X, A)} = \frac{Z'_n(X, A) / S_n(A)}{B'_n(X, A) / S_n(A)} = \frac{Z'_n(X, A)}{B'_n(X, A)}.$$

In other words, we could also have used the latter quotient to define $H_n(X, A)$.

Empty subspaces

As a special case with $A = \emptyset$ we get

$$Z'_n(X, \emptyset) = Z_n(X), \quad B'_n(X, \emptyset) = B_n(X), \quad \text{and} \quad H_n(X, \emptyset) = H_n(X).$$

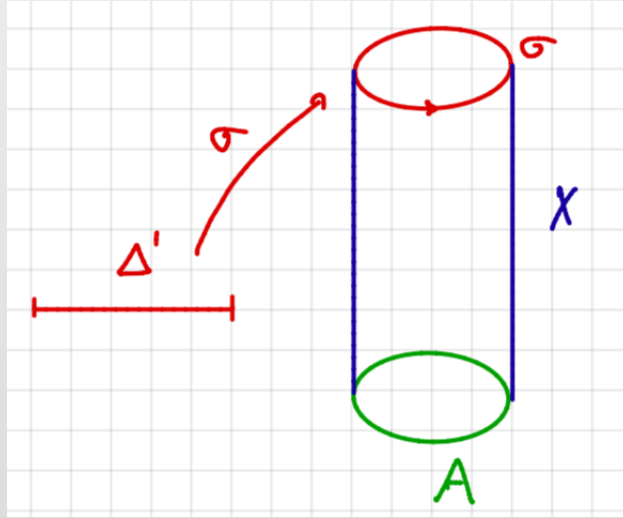
Now let us have a look at two **examples** to see how the images of simplices in $H_n(X)$ and $H_n(X, A)$ can differ.

Example: Relative cycles on the cylinder

Let $X = S^1 \times [0,1]$ be a cylinder over the circle, and let the subspace $A = S^1 \times 0 \subset X$ be the **bottom circle**.

We construct an element in the relative homology $H_1(X,A)$ by taking a **1-simplex**

$$\begin{aligned}\sigma: \Delta^1 &\rightarrow X, \\ (te_1, (1-t)e_0) &\mapsto (\cos(2\pi t), \sin(2\pi t), 1).\end{aligned}$$



Since σ is a closed curve in X , we have $\sigma(e_0) = \sigma(e_1)$. Hence its boundary vanishes:

$$\partial(\sigma) = \sigma(e_1) - \sigma(e_0) = 0.$$

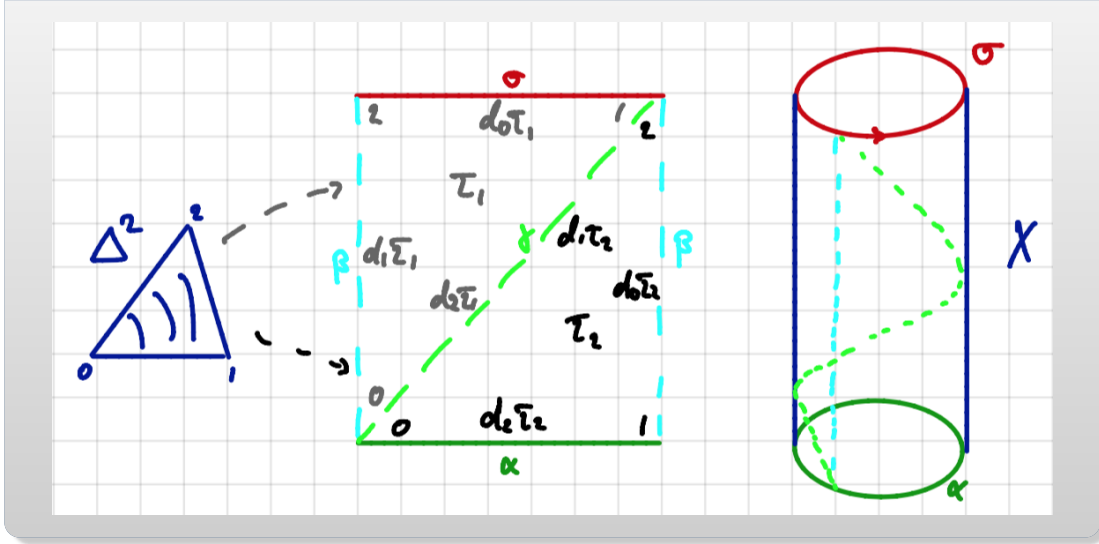
Therefore, $\sigma \in Z_1(X) \subset Z'_1(X,A)$. We will see very soon, that σ , in fact, represents a **nontrivial class in $H_1(X)$** .

However, the image of σ in the relative homology group $H_1(X,A)$ vanishes. For, consider the 2-chains τ_1 and τ_2 as indicated in the picture. Then we have

$$\begin{aligned}\partial(\tau_1 + \tau_2) &= d_0(\tau_1) - d_1(\tau_1) + d_2(\tau_1) + d_0(\tau_2) - d_1(\tau_2) + d_2(\tau_2) \\ &= \sigma - \beta + \gamma + \beta - \gamma + \alpha \\ &= \sigma + \alpha \text{ with } \alpha \in S_1(A).\end{aligned}$$

Hence, **modulo $S_1(A)$** , we have $\sigma \in B_1(X,A)$ and

$$[\sigma] = 0 \text{ in } H_1(X,A).$$



And the second example:

Example: Relative cycles on Δ^n

Let us look the standard n -simplex $X = \Delta^n$ as a space on its own. We would like to study it **relative to its boundary**

$$\partial\Delta^n := \bigcup_i \text{Im } \phi_i^n \approx S^{n-1}$$

which is homeomorphic to the $n - 1$ -dimensional sphere.

There is a special n -simplex in $\text{Sing}_n(\Delta^n) \subset S_n(\Delta^n)$, called the **universal n -simplex**, given by the **identity** map $\iota_n: \Delta^n \rightarrow \Delta^n$. It is **not a cycle**, since its boundary $\partial(\iota_n) \in S_{n-1}(\Delta^n)$ is the alternating sum of the faces of the n -simplex each of which is a **generator** in $S_{n-1}(\Delta^n)$:

$$\partial(\iota_n) = \sum_i (-1)^i \phi_i^n(\Delta^{n-1}) \neq 0.$$

However, each of these singular simplices lies in $\partial\Delta^n$, and hence $\partial(\iota_n) \in S_{n-1}(\partial\Delta^n)$.

Thus the class $\bar{\iota}_n \in S_n(\Delta^n, \partial\Delta^n)$ is a **relative cycle**. We will see later that the relative homology group $H_n(\Delta^n, \partial\Delta^n)$ is an infinite cyclic group generated by $[\bar{\iota}_n]$.

• Long exact sequences

Back to the general case. So let (X, A) be a pair of spaces. We know that the inclusion map $i: A \hookrightarrow X$ induces a homomorphism $H_n(i): H_n(A) \rightarrow H_n(X)$. Moreover, the map of pairs $j: (X, \emptyset) \rightarrow (X, A)$ induces a homomorphism

$$H_n(j): H_n(X) \cong H_n(X, \emptyset) \rightarrow H_n(X, A).$$

We claim that there is yet another interesting map.

Connecting homomorphism

For all n , there is a **connecting homomorphism**, which is often also called **boundary operator** and therefore usually also denoted by ∂ ,

$$\partial: H_n(X, A) \rightarrow H_{n-1}(A), [c] \mapsto [\partial(c)]$$

with $c \in Z'_n(X, A)$.

Let us try to make sense of this definition: We just learned that we can represent an element in $H_n(X, A)$ by an element $c \in Z'_n(X, A)$. Then $\partial(c)$ is an element in $S_{n-1}(A)$. In fact, $\partial(c)$ **is a cycle**, since it is a boundary and therefore

$$\partial(\partial(c)) = 0.$$

In particular, $\partial(c)$ **represents a class** in the homology $H_{n-1}(A)$. Hence we can send $[c]$ under the connecting homomorphism to be the class $[\partial c] \in H_{n-1}(A)$.

It remains to **check that this is well-defined**, i.e., if we choose **another representative** for the class $[c]$ we need to show that we obtain the same class $[\partial(c)]$.

Another representative of $[c]$ in $Z'_n(X, A)$ has the form $c + \partial(b) + a$ with $b \in S_{n+1}(X)$ and $a \in S_n(A)$. Then we get

$$\partial(c + \partial(b) + a) = \partial(c) + \partial(a).$$

But, since $\partial(a) \in B_{n-1}(A)$, we get

$$[\partial(c)] = [\partial(c) + \partial(a)] \text{ in } H_{n-1}(A).$$

Thus, the connecting map is well-defined. And it is a **homomorphism**, since ∂ is a homomorphism.

Hence we get a sequence of homomorphisms

$$H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A).$$

It is an exercise to check that the connecting homomorphism is natural:

The connecting homomorphism is functorial

For any $f \in C((X,A),(Y,B))$, the following diagram commutes

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{H_n(f)} & H_n(Y,B) \\ \partial \downarrow & & \downarrow \partial \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f|_A)} & H_{n-1}(B). \end{array}$$

In fact, the existence of the connecting map, the above sequence and its properties can be deduced by a purely algebraic process, that we will recall below. For, the relative chain complex fits into the **short exact sequence** of chain complexes

$$0 \rightarrow S_*(A) \rightarrow S_*(X) \rightarrow S_*(X,A) \rightarrow 0.$$

Such a sequence induces a **long exact sequence** in homology of the form

$$\begin{array}{ccccccc} & & & \dots & \xrightarrow{H_{n+1}(i)} & H_{n+1}(X,A) & \\ & & & \searrow \partial & & & \\ H_n(A) & \xleftarrow{H_n(i)} & H_n(X) & \xrightarrow{H_n(j)} & H_n(X,A) & & \\ & & \searrow \partial & & & & \\ H_{n-1}(A) & \xleftarrow{H_{n-1}(i)} & \dots & & & & \end{array}$$

A digression to homological algebra

Maps of chain complexes

Let A_* and B_* be two chain complexes. A morphism of chain complexes $f_*: A_* \rightarrow B_*$ is a sequence of homomorphisms $\{f_n\}_{n \in \mathbb{Z}}$

$$f_n: A_n \rightarrow B_n \text{ such that } f_{n-1} \circ \partial_n^A = \partial_n^B \circ f_n \text{ for all } n \in \mathbb{Z}.$$

A homomorphism of chain complexes induces a homomorphism on homology

$$H_n(f): H_n(A_*) \rightarrow H_n(B_*), [a] \mapsto [f_n(a)].$$

Check, as an exercise, that this is well-defined.

Consider a short exact sequence of chain complexes

$$(2) \quad 0 \rightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \rightarrow 0,$$

i.e., for every n , the corresponding sequence of abelian groups is exact.

Since f_* and g_* are homomorphisms of chain complexes, they **induce maps on homology** groups

$$(3) \quad H_n(A_*) \xrightarrow{H_n(f)} H_n(B_*) \xrightarrow{H_n(g)} H_n(C_*).$$

Since $g_n \circ f_n = 0$, we know $H_n(g) \circ H_n(f) = 0$.

But is the sequence **exact** at $H_n(B_*)$, i.e., is $\text{Ker}(H_n(g)) = \text{Im}(H_n(f))$?

Let us look at an extended picture of the short exact sequence:

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \longrightarrow 0 \\ & & \downarrow \partial_A & & \downarrow \partial_B & & \downarrow \partial_C \\ 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \longrightarrow 0 \\ & & \downarrow \partial_A & & \downarrow \partial_B & & \downarrow \partial_C \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} \longrightarrow 0 \end{array}$$

Let $[b] \in H_n(B_*)$ such that $H_n(g)([b]) = 0$. In fact, $[b]$ is represented by a **cycle**, i.e., some $b \in B_n$ with $\partial_B(b) = 0$. Since $H_n(g)([b]) = 0$, there is some $\bar{c} \in C_{n+1}$ such that $\partial_C(\bar{c}) = g_n(b)$. **By exactness** of (2), g_{n+1} is **surjective** and there is some $\bar{b} \in B_{n+1}$ with $g_{n+1}(\bar{b}) = \bar{c}$.

Now we can consider $\partial_B(\bar{b}) \in B_n$, and have $g_n(\partial_B(\bar{b})) = \partial_C(\bar{c})$ in C_n . What is the difference $b - \partial_B(\bar{b})$?

Well, it maps to 0 in C_n . **By exactness** of (2), there is some $a \in A_n$ such that $f_n(a) = b - \partial_B(\bar{b})$. Is a a **cycle**, and hence does it represent a homology class?

We know

$$f_{n-1}(\partial_A(a)) = \partial_B(f_n(a)) = \partial_B(b - \partial_B(\bar{b})) = \partial_B(b) - \partial_B(\partial_B(\bar{b})) = \partial_B(b).$$

But we assumed $\partial_B(b) = 0$. Thus $f_{n-1}(\partial_A(a)) = 0$. Since f_{n-1} is **injective**, this implies $\partial_A(a) = 0$. Hence a is indeed a cycle, and therefore represents a homology class $[a] \in H_n(A_*)$. It also follows

$$H_n(f)([a]) = [b - \partial_B(\bar{b})] = [b].$$

Thus sequence (3) is **exact**.

However, the map $H_n(A_*) \xrightarrow{H_n(f_*)} H_n(B_*)$ **may fail to be injective** and the map $H_n(B_*) \xrightarrow{H_n(g_*)} H_n(C_*)$ **may fail to be surjective**. That means sequence (3) does not fit into a short exact sequence, in general.

But we can connect all these sequences together for varying n and obtain a long exact sequence:

The homology long exact sequence

Let $0 \rightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \rightarrow 0$ be a short exact sequence of chain complexes. Then, for each n , there is a functorial homomorphism

$$\partial: H_n(C_*) \rightarrow H_{n-1}(A_*)$$

such that the sequence

$$\begin{array}{ccccccc} & & & \cdots & \xrightarrow{H_{n+1}(f_*)} & H_{n+1}(C_*) & \\ & & & \searrow \partial & & & \\ H_n(A_*) & \xleftarrow{H_n(f_*)} & H_n(B_*) & \xrightarrow{H_n(g_*)} & H_n(C_*) & & \\ & & \searrow \partial & & & & \\ H_{n-1}(A_*) & \xleftarrow{H_{n-1}(f_*)} & \cdots & & & & \end{array}$$

is **exact**.

Proof: This is a typical example of a **diagram chase**. We will illustrate it by constructing the connecting homomorphism ∂ and leave the rest as an exercise. It is more fun and, in fact, easier to do it yourself than to read it. All we need is to look again at the extended picture (4) of the short exact sequence above.

To construct $\partial: H_n(C_*) \rightarrow H_{n-1}(A_*)$, let $c \in C_n$ be a cycle. Since g_n is **surjective**, there is a $b \in B_n$ with $g_n(b) = c$. Since $\partial_C(c) = 0$ and the diagram **commutes**, we get $g_{n-1}(\partial_B(b)) = \partial_C(g_n(b)) = \partial_C(c) = 0$. Since the horizontal sequences are **exact**, this implies there is an $a \in A_{n-1}$ with $f_{n-1}(a) = \partial_B(b)$. In fact, there is a **unique** such a because f_{n-1} is **injective**.

Moreover, we claim that this a is a cycle. For, since the diagram **commutes**, we have $f_{n-2}(\partial_A(a)) = \partial_B(f_{n-1}(a)) = \partial_B(\partial_B(b)) = 0$. Since f_{n-2} is **injective**, this implies $\partial_A(a) = 0$.

This means a represents a homology class and we define ∂ by sending the class of c to the class of a .

But we need to check that this does not depend on the choices we have made. So let $b' \in B_n$ be another element with $g_n(b') = c$, and let $a' \in A_{n-1}$ be the element that we find as above. Then we need to show $[a'] = [a]$ in $H_{n-1}(A_*)$, i.e., that $a' - a$ is a boundary. So we need an $\bar{a} \in A_n$ such that $\partial_A(\bar{a}) = a' - a$. We know $g_n(b' - b) = c - c = 0$. By **exactness**, there is an $\bar{a} \in A_n$ with $f_n(\bar{a}) = b' - b$. Since the diagram commutes, we have $f_{n-1}(\partial_A(\bar{a})) = \partial_B(b') - \partial_B(b)$. But we also have $f_{n-1}(a' - a) = \partial_B(b') - \partial_B(b)$ by definition of a' and a . Hence, since f_{n-1} is **injective**, we must have $\partial_A(\bar{a}) = a' - a$.

Finally, it is also clear from the construction that if c is a boundary, then a is zero.

This proves the existence of ∂ . Moreover, we know already that the induced homology sequence is exact at $H_n(B_*)$. It remains to check exactness at $H_n(A_*)$ and $H_n(C_*)$. This is left as an exercise. **QED**

Why do we care about long exact sequences?

Well, they are extremely useful. For example, for a pair of space (X, A) , if we can show $H_{n+1}(X, A) = 0$ and $H_n(X, A) = 0$, then $H_n(A) \cong H_n(X)$. Long exact sequences will be one of the **main computational tools** for studying interesting homology groups.

Furthermore, there is the famous Five Lemma (here in one of its variations):

Five Lemma

Suppose we have a commutative diagram

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & f_4 \downarrow & & f_5 \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

with exact rows. Then

- If f_2 and f_4 are surjective and f_5 injective, then f_3 is surjective.
- If f_2 and f_4 are injective and f_1 surjective, then f_3 is injective.

In particular, if f_1, f_2, f_4 , and f_5 are isomorphisms, then f_3 is an isomorphism.

The proof is another diagram chase and left as an exercise. You should definitely do it, it's fun!

Here we rather state two **consequences**:

- Given a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' \longrightarrow 0 \\ & & f' \downarrow & & f \downarrow & & f'' \downarrow \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' \longrightarrow 0 \end{array}$$

in which f' and f'' are isomorphisms. Then f is an isomorphism.

- Back in topology, let $f: (X,A) \rightarrow (Y,B)$ be a map of pairs of spaces. If any two of $A \rightarrow B$, $X \rightarrow Y$ and $(X,A) \rightarrow (Y,B)$ induce isomorphisms, then so does the third. This observation will simplify our life a lot.

LECTURE 6

The Eilenberg-Steenrod Axioms

Singular homology can in fact be uniquely characterized by a quite short list of properties some of which we have already checked. This list of properties is called the **Eilenberg-Steenrod Axioms**. We are now going to formulate them in general and will then discuss the relation to singular homology as we defined it.

First some preparations:

We denote by **Top₂** the category of pairs of topological spaces. Two continuous maps $f_0, f_1: (X, A) \rightarrow (Y, B)$ between pairs are called **homotopic**, denoted $f_0 \simeq f_1$, if there is a continuous map

$$h: X \times [0, 1] \rightarrow Y$$

such that, for all $x \in X$,

$$h(x, 0) = f_0(x), \quad h(x, 1) = f_1(x), \quad \text{and} \quad h(A \times [0, 1]) \subset B.$$

For $A \subset X$, we call

$$A^\circ = \bigcup_{U \subset A} U \text{ with } U \text{ open in } X$$

the **interior** of A and

$$\bar{A} = \bigcap_{A \subset Z} Z \text{ with } Z \text{ closed in } X$$

the **closure** of A .

Eilenberg-Steenrod Axioms

A **homology theory** (for topological spaces) **h consists of**:

- a sequence of functors $h_n: \mathbf{Top}_2 \rightarrow \mathbf{Ab}$ for all $n \in \mathbb{Z}$ and
- a sequence of functorial connecting homomorphisms

$$\partial: h_n(X, A) \rightarrow h_{n-1}(A, \emptyset) =: h_{n-1}(A)$$

which satisfy the following **properties**:

- **Dimension Axiom:** $h_q(\text{pt})$ is nonzero only if $q = 0$.
- **Long exact sequences:** For any pair (X, A) , the sequence

$$\cdots \xrightarrow{\partial} h_n(A) \xrightarrow{h_n(i)} h_n(X) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \xrightarrow{\partial} \cdots$$
 is exact, where we write $h_n(X) := h_n(X, \emptyset)$.
- **Homotopy Axiom:** If $f_0, f_1: (X, A) \rightarrow (Y, B)$ are homotopic, then the induced maps on homology

$$h_n(f_0) = h_n(f_1): h_n(X, A) \rightarrow h_n(Y, B)$$

for all $n \in \mathbb{Z}$.

- **Excision Axiom:** For every pair of spaces (X, A) and every $U \subset A$ with $\bar{U} \subset A^\circ$ the homomorphism

$$h_n(k): h_n(X \setminus U, A \setminus U) \rightarrow h_n(X, A)$$

induced by the inclusion map $k: (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ is an **isomorphism**.

- **Additivity Axiom:** If $X = \sqcup_\alpha X_\alpha$ is a disjoint union, then the inclusion maps $i_\alpha: X_\alpha \hookrightarrow X$ induce an isomorphism for every n

$$\oplus_\alpha h_n(i_\alpha): \bigoplus_\alpha h_n(X_\alpha) \xrightarrow{\cong} h_n(\sqcup_\alpha X_\alpha).$$

We have already shown that singular homology satisfies the dimension axiom and the connecting homomorphism fits into long exact sequences. It remains to check homotopy invariance and excision. But before we do that we will assume these properties for a moment and use them to compute some homology groups.

First an important consequence of the **homotopy axiom**:

Proposition: Homotopy invariance of homology

Let $f: (X, A) \rightarrow (Y, B)$ be a map of pairs which is a **homotopy equivalence**, i.e., there is a map $g: (Y, B) \rightarrow (X, A)$ such that $g \circ f \simeq \text{id}_{(X, A)}$ and $f \circ g \simeq \text{id}_{(Y, B)}$. Then f induces an isomorphism

$$H_n(f): H_n(X, A) \xrightarrow{\cong} H_n(Y, B)$$

in homology for all n .

In other words, **homology is invariant under homotopy** equivalences, not just homeomorphisms.

Recall that, for $n \geq 1$, we write D^n for the n -dimensional unit disk

$$D^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x|^2 = \sum_i x_i^2 \leq 1\}.$$

Recall that D^n is homotopy equivalent to a point. For, the constant map $D^n \rightarrow \{0\}$ is a strong deformation retraction with homotopy

$$h: D^n \times [0,1] \rightarrow D^n, (x,t) \mapsto (1-t)x$$

between the identity map of D^n and the constant map.

As a consequence of the **homotopy axiom** and our computation of $H_n(\text{pt})$ we get:

Corollary for contractible spaces

Recall that a space which is homotopy equivalent to a one-point space is called contractible. For every **contractible** space X , we have

$$H_q(X) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0. \end{cases}$$

Before we look at an application of the axioms, a remark on homology theories with a brief outlook to the future (of your studies in algebraic topology):

A remark on homology theories

In fact, if we require $h_0(\text{pt}) = \mathbb{Z}$ the above properties or axioms characterize singular homology uniquely. In other words, if h satisfies the Eilenberg-Steenrod axioms, then h must be singular homology.

We will see later that we can define variations of singular homology with coefficients different from \mathbb{Z} . If R is a commutative ring with unit and M an R -module, then there are singular homology groups $H_n(X; M)$ which fit into a homology theory which satisfies the dimension axiom with $h_0(\text{pt}) = M$.

We can even go a step further (in a different class) where we drop the dimension assumption allow $h_q(\text{pt}) \neq 0$ for infinitely many n . This leads to **generalized homology theories**, for example **K -theory** or **cobordism**, which are extremely useful for the solution of many fundamental problems, not just in topology. For example, **complex K -theory** can be used to show the theorem me mentioned in the notes of the first lecture: only in dimensions 1, 2, 4, and 8 there is a nice multiplication on \mathbb{R}^n .

• Homology of the sphere

As a fundamental example we are going to compute the homology of the k -dimensional sphere S^n . Actually, we already know one case. For, S^0 is just the disjoint union of two points. Hence $H_q(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $q = 0$ is 0 for all other n .

Theorem: Homology of the sphere

For $n \geq 1$, we have

$$H_q(S^n) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \text{ or } q = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & q = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof of the Theorem: For $n \geq 1$, the n -sphere S^n is **path-connected**. By our previous result, that implies $H_0(S^n) \cong \mathbb{Z}$.

The proof will proceed by **induction** using the long exact sequence in homology for pairs of spaces. This explains why we compute $H_q(S^n)$ and $H_q(D^n, S^{n-1})$ at the same time.

For $n = 1$ and $q = 0$, the pair (D^1, S^0) , with $i: S^0 \hookrightarrow D^1$, is equipped with the **exact sequence**

$$\begin{array}{ccccccc} H_0(S^0) & \xrightarrow{H_0(i)} & H_0(D^1) & \longrightarrow & H_0(D^1, S^0) & \longrightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \\ \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & ? & \longrightarrow & 0. \end{array}$$

The map $H_0(i)$ is induced by the map $S_0(S^0) \rightarrow S_0(D^1)$ which sends a 0-simplex $\Delta^0 \rightarrow S^0$ to the composite $\Delta^0 \rightarrow S^0 \hookrightarrow D^1$.

The image of S^0 in D^1 consists of the two endpoints of D^1 and both points are homologous as 0-simplices of D^1 . Hence they both represent the class of the generator of $H_0(D^1)$. Hence the map $H_0(i)$ **sends each generator of $H_0(S^0)$ to the generator of $H_0(D^1)$** . Writing $(1,0)$ and $(0,1)$ for the generators of $\mathbb{Z} \oplus \mathbb{Z}$, any $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ is of the form $a \cdot (1,0) + b \cdot (0,1)$. Hence (a,b) is sent to $a + b$ under $H_n(i)$.

This implies that $H_0(i)$ is **surjective**. Since the above sequence is **exact**, this implies

$$H_0(D^1, S^0) = 0.$$

For $n \geq 2$, the exact sequence becomes

$$\begin{array}{ccccccc} H_0(S^{n-1}) & \xrightarrow{H_0(i)} & H_0(D^n) & \longrightarrow & H_0(D^n, S^{n-1}) & \longrightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & ? & \longrightarrow & 0. \end{array}$$

Since both S^{n-1} and D^n are **path-connected**, their 0th homology is isomorphic to \mathbb{Z} and the generator of $H_0(S^{n-1})$, the class of any constant map $\kappa_x^0: \Delta^0 \rightarrow S^{n-1}$, is sent to the generator of $H_0(D^n)$, the class of $\kappa_x^0: \Delta^0 \rightarrow D^n$ corresponding to the image point $x \in S^{n-1} \subset D^n$. Hence $H_0(i)$ is surjective and we have again

$$H_0(D^n, S^{n-1}) = 0.$$

This **finishes the argument for H_0** .

For $q = 1$, we start with the exact sequence

$$(5) \quad \begin{array}{ccccccc} H_1(D^1) & \longrightarrow & H_1(D^1, S^0) & \longrightarrow & H_0(S^0) & \xrightarrow{H_0(i)} & H_0(D^1) \\ \parallel & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & ? & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}. \end{array}$$

Since the sequence is exact, this shows that $H_1(D^1, S^0)$ is isomorphic to the **kernel** of

$$H_n(i): \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}, (a, b) \mapsto a + b.$$

Thus

$$(6) \quad H_1(D^1, S^0) \cong \mathbb{Z}.$$

For $n \geq 2$, we get the sequence

$$\begin{array}{ccccccc} H_1(D^n) & \longrightarrow & H_1(D^n, S^{n-1}) & \longrightarrow & H_0(S^{n-1}) & \xrightarrow{\cong} & H_0(D^n) \\ \parallel & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & ? & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z}. \end{array}$$

Since the **right most map is an isomorphism**, we get

$$(7) \quad H_1(D^n, S^{n-1}) = 0 \text{ for all } n \geq 2.$$

In order to study further groups, we consider the subspaces

$$D_+^n := \{(x_0, \dots, x_n) \in S^n : x_0 \geq 0\} \text{ and } D_-^n := \{(x_0, \dots, x_n) \in S^n : x_0 \leq 0\}$$

which correspond to the **upper and lower hemisphere** (including the equator), respectively, of S^n .

For $n \geq 1$, we have the exact sequence

$$\begin{array}{ccccccc} H_1(D_-^n) & \longrightarrow & H_1(S^n) & \xrightarrow{\cong} & H_1(S^n, D_-^n) & \xrightarrow{\partial=0} & H_0(D_-^n) \xrightarrow{\cong} H_0(S^n) \\ \parallel & & & & & & \parallel & & \parallel \\ 0 & & & & & & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z}. \end{array}$$

Since D_-^n is **contractible**, we know $H_1(D_-^n) = 0$. Hence the map $H_1(S^n) \rightarrow H_1(S^n, D_-^n)$ is **injective**. Since the map

$$\mathbb{Z} \cong H_0(D_-^n) \rightarrow H_0(S^n) \cong \mathbb{Z}$$

is an **isomorphism**, the **connecting homomorphism ∂ is 0**. Since the sequence is **exact**, this implies that the map $H_1(S^n) \rightarrow H_1(S^n, D_-^n)$ is also **surjective**.

Thus, in total, we have an **isomorphism**

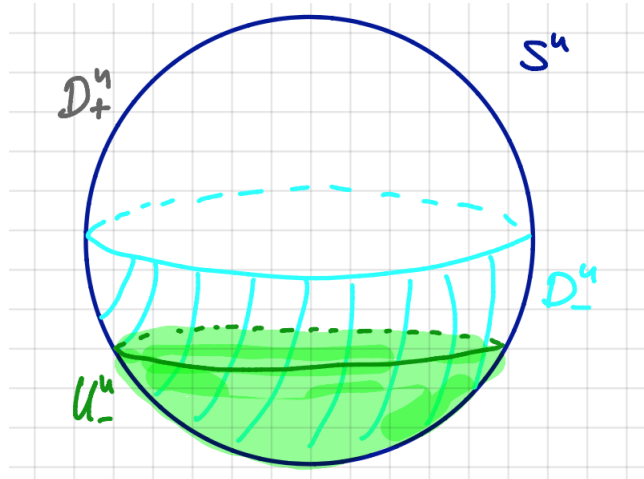
$$H_1(S^n) \xrightarrow{\cong} H_1(S^n, D_-^n).$$

To finish the analysis for $q = 1$, we consider the open subspace

$$U_-^n := \{(x_0, \dots, x_n) \in S^n : x_0 < -1/2\} \subset D_-^n.$$

Its closure is still contained in the open interior of D_-^n , i.e.,

$$\bar{U}_-^n \subset (D_-^n)^\circ$$



Hence we can apply the **excision axiom** to the inclusion of pairs

$$k: (S^n \setminus U_-^n, D_-^n \setminus U_-^n) \hookrightarrow (S^n, D_-^n)$$

and obtain an isomorphism

$$H_q(k): H_q(S^n \setminus U_-^n, D_-^n \setminus U_-^n) \xrightarrow{\cong} H_q(S^n, D_-^n).$$

But we also know

$$(S^n \setminus U_-^n, D_-^n \setminus U_-^n) \simeq (D_+^n, S^{n-1}) \xrightarrow{\cong} (D^n, S^{n-1})$$

where the last **homeomorphism is given by vertical projection**, and the **homotopy equivalence is the natural retraction**.

In particular, we get an **isomorphism**

$$(8) \quad H_q(S^n, D_-^n) \cong H_q(D^n, S^{n-1}).$$

For H_1 , this implies

$$H_1(S^n) \cong H_1(S^n, D_-^n) \cong H_1(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \text{ by (6).} \\ 0 & \text{else by (7).} \end{cases}$$

This **finishes the case $q = 1$** . In particular, we now know $H_1(S^1) \cong \mathbb{Z}$.

Finally, for $q \geq 2$, we proceed by induction. The pair (S^n, D_-^n) yields the exact sequence

$$(9) \quad \begin{array}{ccccccc} H_q(D_-^n) & \longrightarrow & H_q(S^n) & \xrightarrow{\cong} & H_q(S^n, D_-^n) & \longrightarrow & H_{q-1}(D^n) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0. \end{array}$$

Whereas the pair (D^n, S^{n-1}) yields the exact sequence

$$\begin{array}{ccccccc} H_q(D^n) & \longrightarrow & H_q(D^n, S^{n-1}) & \xrightarrow{\cong} & H_{q-1}(S^{n-1}) & \longrightarrow & H_{q-1}(D^n) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0. \end{array}$$

Together with isomorphism (8), we conclude

$$H_q(S^n) \cong H_q(S^n, D_-^n) \cong H_q(D^n, S^{n-1}) \cong H_{q-1}(S^{n-1}).$$

Hence knowing $H_1(S^1) = \mathbb{Z}$ implies $H_2(S^2) = \mathbb{Z}$ and $H_2(D^2, S^1) = \mathbb{Z}$. Continuing by **induction on q** yields the assertion of the theorem. \square

LECTURE 7

Generators for $H_n(S^n)$ and first applications

Generators for $H_n(S^n)$

Last time we calculated the homology groups of S^n and the pair (D^n, S^{n-1}) . To make this calculation a bit more concrete, let us try to figure out the generators of the infinite cyclic groups $H_n(D^n, S^{n-1})$ and $H_n(S^n)$:

- On the standard n -simplex, there is a special n -chain $S_n(\Delta^n)$, called the **fundamental n -simplex**, given by the **identity** map $\iota_n: \Delta^n \rightarrow \Delta^n$. We observed in a previous lecture that ι_n is **not a cycle**, since its boundary $\partial(\iota_n) \in S_{n-1}(\Delta^n)$ is the alternating sum of the faces of the n -simplex each of which is a generator in $S_{n-1}(\Delta^n)$.

$$\partial(\iota_n) = \sum_i (-1)^i \phi_i^n(\Delta^{n-1}) \neq 0.$$

However, each of these singular simplices lies in $\partial\Delta^n$, and hence

$$\partial(\iota_n) \in S_{n-1}(\partial\Delta^n).$$

Thus the image of ι_n in $S_n(\Delta^n, \partial\Delta^n)$ is a **relative cycle**. Let us denote its image also by ι_n and its class in $H^n(\Delta^n, \partial\Delta^n)$ by $[\iota_n]$.

If $H_n(\Delta^n, \partial\Delta^n)$ is nontrivial, then $[\iota_n]$ must be a **nontrivial generator**. For, if $\sigma: \Delta^n \rightarrow \Delta^n$ is any n -simplex of Δ^n which defines a nontrivial class $[\sigma]$ in $H_n(\Delta^n, \partial\Delta^n)$, then

$$[\sigma] = H_n(\sigma)([\iota_n]).$$

This is because ι_n is the identity map and $H_n(\sigma)([\iota_n])$ is defined by composing σ and ι_n . Hence if $[\iota_n]$ was trivial, then any class in $H_n(\Delta^n, \partial\Delta^n)$ would be trivial.

- Now we use this observation to find a generator of $H_n(D^n, S^{n-1})$. The standard n -simplex Δ^n and the unit n -disk D^n are homeomorphic. In order to find a homeomorphism we just need to smoothen out the corners of Δ^n . (Note that we cannot ask for a diffeomorphism, since Δ^n is not a smooth manifold.)

In fact, we can choose a **homeomorphism of pairs**

$$\varphi_n: (\Delta^n, \partial\Delta^n) \xrightarrow{\cong} (D^n, S^{n-1})$$

which maps $\partial\Delta^n$ homeomorphically to S^{n-1} . We will construct a concrete homeomorphism below. For the moment, let us accept that we have such a homeomorphism φ_n for every n .

Then φ_n induces an **isomorphism**

$$H_n(\varphi_n): H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_n(D^n, S^{n-1}) \text{ with } [\iota_n] \mapsto [\varphi_n].$$

Since we now know $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$, we also have $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ and $[\iota_n]$ as a generator. **Thus $[\varphi_n]$ is a generator of $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$.**

- Recall that we showed last time that the connecting homomorphism

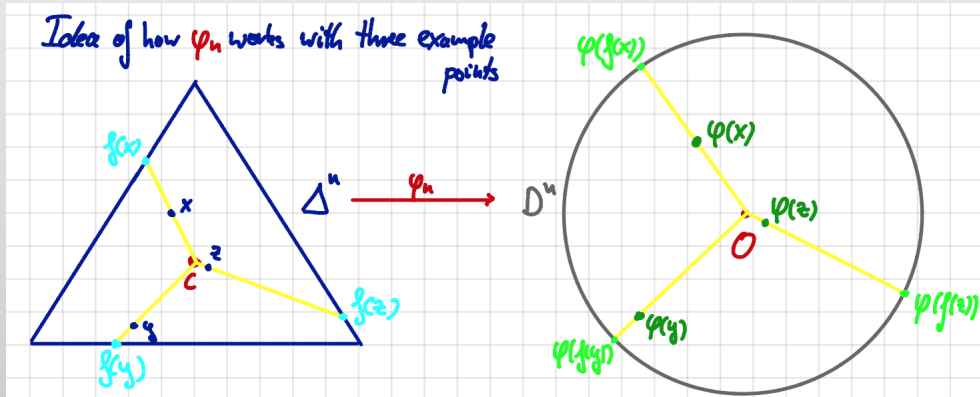
$$\partial: H_n(D^n, S^{n-1}) \xrightarrow{\cong} H_{n-1}(S^{n-1})$$

is an isomorphism. The image of $[\varphi_n]$ under ∂ is a generator. In other words, **$[\partial(\varphi_n)]$ is a generator of $H_{n-1}(S^{n-1})$** for all $n \geq 2$.

Constructing φ_n

For each Δ^n the point $c = (t_0, \dots, t_n)$ with $t_i = \frac{1}{n+1}$ for all i is the **barycenter** of Δ^n .

For every point $x \in \Delta^n$ which is not c , there is a unique ray from c to x . We denote the unique point where this ray hits $\partial\Delta^n$ by $f(x)$. In particular, if $x \in \partial\Delta^n$, then $f(x) = x$.



Now we define the map

$$\varphi_n: \Delta^n \rightarrow D^n, x \mapsto \begin{cases} \frac{x-c}{|f(x)-c|} & \text{if } x \neq c \\ 0 & \text{if } x = c. \end{cases}$$

It is clear that φ_n is continuous except possibly at $x = c$. But since there is a strictly positive lower bound for $|f(x) - c| > 0$, we know $|\varphi_n(x)| \leq M|x - c|$ for some real number M . This implies that φ_n is also continuous at $x = c$. Moreover, φ_n is a bijection, since it is one restricted to each ray. Since Δ^n is compact and φ_n is a continuous bijection, it is a homeomorphism.

Finally, we write down a generator for the unit circle.

A concrete generator of $H_1(S^1)$

We just learned that the class $[\partial(\varphi_2)]$ is a generator of $H_1(S^1)$. We can describe this class as follows:

By definition, $\partial(\varphi_2)$ is the 1-cycle

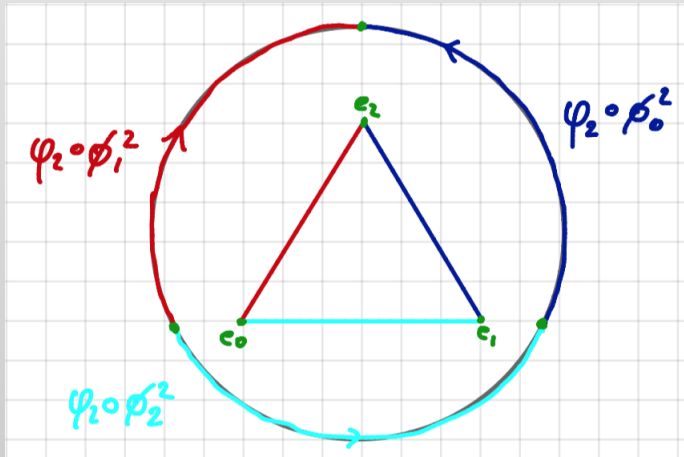
$$\begin{aligned} \partial(\varphi_2) &= d_0\varphi_2 - d_1\varphi_2 + d_2\varphi_2 \\ &= \varphi_2 \circ \phi_0^2 - \varphi_2 \circ \phi_1^2 + \varphi_2 \circ \phi_2^2. \end{aligned}$$

Recall that φ_2 maps $\partial\Delta^2$ homeomorphically to S^1 . With this in mind, the summands look like

$$\varphi_2 \circ \phi_0^2(1-t, t) = e^{i\pi(-\frac{1}{6} + t\frac{2}{3})}$$

$$\varphi_2 \circ \phi_1^2(1-t, t) = e^{i\pi(\frac{7}{6} - t\frac{2}{3})}$$

$$\varphi_2 \circ \phi_2^2(1-t, t) = e^{i\pi(\frac{7}{6} + t\frac{2}{3})}.$$



We proved in Lecture 03 that the 1-simplex

$$\Delta^1 \rightarrow \Delta^1, t \mapsto \varphi_2 \circ \phi_1^2(1-t, t)$$

is **homologous** to the 1-simplex

$$\Delta^1 \rightarrow \Delta^1, t \mapsto \varphi_2 \circ \phi_1^2(t, 1-t)$$

which **reverses the direction** of the walk from one vertex to the other.

In an exercise, we will show that after splitting a path into different steps, the 1-chain associated to the initial path is homologous to the sum of the 1-chains associated to the parts. This result implies that the 1-cycles $\partial\varphi_2$ **is homologous** to the 1-cycle corresponding to the familiar path

$$\gamma: \Delta^1 \rightarrow S^1, (1-t, t) \mapsto e^{2\pi it}$$

which **walks once around the circle**.

In summary, we showed that $[\gamma] = [\partial\varphi_2]$ is the **desired generator of $H_1(S^1)$** .

First applications

The calculation of the homology of spheres has many interesting consequences. We will discuss some of them today and will see many more soon.

We start with a result we advertised in the first lecture:

Theorem: Invariance of dimension

For $n \neq m$, the space \mathbb{R}^n is **not** homeomorphic to \mathbb{R}^m .

Proof: Assume there was a homeomorphism $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then the restricted map

$$f: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{f(0)\}$$

is **also a homeomorphism**, since these are open subsets and $f|_{\mathbb{R}^n \setminus \{0\}}$ and $(f^{-1})|_{\mathbb{R}^m \setminus \{f(0)\}}$ are still continuous mutual inverses.

We showed as an exercise that, for any $k \geq 1$, S^{k-1} is a strong deformation retract of $\mathbb{R}^k \setminus \{0\}$. In particular, we showed $S^{k-1} \simeq \mathbb{R}^k \setminus \{0\}$. Since the translation $\mathbb{R}^n \rightarrow \mathbb{R}^n, y \mapsto y + x$ is a homeomorphism for any $x \in \mathbb{R}^n$, this implies that

$$S^{k-1} \simeq \mathbb{R}^k \setminus \{x\} \text{ for every } x \in \mathbb{R}^k.$$

Hence, if the homeomorphism f existed, we would get an **induced isomorphism**

$$H_q(S^{n-1}) \cong H_q(\mathbb{R}^n \setminus \{0\}) \xrightarrow{\cong} H_q(\mathbb{R}^m \setminus \{f(0)\}) \cong H_q(S^{m-1}).$$

But by our calculation of the $H_q(S^{n-1})$, such an isomorphism can only exist if $n - 1 = q = m - 1$. This **contradicts the assumption** $n \neq m$. **QED**

We can also give a short proof of Brouwer's famous Fixed-Point Theorem:

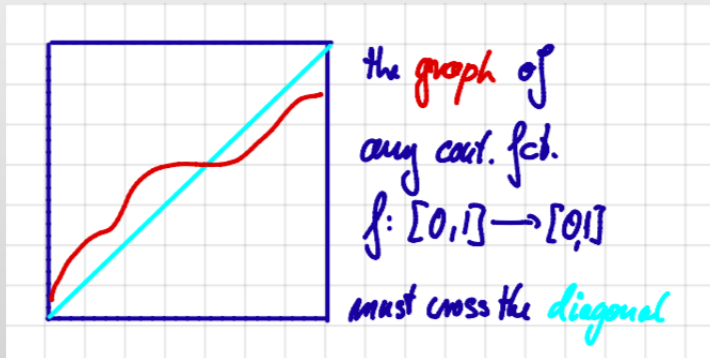
Brouwer Fixed-Point Theorem

Let $f: D^n \rightarrow D^n$ be a **continuous** map of the closed unit disk into itself. Then f must have a **fixed point**, i.e., there is an $x \in D^n$ with $f(x) = x$.

Before we prove the theorem, let us have a look at dimension one, where the result is very familiar:

Brouwer FPT is familiar in dimension one

Note that you have seen this theorem for $n = 1$ in Calculus 1. Let $f: [0, 1] \rightarrow [0, 1]$ be a continuous map. Then it must have a fixed point. For, if not, then $g(x) = f(x) - x$ is a continuous map defined on $[0, 1]$. We have $g(0) \geq 0$ and $g(1) \leq 0$, since $f(0) \geq 0$ and $f(1) \leq 1$.

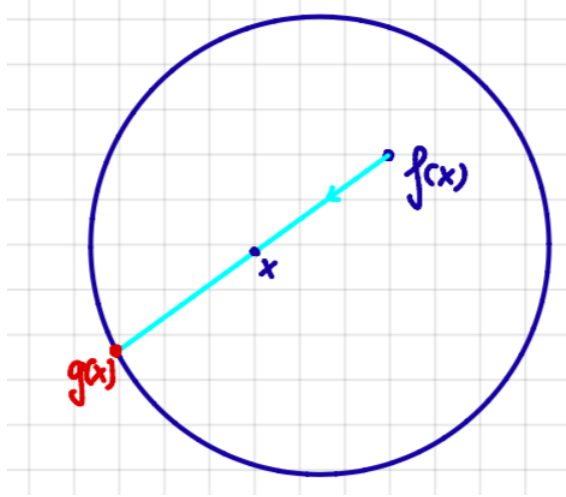


If $g(0) = 0$ or $g(1) = 0$, we are done. But if $g(0) > 0$ and $g(1) < 0$, then the **Intermediate Value Theorem** implies that there is an $x_0 \in (0, 1)$ with $g(x_0) = 0$, i.e., $f(x_0) = x_0$.

Proof of Brouwer's FPT: Since we know the theorem for $n = 1$, we assume $n \geq 2$. Suppose that there exists an f without fixed points, i.e., $f(x) \neq x$ for all $x \in D^n$. Then, for every $x \in D^n$, **the two distinct points** x and $f(x)$

determine a line. Let $g(x)$ be the point where the line segment starting at $f(x)$ and passing through x hits the boundary ∂D^n . This defines a continuous map

$$g: D^n \rightarrow \partial D^n.$$



Let $i: S^{n-1} = \partial D^n \hookrightarrow D^n$ denote the inclusion map. Note that **if** $x \in \partial D^n$, **then** $g(x) = x$. In other words,

$$g \circ i = \text{id}_{S^{n-1}}.$$

Applying the homology functor yields a **commutative diagram**

$$\begin{array}{ccc} \mathbb{Z} \cong H_{n-1}(S^{n-1}) & \xrightarrow{\text{id}_{H_{n-1}(S^{n-1})}} & H_{n-1}(S^{n-1}) \cong \mathbb{Z} \\ & \searrow H_n(i) & \nearrow H_n(g) \\ & H_{n-1}(D^n) = 0. & \end{array}$$

But the **identity homomorphism on \mathbb{Z} cannot factor through 0**. This contradicts the assumption that f has no fixed point. \square

Typical application of homology theory

The previous arguments are in fact a typical examples of proves in Algebraic Topology:

- The topological assumption that a homeomorphism $\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$ exists, is **translated** by applying homology to a **statement about an isomorphism of groups**. For groups, the existence of such an isomorphism is easily checked to be false.
- The geometric assumption that there is no fixed point be expressed in terms of maps and their compositions. Applying the **homology functor translates** this statement into an analogous **statement about groups and homomorphisms and their compositions**. Since the resulting statement about groups is obviously false, the original statement about spaces must be false as well.

• The degree of a map $S^n \rightarrow S^n$

The calculation of the homology of the sphere leads to another important algebraic invariant.

Definition: The degree

For $n \geq 1$, let $f: S^n \rightarrow S^n$ be a continuous map. Then the induced homomorphism

$$\mathbb{Z} \cong H_n(S^n) \xrightarrow{H_n(f)} H_n(S^n) \cong \mathbb{Z}$$

is given by multiplication with an integer, the image of 1.

We denote this integer by $\deg(f)$ and call it the **degree of f** .

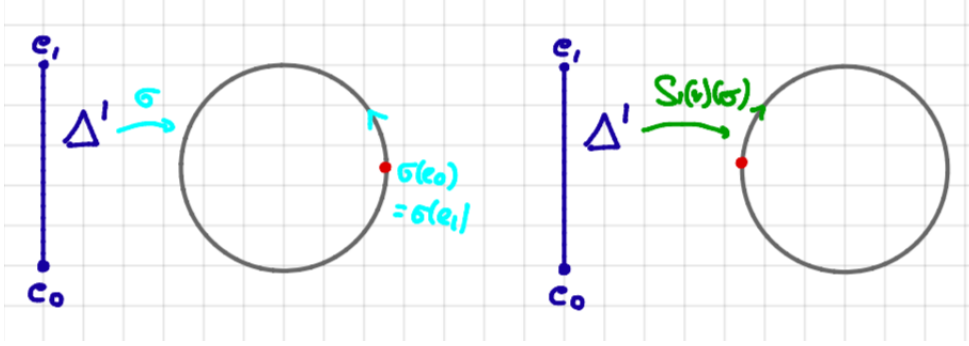
Let us calculate a first example:

Theorem: The degree of a reflection

Let $r: S^n \rightarrow S^n$ be the **reflection map** defined by reversal of the first coordinate

$$r: (x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n).$$

Then $\deg(r) = -1$.



Before we start the proof, let us have a look at what happens for the reflection map

$$r: D^1 = [-1, 1] \rightarrow [-1, 1] = D^1, t \mapsto -t$$

and **its restriction to S^0** . Recall that S^0 consists of just two points, $x = 1$ and $y = -1$ (on the real line \mathbb{R}). The effect of r on S^0 is to **interchange x and y** .

The inclusion maps i_x and i_y induce an isomorphism

$$H_0(\{x\}) \oplus H_0(\{y\}) \xrightarrow{\cong} H_0(S^0).$$

Thus $H_0(r)$ can be viewed as

$$H_0(r): H_0(S^0) \rightarrow H_0(S^0), (a, b) \mapsto (b, a).$$

During the calculation of $H_n(S^n)$, we remarked that the map

$$\epsilon := H_0(i): H_0(S^0) \rightarrow H_0(D^1)$$

induced by the inclusion $i: S^0 \hookrightarrow D^1$ can be identified with the homomorphism

$$\epsilon: \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}, (a, b) \mapsto a + b.$$

Let $\text{Ker}(\epsilon) = \{(a, -a) \in H_0(S^0) : a \in \mathbb{Z}\}$ be the kernel of ϵ . Then we get that the **effect of $H_0(r)$ on $\text{Ker}(\epsilon)$ is given by multiplication by -1** :

$$H_0(r): \text{Ker}(\epsilon) \rightarrow \text{Ker}(\epsilon), (a, -a) \mapsto (-a, a) = -(a, -a).$$

Now we can address the actual proof.

Proof of the Theorem: For $n \geq 1$, let

$$D_+^n := \{(x_0, \dots, x_n) \in S^n : x_n \geq 0\} \text{ and } D_-^n := \{(x_0, \dots, x_n) \in S^n : x_n \leq 0\}$$

be the **upper and lower hemispheres** on S^n , respectively. We also denote by r the reflection map on D_+^n and D_-^n . (Note that we defined D_+^n and D_-^n using a **different coordinate** than for defining r so that $r(D_+^n) \subset D_+^n$ and $r(D_-^n) \subset D_-^n$.)

Then we have a **commutative diagram**

$$\begin{array}{ccccccc} H_1(S^1) & \xrightarrow{\cong} & H_1(S^1, D_+^1) & \xleftarrow{\cong} & H_1(D_-^1, S^0) & \xrightarrow{\cong} & \text{Ker}(\epsilon) \\ H_1(r) \downarrow & & H_1(r) \downarrow & & \downarrow H_1(r) & & \downarrow H_0(r) \\ H_1(S^1) & \xrightarrow{\cong} & H_1(S^1, D_+^1) & \xleftarrow{\cong} & H_1(D_-^1, S^0) & \xrightarrow{\cong} & \text{Ker}(\epsilon). \end{array}$$

The right hand square commutes, since the isomorphism

$$H_1(D_-^1, S^0) \xrightarrow{\cong} \text{Ker}(\epsilon)$$

is part of the exact sequence induced by the pair (D_-^1, S^0) :

$$\begin{array}{ccccccc} H_1(D_-^1, S^0) & \longrightarrow & H_0(S^0) & \xrightarrow{\epsilon} & H_0(D_-^1) & \longrightarrow & H_0(D_-^1, S^0) \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

We know that the left hand and central squares commute, since the inclusion and the reflection commute. We know that the horizontal maps are isomorphisms from the calculation of these groups.

Thus, knowing $H_0(r) = -1$ on $\text{Ker}(\epsilon)$, we see that **$H_1(r)$ is also multiplication by -1 on $H_1(S^1)$.**

Now we can proceed by **induction**: For $n \geq 2$, we have again a **commutative diagram** from the calculation of $H_n(S^n)$:

$$\begin{array}{ccccccc} H_n(S^n) & \xrightarrow{\cong} & H_n(S^n, D_+^n) & \xleftarrow{\cong} & H_n(D_-^n, S^{n-1}) & \xrightarrow{\cong} & H_{n-1}(S^{n-1}) \\ H_n(r) \downarrow & & H_n(r) \downarrow & & \downarrow H_n(r) & & \downarrow H_{n-1}(r) \\ H_n(S^n) & \xrightarrow{\cong} & H_n(S^n, D_+^n) & \xleftarrow{\cong} & H_n(D_-^n, S^{n-1}) & \xrightarrow{\cong} & H_{n-1}(S^{n-1}). \end{array}$$

The right most square commutes by an exercise from last week. The left hand and central squares commute, since the inclusion and the reflection commute.

Assuming the assertion for $n - 1$, i.e., $H_{n-1}(r)$ is multiplication by -1 , we see that $H_n(r)$ is also multiplication by -1 . \square

LECTURE 8

Calculating degrees

In last week's exercises we showed many useful properties of the degree and calculated the degree of some interesting maps. Today, we are going to continue our study of the degree.

But before we move on, another reason why the degree is so important:

Brouwer degree

Let p be an arbitrary point in S^n . We consider p as the **base point** of S^n . Let $C(S^n, S^n)_*$ be the set of **pointed continuous maps**, i.e., maps $f: S^n \rightarrow S^n$ with $f(p) = p$. Pointed homotopy defines an **equivalence relation** on this set. Hence we can define the quotient set

$$[S^n, S^n]_* := C(S^n, S^n)_* / \simeq$$

where we identify f and g if they are homotopic to each other $f \simeq g$. Now the degree defines a function from $C(S^n, S^n)_*$ to the integers \mathbb{Z} . Since the degree is invariant under homotopy, i.e., $f_0 \simeq f_1$ implies $\deg(f_0) = \deg(f_1)$, it induces a function

$$\deg: [S^n, S^n]_* \rightarrow \mathbb{Z}, \quad f \mapsto \deg(f).$$

This function is actually an **isomorphism of abelian groups**. In fancier language, we write $\pi_n(S^n) = [S^n, S^n]_*$, call it the **n th homotopy group of S^n** and say that **the degree completely determines $\pi_n(S^n)$** .

Now let us see what kind of maps between spheres there are. Actually, such maps arise quite naturally. For, every invertible real $n \times n$ -matrix A defines a homeomorphism between $\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$. It extends to a homeomorphism on the **one-point compactification** S^n of \mathbb{R}^n and therefore defines a map

$$A: S^n \rightarrow S^n.$$

A more direct way to produce a map is to assume we have an orthogonal matrix:

Proposition: Orthogonal matrices

Let $O(n)$ denote the group of orthogonal (real) $n \times n$ -matrices, i.e.,

$$O(n) = \{A \in M(n \times n, \mathbb{R}) : A^T A = I\}$$

where I is the identity matrix. The restriction to S^{n-1} of any A in $O(n)$ defines a map

$$A: S^n \rightarrow S^n, x \mapsto Ax.$$

The degree of this map is $\det A$, i.e., $\deg(A) = \det(A) = \pm 1$.

Proof: This follows from the fact that every orthogonal matrix is the **product of reflections** (at appropriate hyperplanes in \mathbb{R}^n). A reflection has determinant -1 , but it also has degree -1 as we have shown before. Since both \deg and \det are **multiplicative**, the result follows. **QED**

Now let $A \in \text{GL}_n(\mathbb{R})$ be an **invertible** $n \times n$ -matrix. It defines a map

$$f: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}, f(x) := Ax.$$

It induces a map

$$H_{n-1}(f): H_{n-1}(\mathbb{R}^n \setminus \{0\}) \rightarrow H_{n-1}(\mathbb{R}^n \setminus \{0\}).$$

Since $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a **deformation retract**, we know

$$\mathbb{Z} \cong H_{n-1}(S^{n-1}) \cong H_{n-1}(\mathbb{R}^n \setminus \{0\}).$$

Hence the effect of $H_{n-1}(A)$ is given by multiplication by an integer.

Proposition: It's the sign

$H_{n-1}(A) = \text{sign}(\det(A))$ where $\text{sign}(\det(A))$ denotes the sign, i.e., 1 or -1 , of $\det(A)$.

Proof: Recall from linear algebra that any invertible matrix A has a **polar decomposition** $A = BC$ with B a symmetric matrix **with only positive eigenvalues** and $C \in O(n)$. Since we already know that the assertion is true if $A \in O(n)$, it suffices to show that B is homotopic to the identity as maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Since all eigenvalues of B are positive, we know $\det(B) > 0$. Hence B and I lie both in the component $\text{GL}_n(\mathbb{R})^+$ of the matrices with $\det > 0$. The continuous

map

$$\Gamma: [0,1] \rightarrow \mathrm{GL}_n(\mathbb{R})^+, t \mapsto tI + (1-t)B$$

defines a **homotopy between I and B** .

To check that $\Gamma(t)$ is in $\mathrm{GL}_n(\mathbb{R})^+$ for all t , we observe that the eigenvalues of $\Gamma(t)$ are all strictly positive. For, let λ be an eigenvalue of B . Then $t + (1-t)\lambda$ is an eigenvalue of $\Gamma(t)$, since all nonzero vectors are eigenvectors of tI . This implies $\det(\Gamma(t)) > 0$. **QED**

For $n > 1$, we know $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$, since $(D^n, S^{n-1}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is a deformation retract. For A as above, we obtain a commutative diagram from the long exact sequences of pairs

$$\begin{array}{ccc} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{H_n(A)} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \downarrow & & \downarrow \\ H_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xrightarrow{H_{n-1}(A)} & H_{n-1}(\mathbb{R}^n \setminus \{0\}). \end{array}$$

The vertical connecting homomorphisms are isomorphisms, since they are isomorphisms for the pair (D^n, S^{n-1}) . Since the diagram commutes, we deduce the following consequence from the previous result:

Corollary

The effect of the map

$$H_n(A): H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

is given by multiplication with $\mathrm{sign}(\det(A))$.

Now we would like to apply this to a situation familiar from Calculus. First a brief observation:

Lemma

Let $U \subset \mathbb{R}^n$ be an open subset and $x \in U$. Then

$$H_n(U, U \setminus \{x\}) \cong \mathbb{Z}.$$

Proof: Let Z be the complement of U in \mathbb{R}^n . Since U is open, Z is closed. Hence $\bar{Z} = Z \subset \mathbb{R}^n \setminus \{x\} = (\mathbb{R}^n \setminus \{x\})^\circ$. Hence we can apply **excision** to the

inclusion of pairs $(U, U \setminus \{x\}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ and get

$$H_n(U, U \setminus \{x\}) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}.$$

Proposition: Degree of smooth maps

Let $U \subset \mathbb{R}^n$ open with $0 \in U$. Let

$$f: U \rightarrow \mathbb{R}^n$$

be a smooth map (or say twice differentiable with continuous second derivatives) with $f^{-1}(0) = 0$ and $Df(0) \in GL_n(\mathbb{R})$.

For such a map f , the effect of the homomorphism

$$H_n(f): H_n(U, U \setminus \{0\}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

is given by multiplication with $\text{sign}(\det(Df(0)))$.

Proof: • By the **Taylor expansion** of a differentiable map, we can write f as

$$f(x) = Ax + g(x) \text{ with } A = Df(0) \text{ and } |g(x)|/|x| \rightarrow 0 \text{ for } x \rightarrow 0.$$

• In particular, we can assume $|g(x)| < |x|/2$ for x small enough. By **excision**, we can **shrink** U to become small enough such that still $0 \in U$ and $|g(x)| < |x|/2$ for all $x \in U$.

• We can further assume that $A = I$ is the identity. For if not, we can replace f with $A^{-1}f$ and use the functoriality of H_n .

• Now we have $|f(x) - x| < |x|/2$ for all $x \in U$. Hence the map

$$h: U \times [0,1] \rightarrow \mathbb{R}^n, (x,t) \mapsto tf(x) + (1-t)x$$

satisfies $h(x,t) \neq 0$ for all (x,t) . This implies that $Dh(0,t)$ is in $GL_n(\mathbb{R})^+$ for all t . Thus h defines a homotopy between f and the identity map and the effect of $H_n(f)$ is the same as the one of the identity map. \square

Local degree

Often the effect of a map can be studied by focussing on the neighborhood of certain interesting points. We would like to exploit this idea for studying the degree.

For $n \geq 1$, let $f: S^n \rightarrow S^n$ be a map with the property that there is a point $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points. (Note that almost all maps have this property.)

We label these points by x_1, \dots, x_m . Now we choose small **disjoint** open neighborhoods U_1, \dots, U_m of each x_i such that each U_i is mapped into an open neighborhood V of y in S^n . (We could choose V first, and then intersect $f^{-1}(V)$ with small open disks around x_i ...).

Since $x_i \in U_i$ and the different U_j s are disjoint, we have

$$f(U_i \setminus \{x_i\}) \subset V \setminus \{y\} \text{ for each } i.$$

For any given i , the obvious inclusions of pairs induce the following diagram:

$$(10) \quad \begin{array}{ccccc} & H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow[\deg(f|_{x_i})]{H_n(f|_{U_i})} & H_n(V, V \setminus \{y\}) & \\ & \cong \swarrow & & \downarrow \cong & \\ & H_n(S^n, S^n \setminus \{x_i\}) & \xleftarrow{p_i} & H_n(S^n, S^n \setminus f^{-1}(y)) & \xrightarrow{H_n(f)} & H_n(S^n, S^n \setminus \{y\}) \\ & \nwarrow \cong & & \uparrow \cong & \\ & H_n(S^n) & \xrightarrow{H_n(f)} & H_n(S^n) & \end{array}$$

By the **excision axiom** applied as in the proof of the lemma below and by an exercise, we know that the diagonal maps on the left and the vertical maps on the right are **isomorphisms**, as indicated in (10).

Definition: Local degree

The source and target of the dotted top horizontal arrow in (10) are identified with \mathbb{Z} . Hence the effect of this homomorphism is given by multiplication by an integer. We denote this integer by $\deg(f|_{x_i})$ and call it the **local degree of f at x_i** .

Let us calculate some **examples**:

- If f is a homeomorphism, then any y has a unique preimage x . In this case, all maps in daigram (10) are isomorphisms and we have

$$\deg(f) = \deg(f|x) = \pm 1.$$

- If f maps each U_i homeomorphically to V , then we have $\deg(f|_{x_i}) = \pm 1$ for each i .

The latter observation can be used to calculate the degree of f in many interesting situations. For we have the following result which connects global and local degrees:

Proposition: Global is the sum of local

With the above assumptions we have

$$\deg(f) = \sum_{i=1}^m \deg(f|_{x_i}).$$

We are going to prove this result in the next lecture. In the diagram above we claimed that some maps are isomorphisms. Here is an explanation why:

Lemma

(a) Let $U \subset S^n$ be an open subset and $x \in U$. Then there is an isomorphism

$$H_n(U, U \setminus \{x\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x\}) \cong \mathbb{Z}.$$

(b) Let x_1, \dots, x_m be m distinct points in S^n and U_1, \dots, U_m disjoint open neighborhoods with $x_i \in U_i$. Then there is an isomorphism

$$\oplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) \cong \oplus_{i=1}^m \mathbb{Z}.$$

Proof: (a) Let Z be the complement of U in \mathbb{R}^n . Since U is open, Z is closed. Hence $\bar{Z} = Z \subset S^n \setminus \{x\} = (S^n \setminus \{x\})^\circ = S^n \setminus \{x\}$. Hence we can apply **excision** to the inclusion of pairs $(U, U \setminus \{x\}) \hookrightarrow (S^n, S^n \setminus \{x\})$ and get the above isomorphism.

(b) Let $U := \cup_i U_i$. Then $Z := S^n \setminus U$ is closed. As above, we can apply **excision** to the inclusion of pairs $(U, U \setminus \{x_1, \dots, x_m\}) \hookrightarrow (S^n, S^n \setminus \{x_1, \dots, x_m\})$ and get an isomorphism

$$H_n(U, U \setminus \{x_1, \dots, x_m\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}).$$

Since U is actually a **disjoint** union and each $x_i \in U_i$, we know the inclusions induce an isomorphism

$$\oplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(U, U \setminus \{x_1, \dots, x_m\}).$$

Together with (a) this proves the assertion. □

LECTURE 9

Local vs global degrees

Last time we defined the local degree of a map. The situation was as follows:

For $n \geq 1$, let $f: S^n \rightarrow S^n$ be a map with the property that there is a point $y \in S^n$ such that $f^{-1}(y) = \{x_1, \dots, x_m\}$ consists of finitely many points.

We choose small **disjoint** open neighborhoods U_1, \dots, U_m of each x_i such that each U_i is mapped into an open neighborhood V of y in S^n . (We could choose V first, and then intersect $f^{-1}(V)$ with small open disks around x_i ...).

Since $x_i \in U_i$ and the different U_j s are disjoint, we have

$$f(U_i \setminus \{x_i\}) \subset V \setminus \{y\} \text{ for each } i.$$

For any given i , the obvious inclusions of pairs induce the following diagram:

$$(11) \quad \begin{array}{ccccc} & & H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow[\deg(f|_{U_i})]{H_n(f|_{U_i})} & H_n(V, V \setminus \{y\}) \\ & \swarrow \cong & \downarrow k_i & & \downarrow \cong \\ H_n(S^n, S^n \setminus \{x_i\}) & \xleftarrow{p_i} & H_n(S^n, S^n \setminus f^{-1}(y)) & \xrightarrow{H_n(f)} & H_n(S^n, S^n \setminus \{y\}) \\ & \nwarrow \cong & \uparrow j & & \uparrow \cong \\ & & H_n(S^n) & \xrightarrow{H_n(f)} & H_n(S^n). \end{array}$$

By the **excision axiom** applied as in the proof of the lemma below and by an exercise, we know that the diagonal maps on the left and the vertical maps on the right are **isomorphisms**, as indicated in (11). Then we made the following definition:

Definition: Local degree

The source and target of the dotted top horizontal arrow in (10) are identified with \mathbb{Z} . Hence the effect of this homomorphism is given by multiplication by an integer. We denote this integer by $\deg(f|x_i)$ and call it the **local degree of f at x_i** .

We have the following result which connects global and local degrees:

Proposition: Global is the sum of local

With the above assumptions we have

$$\deg(f) = \sum_{i=1}^m \deg(f|x_i).$$

Let us finally prove this result.

Proof: • As explained in the lemma below, excision implies that

$$\oplus_i k_i: \oplus_i \mathbb{Z} = H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus f^{-1}(y)) = \oplus_i \mathbb{Z}$$

is an **isomorphism**. Henceforth we are going to identify the groups in diagram (11) with \mathbb{Z} or the direct sum $\sum_i \mathbb{Z}$, respectively.

• We know that $p_i \circ k_i$ is the diagonal isomorphism. Hence k_i corresponds to the **inclusion** of the i th summand, and p_i corresponds to the **projection to the i th summand**.

• Since the **lower triangle commutes**, the composite $p_i \circ j$ satisfies

$$p_i \circ j(1) = 1.$$

Since we also know

$$p_i \circ k_i(1) = 1 \text{ and } \left(\sum_i k_i\right)(1) = (1, \dots, 1)$$

we must have

$$j(1) = (1, \dots, 1)$$

as well.

• Since the **upper square in diagram (11) commutes**, we know

$$H_n(f)(k_i(1)) = \deg(f|x_i).$$

- Together with the above, this shows

$$H_n(f)(j(1)) = \sum_i \deg(f|_{x_i}).$$

- Since the **lower square in diagram (11) commutes** and since the lower horizontal map is given by the degree of f , the asserted formula follows. **QED**

We have used the following lemma in diagram (11) the above proof:

Lemma

- (a) Let $U \subset S^n$ be an open subset and $x \in U$. Then there is an isomorphism

$$H_n(U, U \setminus \{x\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x\}) \cong \mathbb{Z}.$$

- (b) Let x_1, \dots, x_m be m distinct points in S^n and U_1, \dots, U_m disjoint open neighborhoods with $x_i \in U_i$. Then there is an isomorphism

$$\bigoplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{\cong} H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) \cong \bigoplus_{i=1}^m \mathbb{Z}.$$

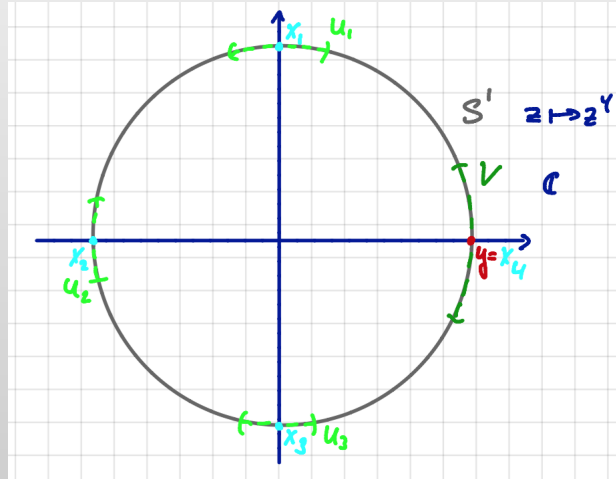
We proved this lemma in a previous lecture. Now let us apply what we learned in an example:

Example: Degree on the unit circle

Let $S^1 \subset \mathbb{C}$ be the unit circle, $k \in \mathbb{Z}$, and let

$$f_k: S^1 \rightarrow S^1, z \mapsto z^k.$$

We claim $\deg(f_k) = k$.



- We know this is true for **for** $k = 0$ when f_0 is the constant map and **for** $k = 1$ when f_1 is the identity.
- We know it also **for** $k = -1$, since $z \mapsto z^{-1}$ is a **reflection** at the real axis.
- It suffices to check the remaining cases for $k > 0$, since the cases **for** $k < 0$ follow from composition with $z \mapsto z^{-1}$ and the **multiplicativity of the degree**.
- So **let** $k > 0$. For any $y \in S^n$, $f_k^{-1}(y)$ consists of k distinct points x_1, \dots, x_k . Each point x_i has an open neighborhood U_i which is **mapped homeomorphically** by f to an open neighborhood V of y . This local homeomorphism is given by **stretching** (by the factor k) and a **rotation in positive direction**.
- Stretching by a factor is homotopic to the identity near x_i . Hence the local degree of the stretching is $+1$.
A rotation is a homeomorphism and its global and local degree at any point agree.
Since the rotation is in the positive direction, it is homotopic to the identity and has therefore degree $+1$.
- Hence $\deg(f|x_i) = 1$.
- Thus we can **conclude by the proposition** that

$$\deg(f) = \sum_{i=1}^k \deg(f|x_i) = k.$$

LECTURE 10

Homotopies of chain complexes

We still need to prove the Homotopy Axiom and Excision Axiom for singular homology. The proof will follow from constructing a homotopy between chain complexes, a concept we are now going to explore.

Recall that a **chain complex** $K_* = (K_*, \partial^K)$ consists of a sequence of abelian groups

$$\cdots \xrightarrow{\partial_{n+2}^K} K_{n+1} \xrightarrow{\partial_{n+1}^K} K_n \xrightarrow{\partial_n^K} K_{n-1} \xrightarrow{\partial_{n-1}^K} \cdots$$

together with homomorphisms $\partial_n^K: K_n \rightarrow K_{n-1}$ with the property that $\partial_{n-1} \circ \partial_n = 0$. Our main example is the singular chain complex.

Just to make sure that we understand the definition, let us look at an example of a sequence of groups that is **not** a chain complex. Consider the sequence of maps

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots$$

where each map consists of multiplication by 2. This is not a chain complex, since $2 \cdot 2 = 4$, i.e., $\partial_{n-1} \circ \partial_n = 4 \neq 0$.

Recall the definition of a map of chain complexes from Lecture 5:

Maps of chain complexes

Let $K_* = (K_*, \partial^K)$ and $L_* = (L_*, \partial^L)$ be two chain complexes. A **morphism of chain complexes** $f_*: K_* \rightarrow L_*$, also called **chain map**, is a sequence of homomorphisms $\{f_n\}_{n \in \mathbb{Z}}$

$$(12) \quad f_n: K_n \rightarrow L_n \text{ such that } f_{n-1} \circ \partial_n^K = \partial_n^L \circ f_n \text{ for all } n \in \mathbb{Z}.$$

A homomorphism of chain complexes induces a homomorphism on homology

$$H_n(f): H_n(K_*) \rightarrow H_n(L_*), [x] \mapsto [f_n(x)].$$

We need to check that this is **well-defined**. Since we hopped over this point in Lecture 5, let us do it now.

There are two things we should check. We need to know that f sends cycles to cycles and boundaries to boundaries.

First, let $x \in K_n$ be a cycle in K , i.e., $\partial_n^K(x) = 0$. Then (12) implies

$$\partial_n^L(f_n(x)) = f_{n-1}(\partial_n^K(x)) = f_{n-1}(0) = 0.$$

Thus $f_n(x)$ is a cycle in L and we get $f_n(Z_n(K_*)) \subset Z_n(L_*)$.

Second, let $x \in K_n$ be a boundary, say $\partial_{n+1}^K(y) = x$. Then (12) implies

$$f_n(x) = f_n(\partial_{n+1}^K(y)) = \partial_{n+1}^L(f_{n+1}(y)).$$

Thus $f_n(x)$ is a boundary in L and we get $f_n(B_n(K_*)) \subset B_n(L_*)$. This shows that f induces a well-defined homomorphism between the homologies of K_* and L_* .

We would like to transfer the notion of homotopies between maps of spaces to the homotopies between maps of chain complexes. This follows the general slogan: **Homotopy is a smart thing to do**.

Why? The notion of an **isomorphism** in a category, e.g. the category of topological spaces or the category of chain complexes, is often **too rigid**. There are too few isomorphism such that **classifying objects** up to isomorphism is too **difficult**. Therefore, one would like to relax the conditions. For many situations, homotopy turns out to provide the **right amount of flexibility** and rigidity at the same time. Moreover, many invariants, in fact all invariants in Algebraic Topology, do not change if we alter a map by a homotopy.

In other words, **our invariants only see the homotopy type**.

Actually, this is exactly what we are going to show for singular homology today. It is also true in Homological Algebra. The homology of a chain complex only depends on the homotopy type of the complex.

So let us define homotopies between chain maps:

Definition: Homotopies of chain maps

Let $f, g: K_* \rightarrow L_*$ be two morphisms of chain complexes. A **chain homotopy** between f and g is a sequence of homomorphisms

$$h_n: K_n \rightarrow L_{n+1}$$

such that

$$(13) \quad f_n - g_n = \partial_{n+1}^L \circ h_n + h_{n-1} \circ \partial_n^K \text{ for all } n \in \mathbb{Z}.$$

$$\begin{array}{ccc} K_{n+1} & \longrightarrow & L_{n+1} \\ \partial_{n+1}^K \downarrow & \nearrow h_n & \downarrow \partial_{n+1}^L \\ K_n & \xrightarrow{f_n - g_n} & L_n \\ \partial_n^K \downarrow & \nearrow h_{n-1} & \downarrow \partial_n^L \\ K_{n-1} & \longrightarrow & L_{n-1} \end{array}$$

If such a homotopy exists, we are going to say that f and g are homotopic and write $f \simeq g$.

We say that f is **null-homotopic** if $f \simeq 0$.

As for topological spaces, this yields an equivalence relation:

Lemma: Homotopy is an equivalence relation

(1) Chain homotopy is an equivalence relation on the set of all morphisms of chain complexes.

(2) If $f \simeq f': K_* \rightarrow L_*$ and $g \simeq g': L_* \rightarrow M_*$, then $g \circ f \simeq g' \circ f'$.

Proof: (1) We need to show that homotopy is reflexive, symmetric and transitive:

- We obtain $f \simeq f$ with $h = 0$ being the zero map.
- If h is a homotopy which gives $f \simeq g$, then $-h$ is a homotopy which shows $g \simeq f$.
- If h is a homotopy which gives $f \simeq g: K_* \rightarrow L_*$ and h' is a homotopy which shows $g \simeq k: K_* \rightarrow L_*$, then $h + h'$ is a homotopy which shows $f \simeq k$. For

$$\begin{aligned} f_n - k_n &= f_n - g_n + g_n - k_n \\ &= \partial_{n+1}^L \circ h_n + h_{n-1} \circ \partial_n^K + \partial_{n+1}^L \circ h'_n + h'_{n-1} \circ \partial_n^K \\ &= \partial_{n+1}^L \circ (h_n + h'_n) + (h_{n-1} + h'_{n-1}) \circ \partial_n^K. \end{aligned}$$

(2) Let h be a homotopy which shows $f \simeq f'$ and k be a homotopy which shows $g \simeq g'$. Composition with g on the left and using that g is a chain map

yields

$$\begin{aligned} g_n \circ (f_n - f'_n) &= g_n \circ (\partial_{n+1}^L \circ h_n + h_{n-1} \circ \partial_n^K) \\ &= \partial_{n+1}^M \circ (g_{n+1} \circ h_n) + (g_n \circ h_{n-1}) \circ \partial_n^K. \end{aligned}$$

This shows that the sequence of maps $g_{n+1} \circ h_n$ defines a homotopy $g \circ f \simeq g \circ f'$.

Composition with f' on the right and using that f' is a chain map yields

$$\begin{aligned} (g_n - g'_n) \circ f'_n &= (\partial_{n+1}^M \circ k_n + k_{n-1} \circ \partial_n^L) \circ f'_n \\ &= \partial_{n+1}^M \circ (k_n \circ f'_n) + (k_{n-1} \circ f'_{n-1}) \circ \partial_n^K. \end{aligned}$$

This shows that the sequence of maps $k_n \circ f'_n$ defines a homotopy $g \circ f' \simeq g' \circ f'$.

Summarizing we have shown

$$g \circ f \simeq g \circ f' \simeq g' \circ f'.$$

By transitivity, this shows the desired result. \square

Now we are ready to show an important fact in homological algebra:

Homology identifies chain homotopies

If $f \simeq g: K_* \rightarrow L_*$ are **homotopic** morphisms of chain complexes, then

$$H_n(f) = H_n(g) \text{ for all } n \in \mathbb{Z}.$$

Proof: This follows immediately from the fact that $f_n - g_n$ is just given by boundaries which, by definition, vanish in homology.

More concretely, let $x \in K_n$ be an arbitrary cycle in K_n and let h be a homotopy which gives $f \simeq g$. Then we get by using the definition of homotopies

$$H_n(f)([x]) = [f_n(x)] = [g_n(x) + \partial_{n+1}^L(h_n(x)) + h_{n-1}(\partial_n^K(x))] = [g_n(x)] = H_n(g)([x])$$

where we use that $\partial_{n+1}^L(h_n(x))$ is obviously a **boundary in L_n** and that $h_{n-1}(\partial_n^K(x)) = 0$, since x is a **cycle in K_n** by assumption. \square

Now we can also mimic the notion of homotopy equivalences.

Chain homotopy equivalences

A morphism of chain complexes $f: K_* \rightarrow L_*$ is called a **homotopy equivalence** if there exists a morphism of chain complexes $g: L_* \rightarrow K_*$ such that $g \circ f \simeq \text{id}_{K_*}$ and $f \circ g \simeq \text{id}_{L_*}$.

If such a homotopy equivalence exists, we write $K_* \simeq L_*$ and say that K_* and L_* are **homotopy equivalent**.

In particular, by adopting language from algebraic topology, if the identity map on a chain complex K_* is homotopy equivalent to the zero map, then we say that K_* is **contractible**. For example, if X is a **contractible space**, then its singular chain complex $S_*(X)$ is a **contractible chain complex**. Note that a chain complex K_* with at least one nonzero homology group cannot be contractible.

As a consequence of what we proved we get:

Chain homotopy equivalences

- If $K_* \simeq L_*$, then $H_n(K_*) \cong H_n(L_*)$.
- Given two chain complexes K_* and L_* we denote the set of morphisms of chain complexes by $\text{Mor}(K_*, L_*)$. Let $[K_*, L_*] := \text{Mor}(K_*, L_*) / \simeq$ denote the set of equivalence classes under the relation given by chain homotopies. Then we can define a new category whose objects are chain complexes and whose sets of morphisms from $K_* \rightarrow L_*$ are homotopy classes of chain maps, i.e., the sets $[K_*, L_*]$. Let us call this category **K**. Since the homotopy relation respects composition, we obtain that homology defines a functor

$$\mathbf{K} \rightarrow \mathbf{Ab}, K_* \mapsto H_n(K_*)$$

where **Ab** denotes the category of abelian groups.

Let us look at **some examples**:

- Let K_* be the chain complex

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \rightarrow \cdots .$$

Since all maps are trivial, we have $H_n(K_*) = K_n$ for all n . Hence K_* has exactly two nonzero homology groups, both being \mathbb{Z} . In particular, it is **not contractible**.

- Let K_* be the chain complex

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \rightarrow 0 \rightarrow \cdots .$$

This complex is actually an exact sequence. Thus $H_n(K_*) = 0$ for all n . Moreover it is **contractible**. We can write down a homotopy by

$$\begin{array}{ccc}
 0 & \xrightarrow{\text{id}} & 0 \\
 \downarrow & \nearrow 0 & \downarrow \\
 \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\
 \downarrow & \nearrow 1 & \downarrow \\
 \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\
 \downarrow & \nearrow 0 & \downarrow \\
 0 & \xrightarrow{\text{id}} & 0.
 \end{array}$$

- Let K_* be the chain complex

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow 0 \rightarrow \cdots.$$

This complex has one nonzero homology group $H_1(K_*) = \mathbb{Z}/2$. It is therefore **not contractible**.

- Let K_* be the chain complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots.$$

The homology of K_* vanishes, since, at each stage, the image and the kernel of the differential is $2\mathbb{Z}/4$. Nevertheless, K_* is **not contractible**. For if there was a homotopy between id_{K_*} and the zero map, it would like this

$$\begin{array}{ccc}
 \mathbb{Z}/4 & \xrightarrow{\text{id}} & \mathbb{Z}/4 \\
 \downarrow 2 & \nearrow h_n & \downarrow 2 \\
 \mathbb{Z}/4 & \xrightarrow{\text{id}} & \mathbb{Z}/4 \\
 \downarrow 2 & \nearrow h_{n-1} & \downarrow 2 \\
 \mathbb{Z}/4 & \xrightarrow{\text{id}} & \mathbb{Z}/4
 \end{array}$$

and satisfy $\text{id} = 2h_n + h_{n-1}2$. But $2h_n + h_{n-1}2$ can only produce even numbers modulo 4. Hence it cannot be the identity map on $\mathbb{Z}/4$.

After all this abstract stuff we should better demonstrate that the notion of chain homotopies is useful for our purposes. We are going to do this by showing that homotopies between maps of spaces induces a chain homotopy. By what we have just seen, this will prove the Homotopy Axiom for singular homology.

LECTURE 11

Homotopy invariance of singular homology

We are going to prove the Homotopy Axiom for singular homology. The proof will follow from constructing a homotopy between chain complexes.

Let $f, g: X \rightarrow Y$ be **two homotopic maps** and let $h: X \times [0,1] \rightarrow Y$ be a homotopy between them. Let $\sigma: \Delta^n \rightarrow X$ be an n -simplex on X . Then h induces a map

$$\Delta^n \times [0,1] \xrightarrow{\sigma \times \text{id}} X \times [0,1] \xrightarrow{h} Y$$

which defines a homotopy between $f \circ \sigma$ and $g \circ \sigma$.

Our goal is to **turn this into a geometrically induced chain homotopy between $S_n(f)$ and $S_n(g)$** . By our result from the previous lecture, this will imply the Homotopy Axiom.

So let us have a closer look at the space $\Delta^n \times [0,1]$. For $n = 1$, it looks just like a square. Via the diagonal we can divide it into two triangles which look like Δ^2 . For $n = 2$, $\Delta^2 \times [0,1]$ looks like a prism which we can divide into three copies of Δ^3 .

In general, $\Delta^n \times [0,1]$ looks like a higher dimensional prism which we can divide into $n + 1$ copies of Δ^{n+1} . We should make this idea more precise:

Simplices on a prism

For every $n \geq 0$ and $0 \leq i \leq n$, we define an injective map

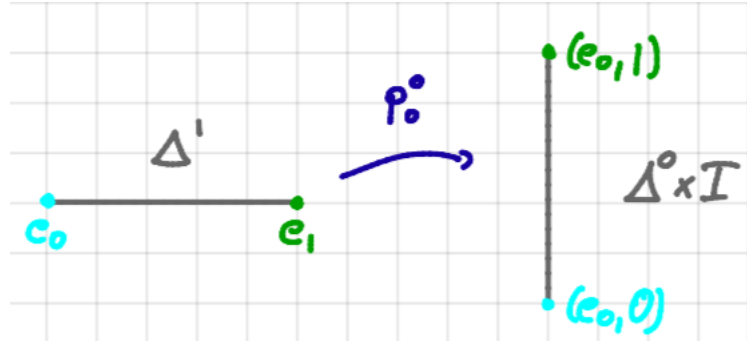
$$p_i^n: \Delta^{n+1} \rightarrow \Delta^n \times [0,1],$$

$$(t_0, \dots, t_{n+1}) \mapsto ((t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}), t_{i+1} + \dots + t_{n+1}).$$

We can consider each p_i^n as an $n + 1$ -simplex on the space $\Delta^n \times [0,1]$. When n is clear we will often drop it from the notation.

For $n = 0$, we have only one map

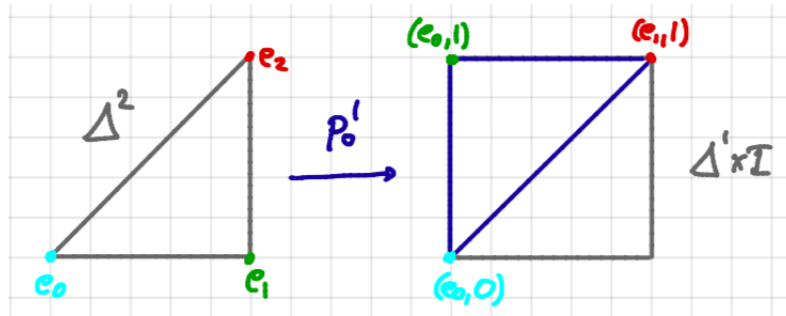
$$p_0^0: \Delta^1 \rightarrow \Delta^0 \times I = \{e_0\} \times [0,1], (t_0, t_1) \mapsto (0, t_1).$$



Let us have a look at **what happens for $n = 1$** : Then the effect of p_0^1 and p_1^1 is given by

$$p_0: \Delta^2 \rightarrow \Delta^1 \times [0,1], (t_0, t_1, t_2) \mapsto (t_0 + t_1, t_1 + t_2)$$

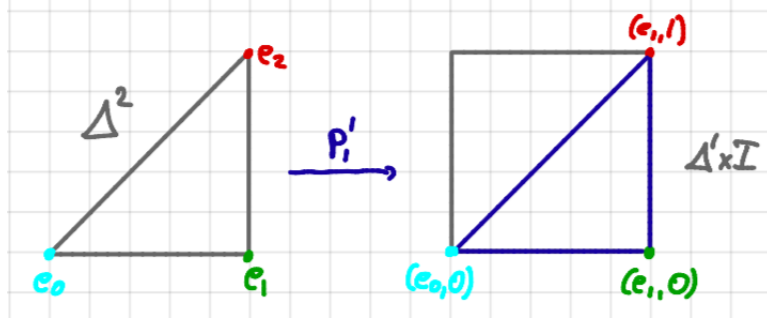
$$\begin{cases} e_0 = (1,0,0) & \mapsto (1,0,0) = (e_0,0) \\ e_1 = (0,1,0) & \mapsto (1,0,1) = (e_0,1) \\ e_2 = (0,0,1) & \mapsto (0,1,1) = (e_1,1) \end{cases}$$



and

$$p_1: \Delta^2 \rightarrow \Delta^1 \times [0,1], (t_0, t_1, t_2) \mapsto (t_0, t_1 + t_2)$$

$$\begin{cases} e_0 = (1,0,0) & \mapsto (1,0,0) = (e_0,0) \\ e_1 = (0,1,0) & \mapsto (0,1,0) = (e_1,0) \\ e_2 = (0,0,1) & \mapsto (0,1,1) = (e_1,1). \end{cases}$$



In general, the effect of p_i^n on the vertex e_k of Δ^{n+1} for $0 \leq k \leq n+1$ is given by

$$(14) \quad p_i^n(e_k) = \begin{cases} (e_k, 0) & \text{if } 0 \leq k \leq i \\ (e_{k-1}, 1) & \text{if } k > i. \end{cases}$$

In fact, we could **define** p_i^n as the unique **affine map which satisfies (14)**.

Let j_0 and j_1 be the two inclusions

$$\begin{aligned} j_0: \Delta^n &\hookrightarrow \Delta^n \times [0, 1], x \mapsto (x, 0) \\ j_1: \Delta^n &\hookrightarrow \Delta^n \times [0, 1], x \mapsto (x, 1) \end{aligned}$$

determined by the endpoints of $[0, 1]$.

For the next result, recall our formulae for the **face maps** on standard simplices:

For $0 \leq i \leq n+1$ which can be described as

$$\phi_i^{n+1}(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$$

with the 0 inserted at the i th coordinate.

Using the **standard basis**, ϕ_i^{n+1} can be described as the affine map (a translation plus a linear map)

$$(15) \quad \phi_i^{n+1}: \Delta^n \hookrightarrow \Delta^{n+1} \text{ determined by } \phi_i^{n+1}(e_k) = \begin{cases} e_k & k < i \\ e_{k+1} & k \geq i. \end{cases}$$

Lemma: Prism and face maps

We have the following identifications of maps:

$$(16) \quad p_0^n \circ \phi_0^{n+1} = j_1,$$

$$(17) \quad p_n^n \circ \phi_{n+1}^{n+1} = j_0,$$

$$(18) \quad p_i^n \circ \phi_i^{n+1} = p_{i-1}^n \circ \phi_i^{n+1} \text{ for } 1 \leq i \leq n,$$

$$(19) \quad p_{j+1}^n \circ \phi_i^{n+1} = (\phi_i^n \times \text{id}) \circ p_j^{n-1} \text{ for } j \geq i.$$

$$(20) \quad p_j^n \circ \phi_{i+1}^{n+1} = (\phi_i^n \times \text{id}) \circ p_j^{n-1} \text{ for } j < i,$$

Proof: (16) We check the effect of $p_0^n \circ \phi_0^{n+1}$

$$\begin{aligned} p_0^n(\phi_0^{n+1}(t_0, \dots, t_n)) &= p_0(0, t_0, \dots, t_n) \\ &= \left((t_0, \dots, t_n), \sum_{i=0}^n t_i \right) \\ &= (t_0, \dots, t_n, 1) = j_1(t_0, \dots, t_n). \end{aligned}$$

(17) Similarly, we calculate

$$\begin{aligned} p_n^n(\phi_{n+1}^{n+1}(t_0, \dots, t_n)) &= p_n^n(t_0, \dots, t_n, 0) \\ &= (t_0, \dots, t_n, 0) = j_0(t_0, \dots, t_n). \end{aligned}$$

(18) We calculate and compare:

$$\begin{aligned} p_i^n(\phi_i^{n+1}(t_0, \dots, t_n)) &= p_i^n(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n) \\ &= ((t_0, \dots, t_{i-1}, 0 + t_i, t_{i+1}, \dots, t_n), 0 + t_i + \dots + t_n) \\ &= \left((t_0, \dots, t_n), \sum_{j=i}^n t_j \right), \end{aligned}$$

and

$$\begin{aligned}
 & p_{i-1}^n(\phi_i^{n+1}(t_0, \dots, t_n)) \\
 &= p_{i-1}^n(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n) \\
 &= ((t_0, \dots, t_{i-2}, t_{i-1} + 0, t_i, t_{i+1}, \dots, t_n), t_i + \dots + t_n) \\
 &= \left((t_0, \dots, t_n), \sum_{j=i}^n t_j \right).
 \end{aligned}$$

Hence both maps agree.

(19) For $j \geq i$, the assertion amounts to showing that the following diagram commutes:

$$\begin{array}{ccc}
 & \Delta^{n+1} & \xrightarrow{p_j^{n-1}} \Delta^n \times [0,1] \\
 \phi_i^{n+1} \nearrow & & \parallel \\
 \Delta^n & & \\
 p_{j+1}^n \searrow & & \\
 & \Delta^{n-1} \times [0,1] & \xrightarrow{\phi_i^n \times \text{id}} \Delta^n \times [0,1].
 \end{array}$$

To check this, we are going to use formulae (14) and (15). Since the affine maps involved are determined by their effect on the e_k s, this will suffice to prove the formulae.

For $k < i$, we get

$$\begin{aligned}
 p_{j+1}^n \circ \phi_i^{n+1}(e_k) &= p_{j+1}^n(e_k) = (e_k, 0) \\
 &= (\phi_i^n \times \text{id})(e_k, 0) = (\phi_i^n \times \text{id}) \circ p_j^{n-1}(e_k).
 \end{aligned}$$

For $i \leq k \leq j$, we get

$$\begin{aligned}
 p_{j+1}^n \circ \phi_i^{n+1}(e_{k+1}) &= p_{j+1}^n(e_k) = (e_{k+1}, 0) \\
 &= (\phi_i^n \times \text{id})(e_k, 0) = (\phi_i^n \times \text{id}) \circ p_j^{n-1}(e_k).
 \end{aligned}$$

And for $k > j$, we get

$$\begin{aligned}
 p_{j+1}^n \circ \phi_i^{n+1}(e_k) &= p_{j+1}^n(e_{k+1}) = (e_k, 1) \\
 &= (\phi_i^n \times \text{id})(e_{k-1}, 1) = (\phi_i^n \times \text{id}) \circ p_j^{n-1}(e_k).
 \end{aligned}$$

(20) follows by a similar argument. □

Definition: Induced prism operator

- For every $n \geq 0$ and $0 \leq i \leq n$, the map p_i^n induces a group homomorphism

$$P_i^n: S_n(X) \rightarrow S_{n+1}(X \times [0, 1])$$

which is defined on generators by composition with p_i^n

$$P_i^n(\sigma) = (\sigma \times \text{id}) \circ p_i^n: \Delta^{n+1} \xrightarrow{p_i^n} \Delta^n \times [0, 1] \xrightarrow{\sigma \times \text{id}} X \times [0, 1]$$

and extended \mathbb{Z} -linearly.

- This construction descends to a map P_i^n on **relative chains** for any pair (X, A) .
- We define a group homomorphism, often called **prism operator**,

$$P^n: S_n(X, A) \rightarrow S_{n+1}(X \times I, A \times I), \quad P^n = \sum_{i=0}^n (-1)^i P_i^n.$$

Let j_t^X denote the inclusion $X \hookrightarrow X \times [0, 1]$, $x \mapsto (x, t)$. The prism operator is the desired chain homotopy:

Chain homotopy lemma

The homomorphisms P^n provide a **chain homotopy** between the two morphisms of chain complexes

$$S_*(j_0^X) \simeq S_*(j_1^X): S_*(X, A) \rightarrow S_*(X \times I, A \times I).$$

Proof: We need to show $S_n(j_0^X) - S_n(j_1^X) = \partial_{n+1} \circ P^n + P^{n-1} \circ \partial_n$. Let σ be an n -simplex. Then we calculate

$$\begin{aligned}
P^{n-1} \circ \partial_n(\sigma) &= P^{n-1} \left(\sum_{i=0}^n (-1)^i \sigma \circ \phi_i^n \right) \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} (\sigma \times \text{id}) \circ (\phi_i^n \times \text{id}) \circ p_j^{n-1} \\
&\quad + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} (\sigma \times \text{id}) \circ (\phi_i^n \times \text{id}) \circ p_j^{n-1} \\
&= - \sum_{0 \leq j < i \leq n} (-1)^{i+j+1} (\sigma \times \text{id}) \circ p_{j+1}^n \circ \phi_i^{n+1} \text{ by (20)} \\
&\quad - \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j+1} (\sigma \times \text{id}) \circ p_j^n \circ \phi_{i+1}^{n+1} \text{ by (19)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\partial_{n+1} \circ P^n(\sigma) &= \partial_{n+1} \left(\sum_{j=0}^n (-1)^j (\sigma \times \text{id}) \circ p_j^n \right) \\
&= \sum_{j=0}^n \sum_{i=0}^{n+1} (-1)^{i+j} (\sigma \times \text{id}) \circ p_j^n \circ \phi_i^{n+1} \\
&= \sum_{0 \leq i < j \leq n} (-1)^{i+j} (\sigma \times \text{id}) \circ p_j^n \circ \phi_i^{n+1} \quad (i < j) \\
&\quad + \sum_{i=0}^n (\sigma \times \text{id}) \circ p_i^n \circ \phi_i^{n+1} \quad (i = j) \\
&\quad - \sum_{i=1}^{n+1} (\sigma \times \text{id}) \circ p_{i-1}^n \circ \phi_i^{n+1} \quad (i = j+1) \\
&\quad + \sum_{1 \leq j+1 \leq i \leq n+1} (-1)^{i+j} (\sigma \times \text{id}) \circ p_j^n \circ \phi_i^{n+1} \quad (i > j+1) \\
&= \sum_{0 \leq i \leq j' \leq n-1} (-1)^{i+j'+1} (\sigma \times \text{id}) \circ p_{j'+1}^n \circ \phi_i^{n+1} \\
&\quad + (\sigma \times \text{id}) \circ j_1 - (\sigma \times \text{id}) \circ j_0 \\
&\quad + \sum_{0 \leq j \leq i' \leq n} (-1)^{i'+j+1} (\sigma \times \text{id}) \circ p_j^n \circ \phi_{i'+1}^{n+1}.
\end{aligned}$$

For the final step we used again the trick to relabel the indices and wrote $j' = j - 1$ and $i' = i - 1$. By comparing the two calculations, we see that all summands cancel out except for $(\sigma \times \text{id}) \circ j_1 - (\sigma \times \text{id}) \circ j_0$.

Thus we can conclude:

$$\begin{aligned} \partial_{n+1} \circ P^n(\sigma) + P^{n-1} \circ \partial_n(\sigma) &= (\sigma \times \text{id}) \circ j_1^X - (\sigma \times \text{id}) \circ j_0^X \\ &= j_1^X \circ \sigma - j_0^X \circ \sigma \\ &= S_n(j_1^X)(\sigma) - S_n(j_0^X)(\sigma). \quad \square \end{aligned}$$

As a consequence we get the Homotopy Axiom:

Theorem: Homotopy Invariance

If $f \simeq g: (X, A) \rightarrow (Y, B)$, then

$$S_n(f) \simeq S_n(g): S_n(X, A) \rightarrow S_n(Y, B)$$

for all n . Hence $f \simeq g$ implies $H_n(f) = H_n(g)$.

Proof: Let h be a homotopy between f and g . We can write this as

$$f = h \circ j_1^X \text{ and } g = h \circ j_0^X.$$

Then the **previous lemma** yields

$$\begin{aligned} S_n(f) - S_n(g) &= S_n(h) \circ S_n(j_1^X) - S_n(h) \circ S_n(j_0^X) \\ &= S_n(h) \circ (\partial_{n+1} \circ P^n) + S_n(h) \circ (P^{n-1} \circ \partial_n) \\ &= \partial_{n+1} \circ (S_{n+1}(h) \circ P^n) + (S_n(h) \circ P^{n-1}) \circ \partial_n \text{ since } S_*(h) \text{ is a } \mathbf{chain map}. \end{aligned}$$

Thus the sequence of homomorphisms $S_{n+1}(h) \circ P^n$ is a **chain homotopy** between $S_n(f)$ and $S_n(g)$.

Applying the theorem about chain homotopic maps and their induced maps on homology implies the last statement. \square

We have already used homotopy invariance of singular homology at numerous occasions. Here is yet another one:

Proposition: Homology of weak retracts

Let $i: A \hookrightarrow X$ be a **weak retract**, i.e., assume there is continuous map $\rho: X \rightarrow A$ such that $\rho \circ i \simeq \text{id}_A$. Then

$$H_n(X) \cong H_n(A) \oplus H_n(X, A) \text{ for all } n.$$

Proof: Functoriality and homotopy invariance tell us that $\rho \circ i \simeq \text{id}_A$ implies

$$H_n(\rho) \circ H_n(i) = H_n(\rho \circ i) = H_n(\text{id}_A) = \text{id}_{H_n(A)} \text{ for all } n.$$

Hence $H_n(i)$ is **injective** for all n . That means that the sequence

$$0 \rightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X)$$

is **exact**. Since $H_{n-1}(i)$ is injective, the exactness of the sequence

$$\cdots \rightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(X) \rightarrow \cdots$$

implies that the connecting homomorphism $\partial: H_n(X, A) \rightarrow H_{n-1}(A)$ is the zero map. Thus we get a **short exact sequence**

$$(21) \quad 0 \rightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \rightarrow 0.$$

Since $H_n(\rho)$ is a **left-inverse** of $H_n(i)$, this sequence **splits**, i.e.,

$$H_n(X) \xrightarrow{\cong} H_n(A) \oplus H_n(X, A), \quad [c] \mapsto (H_n(\rho)([c]), H_n(j)([c])).$$

is an **isomorphism**.

We can describe an **inverse** of this map as

$$\begin{aligned} H_n(A) \oplus H_n(X, A) &\rightarrow H_n(X), \\ ([a], [b]) &\mapsto H_n(i)([a]) + [c'] - H_n(i \circ \rho)([c']) \end{aligned}$$

where $[c']$ is any class with $H_n(j)([c']) = [b]$.

We need to check that **the choice of $[c']$ does not matter**. So let $[c'']$ be another class with $H_n(j)([c'']) = [b]$. Then we have

$$H_n(j)([c'] - [c'']) = 0, \text{ i.e., } [c'] - [c''] \in \text{Ker}(H_n(j)).$$

By the exactness of (21), this implies that there exists a class $[\tilde{a}] \in H_n(A)$ with $H_n(i)([\tilde{a}]) = [c'] - [c'']$. Thus

$$\begin{aligned}
 & [c'] - H_n(i \circ \rho)([c']) - ([c''] - H_n(i \circ \rho)([c''])) \\
 &= [c'] - [c''] - H_n(i \circ \rho)([c'] - [c'']) \\
 &= H_n(i)([\tilde{a}]) - H_n(i \circ \rho)(H_n(i)([\tilde{a}])) \\
 &= H_n(i)([\tilde{a}]) - (H_n(i) \circ H_n(\rho) \circ H_n(i))([\tilde{a}]) \\
 &= H_n(i)([\tilde{a}]) - H_n(i)([\tilde{a}]) \text{ since } H_n(\rho) \circ H_n(i) = \text{id}_{H_n(A)} \\
 &= 0. \quad \square
 \end{aligned}$$

Homotopy invariance revisited

Let **hoTop** denote the **homotopy category** of **Top**, i.e., the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps:

$$\text{Mor}_{\mathbf{hoTop}}(X, Y) = \text{Map}_{\mathbf{Top}}(X, Y) / \simeq =: [X, Y].$$

where $\text{Map}_{\mathbf{Top}}(X, Y)$ denotes the set of continuous maps from X to Y . The result we have just shown implies that singular homology descends to a functor on **hoTop**:

$$\begin{array}{ccc}
 \mathbf{Top} & \xrightarrow{H_n} & \mathbf{Ab} \\
 \downarrow & \nearrow H_n & \\
 \mathbf{hoTop} & &
 \end{array}$$

This observation applies to almost all algebraic invariants in Topology. In other words, invariants in Algebraic Topology distinguish neither between homotopic maps nor between homotopy equivalent spaces.

However, many topological properties are not invariant under homotopy. For example, **compactness** is not invariant under homotopy. In other words, if X is **compact**, then it may well be the case that a space which is homotopy equivalent to X is not compact. To convince ourselves of this fact, it suffices to take $X = \{0\}$ and $Y = \mathbb{R}^n$. A bit more interesting is $X = S^n$ and $Y = \mathbb{R}^{n+1} \setminus \{0\}$. While X is **compact**, Y is **not**, but the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence.

This observation should make us aware of the scope of our abilities. The **tools** we develop in this class **are great**. But they are **not the end of the story...**

LECTURE 12

Locality and the Mayer-Vietoris sequence

We are going to discuss the Excision Axiom for singular homology and some consequences. Let us first recall what it says:

Excision Axiom of singular homology

Let (X, A) be a pair of spaces and let $Z \subset A$ be a subspace the closure of which is contained in the interior of A , in formulae $\bar{Z} \subseteq A^\circ$. Then the inclusion map $k: (X - Z, A - Z) \hookrightarrow (X, A)$ induces an **isomorphism**

$$H_n(k): H_n(X - Z, A - Z) \rightarrow H_n(X, A) \text{ for all } n.$$

We are going to deduce the excision property of homology from the following **locality principle**.

Let X be a topological space and let $\mathcal{A} = \{A_j\}_{j \in J}$ be a **cover** of X , i.e., a collection of subsets $A_j \subseteq X$ such that X is the **union of the interiors** of the A_j s.

\mathcal{A} -small chains

- An n -simplex $\sigma: \Delta^n \rightarrow X$ is called **\mathcal{A} -small** if the image of σ is contained in one of the A_j s.
- An n -chain $c = \sum_i n_i \sigma_i$ if X is called **\mathcal{A} -small** if, for every i , there is a A_j such that $\sigma_i(\Delta^n) \subset A_j$.
- We are going to denote the subgroup of \mathcal{A} -small n -chains by

$$S_n^{\mathcal{A}}(X) := \{c \in S_n(X) : c \text{ is } \mathcal{A}\text{-small}\}.$$

- For a subspace $A \subset X$, we write

$$S_n^{\mathcal{A}}(X, A) := \frac{S_n^{\mathcal{A}}(X)}{S_n^{\mathcal{A}}(A)}.$$

If, for each j , $\iota_j: A_j \hookrightarrow X$ denotes the inclusion map, then we can describe $S_n^{\mathcal{A}}(X)$ also as

$$S_n^{\mathcal{A}}(X) = \text{Im} \left(\bigoplus_{j \in J} S_n(A_j) \xrightarrow{\oplus_j S_n(\iota_j)} S_n(X) \right).$$

The point of \mathcal{A} -small chains is that we can use their chain complex to compute singular homology:

Locality Principle/Small Chain Theorem

For any cover \mathcal{A} of X , the inclusion of chain complexes

$$S_*^{\mathcal{A}}(X, \mathcal{A}) \subset S_*(X, \mathcal{A})$$

induces an **isomorphism in homology**.

The **proof** of this theorem takes quite an effort and we will postpone it for a moment. Instead we will now explain how the excision property follows from the theorem.

• Proof of the Excision Axiom using small chains:

Since $\bar{Z} \subseteq A^\circ$, we have $(X - Z)^\circ \cup A^\circ = X$. Thus, if we set $B := X - Z$, $\mathcal{A} = \{A, B\}$ is a cover of X .

Moreover, we can rewrite

$$(X - Z, A - Z) = (B, A \cap B).$$

Hence our **goal** is to show that

$$S_*(B, A \cap B) \rightarrow S_*(X, \mathcal{A})$$

induces an **isomorphism in homology**.

The inclusion of chain complexes $S_*^{\mathcal{A}}(X) \subset S_*(X)$ induces a morphism of short exact sequences of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*^{\mathcal{A}}(X) & \longrightarrow & S_*^{\mathcal{A}}(X)/S_*(A) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X)/S_*(A) \longrightarrow 0. \end{array}$$

The **middle vertical map** induces an isomorphism in homology by the **Small Chain Theorem**. The induced long exact sequences in homology and

the **Five-Lemma** imply that the **right-hand vertical map** induces an isomorphism in homology as well. Thus we are **reduced to compare** $S_*(B, A \cap B)$ and $S_*^A(X)/S_*(A)$.

Now we observe

$$S_*^A(X) = S_*(A) + S_*(B) \subset S_*(X)$$

and hence

$$\frac{S_*(B)}{S_*(A \cap B)} = \frac{S_*(B)}{S_*(A) \cap S_*(B)} \xrightarrow{\cong} \frac{S_*(A) + S_*(B)}{S_*(A)} = \frac{S_*^A(X)}{S_*(A)}$$

where the middle isomorphism follows from the **general comparison** of quotients of sums and intersections of abelian groups.

Thus the chain map

$$S_*(B, A \cap B) \rightarrow S_*^A(X)/S_*(A)$$

induces an isomorphism in homology and the excision axiom holds. **QED**

• The Mayer-Vietoris sequence

The above proof inspires us to look at the following situation which will lead to an important computational tool.

Assume that $\mathcal{A} = \{A, B\}$ is a **cover** of X . Consider the diagram

$$\begin{array}{ccc} A \cap B & \xrightarrow{j_A} & A \\ j_B \downarrow & & \downarrow i_A \\ B & \xrightarrow{i_B} & X. \end{array}$$

For every n , these maps induce homomorphisms in homology

$$\alpha_n: H_n(A \cap B) \rightarrow H_n(A) \oplus H_n(B), \alpha_n = \begin{bmatrix} H_n(j_A) \\ -H_n(j_B) \end{bmatrix}$$

$$x \mapsto (H_n(j_A)(x), -H_n(j_B)(x))$$

and

$$\beta_n: H_n(A) \oplus H_n(B) \rightarrow H_n(X), \beta_n = \begin{bmatrix} H_n(i_A) & H_n(i_B) \end{bmatrix}$$

$$(a, b) \mapsto H_n(i_A)(a) + H_n(i_B)(b).$$

Theorem: Mayer-Vietoris sequence

For any cover $\mathcal{A} = \{A, B\}$ of X , there are natural homomorphisms

$$\partial_n^{MV} : H_n(X) \rightarrow H_{n-1}(A \cap B) \text{ for all } n$$

which fit into an **exact sequence**

$$\begin{array}{ccccccc} & & \cdots & \xrightarrow{\beta_{n+1}} & H_{n+1}(X) & & \\ & & \searrow \partial_{n+1}^{MV} & & \swarrow & & \\ H_n(A \cap B) & \xleftarrow{\alpha_n} & H_n(A) \oplus H_n(B) & \xrightarrow{\beta_n} & H_n(X) & & \\ & & \swarrow \partial_n^{MV} & & \searrow & & \\ H_{n-1}(A \cap B) & \xleftarrow{\alpha_{n-1}} & \cdots & & & & \end{array}$$

Proof: From the proof of the Excision Axiom we remember that there is a short exact sequence of chain complexes

$$0 \rightarrow S_*(A \cap B) \xrightarrow{\begin{bmatrix} S_*(j_A) \\ -S_*(j_B) \end{bmatrix}} S_*(A) \oplus S_*(B) \xrightarrow{\begin{bmatrix} S_*(i_A) & S_*(i_B) \end{bmatrix}} S_*^A(X) \rightarrow 0.$$

Note that the exactness at the right-hand term was part of the proof of the Excision Axiom and the exactness at the middle term can be easily checked by looking long enough at the **commutative diagram**

$$\begin{array}{ccccc} \Delta^n & & & & \\ & \searrow & & \searrow & \\ & A \cap B & \xrightarrow{j_A} & A & \\ & \downarrow j_B & & \downarrow i_A & \\ & B & \xrightarrow{i_B} & X & \end{array}$$

It induces a long exact sequence in homology

$$\begin{array}{ccccc}
 & & \cdots & \xrightarrow{\beta_{n+1}^A} & H_{n+1}^A(X) \\
 & & \nearrow \partial_{n+1}^A & & \\
 H_n(A \cap B) & \xleftarrow{\alpha_n} & H_n(A) \oplus H_n(B) & \xrightarrow{\beta_n^A} & H_n^A(X) \\
 & \nwarrow \partial_n^A & & & \\
 H_{n-1}(A \cap B) & \xleftarrow{\alpha_{n-1}} & \cdots & &
 \end{array}$$

By definition of small chains, the homomorphism β_n factors through small chains, in other words, it is induced by the composition

$$S_n(A) \oplus S_n(B) \xrightarrow{[S_n(i_A) \quad S_n(i_B)]} S_n^A(X) \hookrightarrow S_n(X).$$

Thus we can apply the inverse of the isomorphism of the **Small Chain Theorem** and define ∂_n^{MV} to be

$$\partial_n^{MV} : H_n(X) \xrightarrow{\cong} H_n^A(X) \xrightarrow{\partial_n^A} H_{n-1}(A \cap B).$$

Then the following sequence

$$\begin{array}{ccccc}
 H_n(A) \oplus H_n(B) & \xrightarrow{\beta_n} & H_n(X) & \xrightarrow{\partial_n^{MV}} & H_{n-1}(A \cap B) \\
 & \searrow \beta_n^A & \uparrow \cong & \nearrow \partial_n^A & \\
 & & H_n^A(X) & &
 \end{array}$$

is exact at $H_n(X)$, since the triangles commute.

This yields the sequence of homomorphisms and the desired long exact sequence. **QED**

The MVS is an extremely useful tool

The Mayer-Vietoris sequence (MVS) is an important computational tool. Its power relies on the simple idea: If you want to understand a big space, split it up into smaller spaces you understand and then put the information back together.

The MVS tells us how the homology of X is built out of homologies of the cover by A and B .

Let us apply this new insight to some **concrete examples**:

• Let us calculate the homology of $X = S^1$ yet another time. Let $x = (0,1)$ and $y = (0, -1)$ on S^1 . We set $A = S^1 - \{x\}$ and $B = S^1 - \{y\}$. Then A and B are **two open subsets** which cover S^1 . We observe that both A and B are contractible.

The intersection $A \cap B$ contains the points $p = (-1,0)$ and $q = (1,0)$. In fact, the inclusion

$$\{p, q\} \hookrightarrow A \cap B$$

is a **deformation retract**.

Since $\mathcal{A} = \{A, B\}$ is a cover of S^1 , we can write down the corresponding MVS. For $n \geq 2$, all the homology groups hitting and being hit by $H_n(S^1)$ are zero, since $H_n(A) \oplus H_n(B) = H_n(\{x\}) \oplus H_n(\{y\}) = 0$ and $H_{n-1}(A \cap B) = H_{n-1}(\{p, q\}) = 0$. Thus

$$H_n(S^1) = 0 \text{ for all } n \geq 2.$$

Since S^1 is path-connected, we know $H_0(S^1) = \mathbb{Z}$. It remains to check $n = 1$.

The MVS for $n = 1$ looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(S^1) & \longrightarrow & H_0(A \cap B) & \longrightarrow & H_0(A) \oplus H_0(B) \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & H_1(S^1) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

where we obtain the lower right-hand map by observing that all summands are of the form $H_0(\text{pt})$ and hence each generator in $H_0(A \cap B)$ is sent to $(1, -1)$ by $[H_0(j_A), -H_0(j_B)]$. Thus $H_1(S^1)$ is the kernel of this map:

$$H_1(S^1) \cong \text{Ker} \left(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right) = \{(x, -x) \in \mathbb{Z} \oplus \mathbb{Z}\} \cong \mathbb{Z}.$$

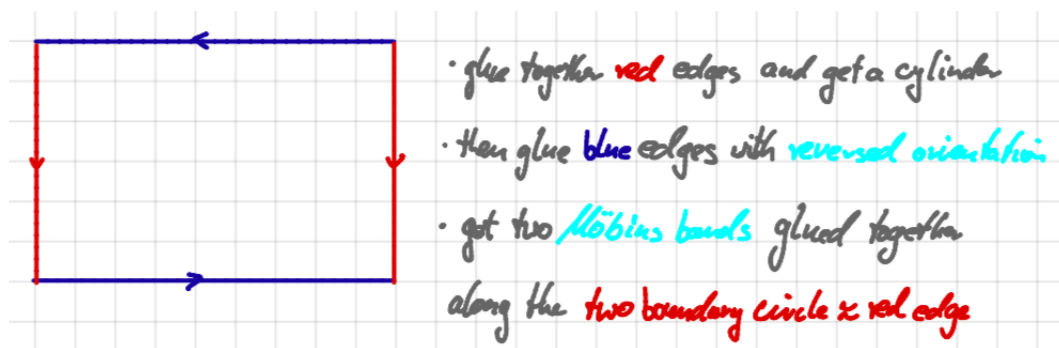
• For $n \geq 2$, let $A = S^n - \{S\}$ and $B = S^n - \{N\}$ where N and S are the **north- and south-pole** of S^n , respectively. We observe that both A and B are **contractible**. Moreover, the inclusion of $j: S^{n-1} \hookrightarrow A \cap B$ as the equator is a strong deformation retract. In particular, j is a homotopy equivalence.

Together with the inverse of the isomorphism $H_{q-1}(j)$, we get

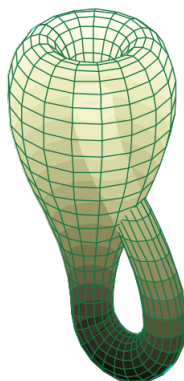
$$H_{q-1}(j)^{-1} \circ \partial_q^{MV}: H_q(S^n) \xrightarrow{\cong} H_{q-1}(S^{n-1})$$

is an isomorphism for all $q \geq 2$. Since we know $H_q(S^1)$ for all q , this yields $H_q(S^n)$ by induction.

• Let K be the **Klein bottle** which can be constructed from a square by gluing the edges as indicated in the following picture:



The outcome of this procedure is the **twisted surface** whose 3-dimensional shadow we see in the next picture which is taken from wikipedia.org:



(Note that we should really think of K as an object in \mathbb{R}^4 where it does not self-intersect.)

We observe that K can be constructed by taking **two Möbius bands** A and B and gluing them together by a homeomorphism between their boundary circles. Hence $K = A \cup B$ and $A \cap B \approx S^1$. In the exercises we are going to calculate the homology of the Möbius strip. It is given by $H_0(M) = H_1(M) = \mathbb{Z}$ and $H_2(M) = 0$.

We would like to use this information to calculate the homology of K .

Since K is **path-connected** as a quotient of a path-connected space, we know $H_0(K) = \mathbb{Z}$.

Now we apply the **MVS**: We observe that $H_n(A)$, $H_n(B)$ and $H_n(A \cap B)$ vanish for $n \geq 2$. Hence $H_n(K) = 0$ for all $n \geq 3$.

The remaining MVS looks like this:

$$0 \rightarrow H_2(K) \xrightarrow{\partial^{MV}} H_1(A \cap B) \xrightarrow{\varphi_1} H_1(A) \oplus H_1(B) \rightarrow H_1(K) \rightarrow 0.$$

The 0 on the right-hand side is justified by the fact that

$$H_0(A \cap B) \cong \mathbb{Z} \xrightarrow{\varphi_0} \mathbb{Z} \oplus \mathbb{Z} \cong H_0(A) \oplus H_0(B)$$

is **injective**.

The map φ_1 is given by

$$\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z} \oplus \mathbb{Z}, 1 \mapsto (2, -2),$$

since

$$H_1(A \cap B) = H_1(S^1) \rightarrow H_1(M) = H_1(A)$$

wraps the circle around the boundary of M **twice**, and

$$H_1(A \cap B) = H_1(S^1) \rightarrow H_1(M) = H_1(B)$$

does that too, but with **reversed orientation**. (We will understand this fact better after we have done the **exercises**.) Hence on the second factor we use the map $z \mapsto z^{-2}$ to produce a Möbius band.

In particular, φ_1 is **injective** and hence

$$\mathbf{H}_2(\mathbf{K}) = \mathbf{0}.$$

Moreover, $H_1(K)$ is the **cokernel** of φ_1 . If we choose the **basis**

$$\{b_1 := (1, 0), b_2 := (1, -1)\} \text{ for } \mathbb{Z} \oplus \mathbb{Z},$$

then we see that φ_1 maps $1 \in \mathbb{Z}$ to $2b_1$ in $\mathbb{Z}b_1 \oplus \mathbb{Z}b_2$. Hence the cokernel of φ_1 is isomorphic to $\mathbb{Z}b_1 \oplus \mathbb{Z}b_2/2b_1$. Thus

$$\mathbf{H}_1(\mathbf{K}) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

LECTURE 13

Cell complexes

We return to an important type of topological spaces, called CW- or cell complexes, that is particularly convenient for our purposes in many respects. It will turn out that this type of spaces both appears very frequently and is quite accessible for calculations. In particular, we will learn next week that the homology of a cell complex is quite easy to compute.

The idea of creating a cell complex is to **successively glue cells** to what has already been built. The **general procedure** for doing this is the following:

Gluing a space along a map

Suppose we have a space X and a pair (B, A) of spaces. We **define a space** $X \cup_f B$, often also denoted $X \cup_A B$ if the map f is either understood or just the inclusion, which fits into the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{\varphi} & X \cup_f B \end{array}$$

by

$$X \cup_f B := (X \sqcup B) / (a \sim f(a) \text{ for all } a \in A).$$

We say that $X \cup_f B$ arises from **attaching B to X along f** , or along A , and f is called an **attaching map**.

By its construction, there are two types of equivalence classes in $X \cup_f B$:

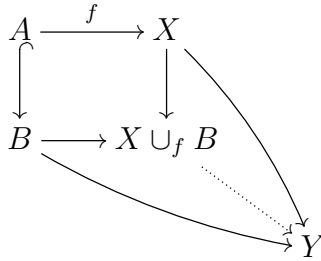
- classes which consist of single points of $B - A$,
- classes which consist of sets $\{x\} \sqcup f^{-1}(x)$ for any point $x \in X$.

Note that the **lower horizontal map** $\varphi: B \rightarrow B \cup_f X$ arises as **part of the construction**. It is given by

$$\varphi: B \rightarrow B \cup_f X, b \mapsto \begin{cases} b & \text{if } b \in B - A \\ [b] & \text{if } b \in A. \end{cases}$$

In particular, this shows that $\varphi|_{B-A}$ is a **homeomorphism**.

The topology of $X \cup_f B$ is the **quotient topology** and is characterized by the **universal property**: whenever there is a diagram of solid arrows of the form



then **there is a unique dotted arrow** which makes all triangles commute. We can reformulate this fact by saying that $X \cup_f B$ is the **pushout** of the solid diagram.

For **example**:

- if $X = *$ consists of just a point, then

$$X \cup_f B = * \cup_f B = B/A;$$

- if $A = \emptyset$, then $X \cup_f B = X \sqcup B$ is just a disjoint union.

A more important example is the following:

Attaching a cell

We consider the pair (D^n, S^{n-1}) of an n -disk and its boundary. We are going to think of D^n as an **n -cell**.

Suppose we are given a map $f: S^{n-1} \rightarrow X$. Then we can **attach an n -cell** to X via f as

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X \cup_f D^n. \end{array}$$

We could speed up this process by attaching several cells at once:

$$\begin{array}{ccc} \coprod_{\alpha \in J} S_{\alpha}^{n-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in J} D_{\alpha}^n & \longrightarrow & X \cup_f \coprod_{\alpha \in J} D_{\alpha}^n. \end{array}$$

Let us look at some **examples**:

- Let us start with $n = 0$ and write (D^0, S^{-1}) for $(*, \emptyset)$. Attaching 0-cells to a space X just means **adding a set of discrete points** to X :

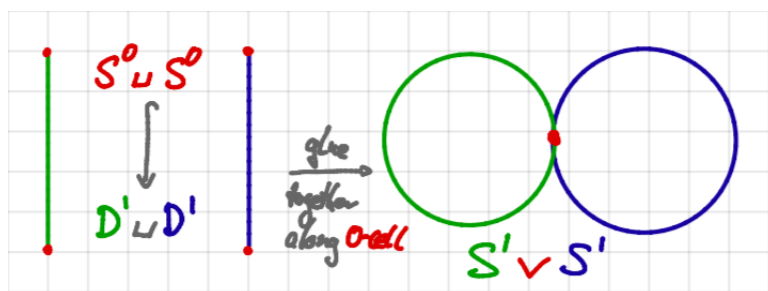
$$X \cup_f \coprod_{\alpha \in J} D_{\alpha}^0 = X \sqcup J$$

where J is a set with the discrete topology.

- Now let us attach two 1-cells to a point $X = *$:

$$\begin{array}{ccc} S^0 \sqcup S^0 & \xrightarrow{f} & * \\ \downarrow & & \downarrow \\ D^1 \sqcup D^1 & \longrightarrow & * \cup_f (D^1 \sqcup D^1). \end{array}$$

Since there is only one choice for f , we get a **figure eight**: we start with two 1-disks D^1 and then we identify all four boundary points with the 0-cell. We **denote this space by** $S^1 \vee S^1$.

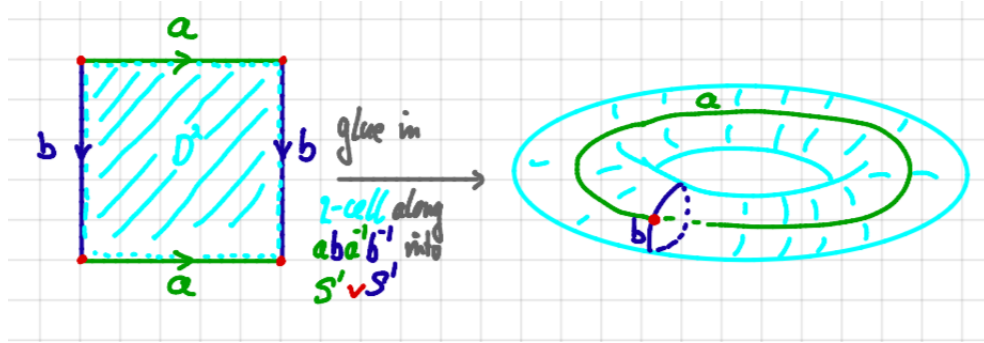


- We continue with this space and attach one 2-cell: We can think of $S^1 \vee S^1$ as an empty square where we glue together the horizontal edges and the vertical edges. Then we glue in a 2-cell into the square by attaching its

boundary to the edges a , b , a^{-1} , and b^{-1} , i.e., by walking clockwise:

$$\begin{array}{ccc} S^1 & \xrightarrow{f=aba^{-1}b^{-1}} & S^1 \vee S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & (S^1 \vee S^1) \cup_f D^2 = T^2. \end{array}$$

The result of this procedure is a two-dimensional **torus**.



This example motivates the following key concept:

Cell complex

A **cell complex**, or CW-complex, is a space X equipped with a sequence of subspaces

$$\emptyset = \text{Sk}_{-1}X \subseteq \text{Sk}_0X \subseteq \text{Sk}_1X \subseteq \text{Sk}_2X \subseteq \cdots X$$

such that

- X is the union of the Sk_nX s,
- for all n , Sk_nX arises from Sk_{n-1} by **attaching n -cells**, i.e., there is a **pushout diagram**

$$\begin{array}{ccc} \coprod_{\alpha \in J_n} S_{\alpha}^{n-1} & \xrightarrow{f_n} & \text{Sk}_{n-1}X \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in J_n} D_{\alpha}^n & \xrightarrow{\varphi_n} & \text{Sk}_nX. \end{array}$$

The space Sk_nX is called the **n -skeleton** of X .

In our **example of the torus** T^2 the skeleta are

$$\text{Sk}_0T^2 = *, \text{Sk}_1T^2 = S^1 \vee S^1, \text{Sk}_2T^2 = T^2.$$

Before we study more examples, we fix more terminology and list some facts which should help clarify the picture:

- The **topology** of a cell complex is determined by its skeleta, i.e., a subset $U \subset X$ is **open** (closed) if and only if $U \cap \text{Sk}_n X$ is open (closed) for all n .
- In fact, the topology on X is determined by its cells, i.e., U is **open** (**closed**) in X if and only if its intersection with each cell is open (closed), or equivalently, if $\varphi_\alpha^{-1}(U)$ is open (closed) in each D_α^n . This topology is called the **weak topology** and explains the **W** in *CW-complex*.
- That implies that a map $g: X \rightarrow Y$ is continuous if and only if its restriction to each skeleton is continuous, or equivalently, if and only if

$$g \circ \varphi_\alpha: D_\alpha^n \rightarrow Y \text{ is continuous for all } D_\alpha^n.$$

- For any n -cell D_α^n , the induced map $\varphi_\alpha: D_\alpha^n \rightarrow X$ is called the **characteristic map** of the cell. As we explained before, the restriction to the **open interior** $(D_\alpha^n)^\circ = D_\alpha^n - S_\alpha^{n-1}$

$$(\varphi_\alpha)|_{(D_\alpha^n)^\circ} \rightarrow X$$

is a **homeomorphism onto its image**.

- We will call the image of D_α^n under φ_α in X a **closed n -cell** of X . We will refer to n as the **dimension of the cell**. Since D^n is compact, it is a compact subset.
- The image of the **interior** $(D_\alpha^n)^\circ$ of D_α^n in X is often called an **n -cell** or **open n -cell** of X and will be denoted by e_α^n . Note that this subset is **not** necessarily an open subset of X .
- The **C** in *CW-complex* stands for **closure finite** which means that, for every cell, $\varphi_\alpha(S_\alpha^{n-1})$ is contained in **finitely** many cells (of dimension at most $n - 1$).
- A cell complex X is called **finite-dimensional** if there is an n such that $X = \text{Sk}_n X$. The smallest such n is called the **dimension of X** , i.e., the unique n such that $\text{Sk}_n X = X$ and $\text{Sk}_{n-1} X \subsetneq X$.
- A cell complex is called **of finite type** if each indexing set J_n is finite, i.e., if only finitely many cells are attached in each step.

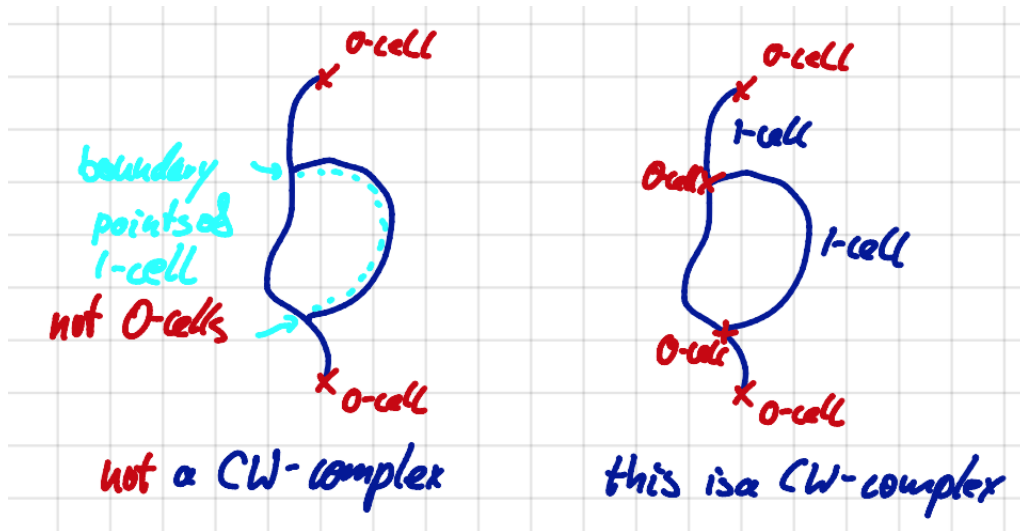
- A cell complex is called **finite** if it is finite-dimensional and of finite type, i.e., if it has only finitely many cells.
- The dimension of a cell complex is a topological invariant, i.e., it is invariant under homeomorphisms. Moreover, every cell complex is **Hausdorff**.
- However, a cell complex is **compact** if and only if it is **finite**.
- Note that every nonempty cell complex must have at least one 0-cell.
- The cell structure of a cell complex is in general not unique. Often there are many different cell structures. We will observe this for example for the n -sphere.

Here is an important theorem which demonstrates the wide range and importance of cell complexes:

Compact smooth manifolds are cell complexes

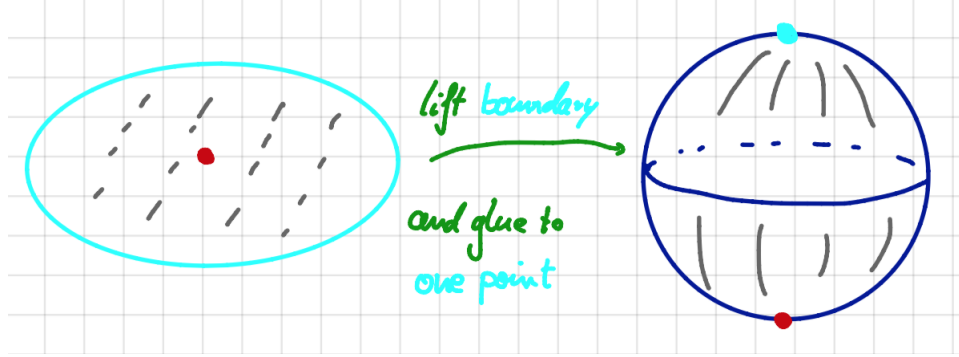
Every compact smooth manifold can be given the structure of a cell complex.

Here some important examples:



- A simple **example** is given by surfaces of a three-dimensional **cube**: it has eight 0-cells, twelve 1-cells, six 2-cells.

- Similarly, every **n -simplex** is a cell complex. For example, Δ^3 has four 0-cells, six 1-cells, four 2-cells, and one 3-cell.
- The **sphere** S^n is a cell complex with just two cells: **one 0-cell** e^0 (that is a point) and **one n -cell** which is attached to e^0 via the constant map $S^{n-1} \rightarrow e^0$. Geometrically, this corresponds to expressing S^n as $D^n / \partial D^n$: we take the open n -disk $D^n \setminus \partial D^n$ and collapse the boundary ∂D^n to a single point which is, say, the north pole $N = e^0$.



- The n -sphere $X = S^n$ can also be equipped with a **different cell structure**:

We start with two 0-cells which give us the 0-skeleton

$$\text{Sk}_0 X \xrightarrow{\cong} S^0.$$

Now we attach two 1-cells via the homeomorphism $f: S^0 \xrightarrow{\cong} \text{Sk}_0 X$. This gives us one 1-cell as the **upper half-circle** and one 1-cell as the **lower half-circle** and

$$S^1 \xrightarrow{\cong} \text{Sk}_1 X.$$

Then we attach two 2-cells as the **upper and lower hemisphere** along the map $S^1 \xrightarrow{\cong} \text{Sk}_1$, i.e., this gives us

$$\text{Sk}_2 X \xrightarrow{\cong} S^2$$

with $\text{Sk}_1 X \approx S^1$ as the equator of S^2 . Now we continue this procedure until we reach S^n .

Hence, in this cell structure on S^n , there are **exactly two k -cells in each dimension** $k = 0, \dots, n$.

- **Real projective space** \mathbb{RP}^n is a cell complex with **one cell in each dimension** up to n . To show this we proceed inductively. We know that \mathbb{RP}^0 consists of a single point, since it is S^0 whose two antipodal points are identified.

Now we would like to understand how \mathbb{RP}^n can be constructed from \mathbb{RP}^{n-1} :

We embed D^n as the **upper hemisphere** into S^n , i.e., we consider D^n as $\{(x_0, \dots, x_n) \in S^n : x_0 \geq 0\}$. Then

$$\mathbb{RP}^n = S^n / (x \sim -x) = D^n / (x \sim -x \text{ for } \text{boundary points } x \in \partial D^n).$$

But ∂D^n is just S^{n-1} . Hence the quotient map

$$S^{n-1} \rightarrow S^{n-1} / \sim = \mathbb{RP}^{n-1}$$

attaches an n -cell e^n , the open interior of D^n , at \mathbb{RP}^{n-1} .

Thus we obtain \mathbb{RP}^n from \mathbb{RP}^{n-1} by attaching one n -cell via the quotient map $S^{n-1} \rightarrow \mathbb{RP}^{n-1}$.

Summarizing, we have shown that \mathbb{RP}^n is a cell complex with one cell in each dimension from 0 to n .

- We can continue this process and build the **infinite projective space** $\mathbb{RP}^\infty := \bigcup_n \mathbb{RP}^n$. It is a cell complex with one cell in each dimension. We can think of \mathbb{RP}^∞ as the space of lines in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

- **Complex projective space** \mathbb{CP}^n is a cell complex.

Let $(z_0 : \dots : z_n)$ denote the homogeneous coordinates of a point in \mathbb{CP}^n . Let $\varphi : D^{2n} \rightarrow \mathbb{CP}^n$ be given by

$$(z_0, \dots, z_{n-1}) \mapsto (z_0 : z_1 : \dots : z_{n-1} : 1 - (\sum_{i=0}^{n-1} |z_i|^2)^{1/2}).$$

Then φ sends ∂D^{2n} to the points with $z_n = 0$, i.e., into \mathbb{CP}^{n-1} .

Let f denote the restriction of φ to $S^{2n-1} = \partial D^{2n}$. Then φ **factors** through $D^{2n} \cup_f \mathbb{CP}^{n-1}$, i.e., we get a commutative diagram with an induced dotted arrow

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{f} & \mathbb{CP}^{n-1} \\ \downarrow & & \downarrow \\ D^{2n} & \longrightarrow & \mathbb{CP}^{n-1} \cup_f D^{2n} \\ & \searrow \varphi & \downarrow g \\ & & \mathbb{CP}^n. \end{array}$$

The induced map

$$g : D^{2n} \cup_f \mathbb{CP}^{n-1} \rightarrow \mathbb{CP}^n$$

Since we can rescale the n th coordinate, this map is **bijjective**. Hence it is a continuous bijection defined on a **compact** space. We learned earlier that this implies that g is a **homeomorphism**.

We conclude that \mathbb{CP}^n is a cell complex with exactly **one i -cell in each even dimension** up to $2n$.

- Again we could continue this process and build **infinite complex projective space** \mathbb{CP}^∞ which is a cell complex with one i -cell in each even dimension.

Finally, we would like to have a **good notion of subspace** in a cell complex which respects the cell structure. It turns out that it is not sufficient to just require to have a subspace. Though not much more is actually required. For, a subspace $A \subset X$ is **subcomplex**, or sub-CW-complex, if it is **closed** and a **union of cells** of X .

These conditions imply that A is a cell complex on its own. For, since A is closed the characteristic maps of each cell of A has image in A and so does each attaching map. Hence the cells with their characteristic maps which lie in A provide A with a cell structure.

A more technical definition sounds like this:

Subcomplexes

Let X be a cell complex with attaching maps $\{f_\alpha: S_\alpha^{n-1} \rightarrow \text{Sk}_{n-1}X : \alpha \in J_n, n \geq 0\}$.

A **subcomplex** A of X is a closed subspace $A \subseteq X$ such that for all $n \geq 0$, there is a **subset** $J'_n \subset J_n$ so that $\text{Sk}_n A := A \cap \text{Sk}_n X$ turns A into a cell complex with attaching maps $\{f_\beta: \beta \in J'_n, n \geq 0\}$.

A pair (X, A) which consists of a CW-complex X and a **subcomplex** A is called a **CW-pair**.

Examples of CW-pairs are given by

- each skeleton $\text{Sk}_n X$ of a cell complex X ;
- $\mathbb{RP}^k \subset \mathbb{RP}^n$ for every $k \leq n$;
- $\mathbb{CP}^k \subset \mathbb{CP}^n$ for every $k \leq n$;

- the spheres $S^k \subset S^n$ for every $k \leq n$ **but only** for the **second cell structure** with two i -cells in each dimension.

With the **first cell structure on S^n** with one 0-cell and one n -cell, S^k is **not a subcomplex** of S^n .

The next step is to study the homology of cell complexes...

LECTURE 14

Homology of cell complexes

We are going to show that there is a relatively simple procedure to determine the homology of a cell complex.

Before we start this endeavour we need an auxiliary result which is a consequence of the excision property of singular homology:

Lemma: Homology after collapsing a subspace

Let $A \subset X$ be a subspace. Suppose there is another subspace B of X such that

- (a) $\bar{A} \subseteq B^\circ$ and
- (b) $A \hookrightarrow B$ is a **deformation retract**.

Then

$$H_n(X, A) \xrightarrow{\cong} H_n(X/A, *)$$

is an **isomorphism** for all n .

Proof: We have a **commutative diagram**

$$\begin{array}{ccccc} (X, A) & \xrightarrow{i} & (X, B) & \xleftarrow{j} & (X - A, B - A) \\ \downarrow & & \downarrow & & \downarrow k \\ (X/A, *) & \xrightarrow{\bar{i}} & (X/A, B/A) & \xleftarrow{\bar{j}} & (X/A - *, B/A - *) \end{array}$$

Our goal is to show that the left-hand vertical map induces an isomorphism in homology. We will achieve this by showing that all the other maps **induce isomorphisms in homology**:

- The **map** k is a **homeomorphism of pairs** and hence induces an isomorphism in homology.
- The **map** j induces an isomorphism in homology by the assumption (a) and **excision**.

- The **map** i induces a homomorphism of long exact sequences

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) \longrightarrow \cdots \\
 & & \downarrow \cong & & \parallel & & \downarrow \\
 \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, B) \longrightarrow \cdots
 \end{array}$$

By **assumption (b)**, the left-hand vertical arrow is an isomorphism for all n . By the **Five-Lemma** this implies that i induces an isomorphism in homology.

- For the **map** \bar{i} , we observe that the retraction $\rho: B \rightarrow A \hookrightarrow B$ induces a map $\bar{\rho}: B/A \rightarrow A/A = * \hookrightarrow B/A$.

Moreover, the homotopy $B \times I \rightarrow B$ between ρ and the identity of B is constant on A . Thus it induces a homotopy $B/A \times I \rightarrow B/A$ between $\bar{\rho}$ and the identity of B/A .

In other words, $*$ $\rightarrow B/A$ is a **deformation retract**. Hence the long exact sequence and the Five-Lemma imply that \bar{i} induces an isomorphism in homology.

- Finally, we have $\bar{*} \subset (B/A)^\circ$ by definition of the quotient topology. Hence **map** \bar{j} induces an isomorphism in homology by **excision**.

QED

Corollary: Homology of a bouquet of spheres

For any indexing set J , let us write $\bigvee_{\alpha \in J} S_\alpha^k$ for the quotient

$$\coprod_{\alpha \in J} S_\alpha^{k-1} \hookrightarrow \coprod_{\alpha \in J} D_\alpha^k \rightarrow \bigvee_{\alpha \in J} S_\alpha^k.$$

The homology of this space, often called bouquet of k -spheres, is given by

$$H_q\left(\bigvee_{\alpha \in J} S_\alpha^{k,*}\right) \cong \begin{cases} \mathbb{Z}[J] & \text{if } q = k \\ 0 & \text{if } q \neq k \end{cases}$$

where $\mathbb{Z}[J]$ denotes the free abelian group on the set J .

(Note that the relative homology group in the statement is an example of a reduced homology that we introduced in last week's exercises.)

Proof: Each summand S_α^{k-1} is a subspace of D_α^k for which there is an open neighborhood U_α such that $S_\alpha^{k-1} \hookrightarrow U_\alpha$ is a deformation retract (we could even

take $U_\alpha = D_\alpha^n - \{0\}$. Hence we can **apply the previous result** to conclude

$$H_*(\coprod_{\alpha} D_{\alpha}^k, \coprod_{\alpha} S_{\alpha}^{k-1}) \xrightarrow{\cong} H_*(\bigvee_{\alpha} S_{\alpha}^k, *).$$

Hence we are reduced to calculate the relative homology on the left-hand side.

To do this, we can apply the **long exact sequence of a pair** to deduce that

$$\partial: H_q(\coprod_{\alpha} D_{\alpha}^k, \coprod_{\alpha} S_{\alpha}^{k-1}) \xrightarrow{\cong} H_{q-1}(\coprod_{\alpha} S_{\alpha}^{k-1}, *)$$

is an **isomorphism for all q** . Finally, we know that the latter group is isomorphic to $\bigoplus_{\alpha \in J} \mathbb{Z} = \mathbb{Z}[J]$ when $q = k$ and 0 otherwise. \square

Now we would like to apply this observation to a cell complex X . If we write $X_k = \text{Sk}_k X$ for the **k -skeleton** of X , then we get the following commutative diagram

$$(22) \quad \begin{array}{ccccc} \coprod_{\alpha} S_{\alpha}^{k-1} & \hookrightarrow & \coprod_{\alpha} D_{\alpha}^k & \longrightarrow & \bigvee_{\alpha} S_{\alpha}^k \\ \downarrow f & & \downarrow \varphi & & \downarrow \bar{\varphi} \\ X_{k-1} & \hookrightarrow & X_k = X_{k-1} \cup_f (\coprod_{\alpha} D_{\alpha}^k) & \longrightarrow & X_k / X_{k-1}. \end{array}$$

where the right-hand vertical map is induced by φ and taking quotients. Since the **restriction** of φ to the **open interior** of the n -disks is a **homeomorphism** onto its image, this implies that the **dotted arrow $\bar{\varphi}$ is a homeomorphism**.

Hence we deduce from the previous result on bouquets of spheres:

$$H_q(X_k, X_{k-1}) \cong H_q(X_k / X_{k-1}, *) \cong \begin{cases} \mathbb{Z}[J_n] & \text{if } q = k \\ 0 & \text{if } q \neq k \end{cases}$$

where J_n denotes the indexing set of the attached k -cells.

In other words, the relative homology group **$H_k(X_k, X_{k-1})$ keeps track of the k -cells** of X .

This group will play a crucial role for us today. Let us analyse some consequences of what we have found out about this group.

Let us look at a piece of the long exact sequence of the pair (X_k, X_{k-1}) :

$$H_{q+1}(X_k, X_{k-1}) \rightarrow H_q(X_{k-1}) \rightarrow H_q(X_k) \rightarrow H_q(X_k, X_{k-1}).$$

For $q \neq k$, the last term $H_q(X_k, X_{k-1}) = 0$ vanishes and hence the map

$$H_q(X_{k-1}) \rightarrow H_q(X_k) \text{ is surjective.}$$

For $q \neq k-1$, the first term $H_{q+1}(X_k, X_{k-1}) = 0$ vanishes and hence the map

$$H_q(X_{k-1}) \rightarrow H_q(X_k) \text{ is injective.}$$

Hence we have shown that the inclusion $X_{k-1} \hookrightarrow X_k$ induces an **isomorphism**

$$(23) \quad H_q(X_{k-1}) \xrightarrow{\cong} H_q(X_k) \text{ for } q \neq k, k-1.$$

Hence, **for a fixed** $q > 0$, we can observe how $H_q(X_k)$ varies when we **let** X_k **go through all skeleta** of X :

- $H_q(X_0) = 0$ since X_0 is a **discrete set** and the higher homology groups of points vanish.
- For $k = 0, \dots, q-1$, $H_q(X_k) = 0$ remains trivial by (23).
- As a consequence, we observe that $H_n(X_k) = 0$ whenever $n > k$.
- **For** $k = q$, $H_q(X_q)$ is a subgroup of the free abelian group $H_q(X_q, X_{q-1})$, and therefore it is **free abelian** as well.
- **For** $k = q+1$, $H_q(X_{q+1})$ may not be free anymore, i.e., there might be **relations** induced by the exact sequence

$$H_{q+1}(X_{q+1}, X_q) \rightarrow H_q(X_q) \rightarrow H_q(X_{q+1}) \rightarrow 0.$$

- **For** $k \geq q+1$, $H_q(X_k)$ **remains stable**, i.e., the inclusions of skeleta induce a sequence of isomorphisms

$$H_q(X_{q+1}) \xrightarrow{\cong} H_q(X_{q+2}) \xrightarrow{\cong} \dots$$

- If X is **finite-dimensional**, there is a d such that $X = X_d$. The above sequence of isomorphisms then implies the inclusion $X_{q+1} \hookrightarrow X$ induces an **isomorphism**

$$H_q(X_k) \cong H_q(X) \text{ for } q < k.$$

- Still, for X **finite-dimensional**, since $H_q(X_{q+1}) \xrightarrow{\cong} H_q(X)$ and since

$$H_q(X_q) \rightarrow H_q(X_{q+1}) \rightarrow H_q(X_{q+1}, X_q) = 0$$

is **exact**, we see that

$$H_q(X_q) \rightarrow H_q(X) \text{ is } \mathbf{surjective}.$$

- If X is infinite-dimensional, the group $H_q(X_k)$ still maps **isomorphically into** $H_q(X)$ for $q < k$. For, the image of a standard simplex is **compact** and therefore lands in a **finite subcomplex**. Hence the union of the images of a finite collection of standard simplices is still **compact** and therefore also lands in a **finite subcomplex**. Hence it lands in a **finite skeleton**. Thus any q -chain in X is the image of a chain in a finite skeleton. For the same reason, if $c \in S_q(X)$ is a **boundary**, then it is a boundary in $S_q(X_m)$ for some $m \geq q$.
- In summary, all the q -dimensional homology of X is created in the q -skeleton X_q , and all the relations in $H_q(X)$ occur in the $q + 1$ -skeleton X_{q+1} .

The key points of this discussion are:

Proposition: The homology is governed by the skeleta

For any $k, q \geq 0$ and cell complex X , we have

- $H_q(X_k) = 0$ for $k < q$ and
- $H_q(X_k) \xrightarrow{\cong} H_q(X)$ for $k > q$.

In particular, $H_q(X) = 0$ if q is bigger than the dimension of the cell complex X .

Now we would like to find an efficient way to calculate the homology of our cell complex X . Apparently, the group $H_n(X_n, X_{n-1})$ carries crucial information about X . Therefore, we are going to give it a new name:

Cellular n -chains

The group of **cellular n -chains** in a cell complex X is defined to be

$$C_n(X) := H_n(X_n, X_{n-1}).$$

We claim that these groups sit inside a sequence of homomorphisms who form a **chain complex**. The differential

$$d_n: C_n(X) \rightarrow C_{n-1}(X)$$

is defined as the composite

$$\begin{array}{ccc}
 C_n(X) = H_n(X_n, X_{n-1}) & \xrightarrow{d_n} & H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}(X) \\
 & \searrow \partial_n & \nearrow j_{n-1} \\
 & H_{n-1}(X_{n-1}) &
 \end{array}$$

where ∂_n is the connecting homomorphism in the long exact sequence of pairs and j_{n-1} is the homomorphism induced by the inclusion $(X_{n-1}, \emptyset) \hookrightarrow (X_{n-1}, X_{n-2})$.

To show that $d_n \circ d_{n+1} = 0$ we consider the commutative diagram:

$$\begin{array}{ccccc}
 C_{n+1}(X) = H_{n+1}(X_{n+1}, X_n) & & & & H_{n-1}(X_{n-2}) = 0 \\
 \downarrow \partial_{n+1} & \searrow d_{n+1} & & & \downarrow \\
 H_n(X_n) & \xrightarrow{j_n} & C_n(X) = H_n(X_n, X_{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X_{n-1}) \\
 \downarrow & & \searrow d_n & & \downarrow j_{n-1} \\
 H_n(X_{n+1}) & & & & C_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) \\
 \downarrow & & & & \\
 0 = H_n(X_{n+1}, X_n) & & & &
 \end{array}$$

$=0$

Since j and ∂ are part of long exact sequences, we know $j \circ \partial = 0$ and get

$$d_n \circ d_{n+1} = (j_{n-1} \circ \partial_n) \circ (j_n \circ \partial_{n+1}) = 0.$$

Cellular chain complex

Thus $(C_*(X), d)$ is a chain complex. It is called the **cellular chain complex**.

Now we would like to determine the homology of this chain complex.

- To do this we need to understand the kernel of d :

$$\text{Ker}(d_n) = \text{Ker}(j_{n-1} \circ \partial_n).$$

Since j_{n-1} is **injective**, we get

$$\text{Ker}(d_n) = \text{Ker}(\partial_n) = \text{Im}(j_n) = H_n(X_n)$$

where the middle identity is implied by the **exactness of the long exact sequence** these maps are part of, and the last identity is implied by the fact that $j_n: H_n(X_n) \rightarrow H_n(X_n, X_{n-1})$ is **injective**.

- For the image of d , we use again that j_n is injective and get

$$\operatorname{Im}(d_{n+1}) = j_n(\operatorname{Im}(\partial_{n+1})) \cong \operatorname{Im}(\partial_{n+1}) \subseteq H_n(X_n).$$

Since the left-hand column in the above big diagram is exact, we know

$$H_n(X_n)/\operatorname{Im}(\partial_{n+1}) \cong H_n(X_{n+1}).$$

In other words, we just proved:

$$H_n(C_*(X)) = H_n(X_n)/\operatorname{Im}(\partial_{n+1}) \cong H_n(X_{n+1}).$$

But we had already showed $H_n(X_{n+1}) \cong H_n(X)$. Hence we proved the following important result:

Theorem: Cellular Homology

For a **cell complex** X , there is an **isomorphism**

$$H_*(C_*(X)) \cong H_*(X)$$

which is **functorial** with respect to **filtration-preserving maps** between cell complexes.

In this theorem we are referring to maps which preserve the skeleton structure of cell complexes. We should better make this concept precise:

Maps between cell complexes

- A **filtration** on a space X is a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq X_{n+1} \subseteq \dots \subseteq X.$$

such that X can be written as the union of these subspaces. If X is a space together with a filtration, we call X a **filtered space**.

- For example, every **cell complex** has a filtration by its **skeleta**.
- Let X and Y be filtered spaces. A continuous map $f: X \rightarrow Y$ is called **filtration-preserving** if $f(X_p) \subset Y_p$ for all p .
- A map between cell complexes is called **cellular** if it preserves the filtration by skeleta.

In other words, if we are given two cell complexes and care about their cell structure, we should only consider filtration-preserving maps between them.

An immediate and very useful consequence of the above theorem is:

Corollary: Homology of even cell complexes

Let X be a cell complex with only even cells, i.e., the inclusion $X_{2k} \hookrightarrow X_{2k+1}$ is an isomorphism for all k . Then

$$H_*(X) \cong C_*(X).$$

In particular, $H_n(X)$ is free abelian for all n , $H_n(X) = 0$ for odd n , and the rank of $H_n(X)$ for even n is the number of n -cells.

For **example**, recall that complex projective n -space \mathbb{CP}^n has exactly one cell in each even dimension up to $2n$. Hence as an application we can read off the **homology of complex projective space**:

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{for } 0 \leq k \leq 2n \text{ and } k \text{ even} \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

Note to the theorem and corollary

We should keep in mind that the homology of X is independent of any cell structures. We defined it long before we knew that cell complexes exist.

The theorem shows that knowing a cell structure on X can nevertheless be very helpful for computing $H_*(X)$.

Moreover, we learned that the cell structure on any given cell complex may not be unique. We saw for example two different cell structures on S^n .

However, the theorem tells us that any cell structure one can construct on X has to obey certain constraints what are induced by the fact the homology of the cellular chain complex is $H_*(X)$.

LECTURE 15

Computations of cell homologies and Euler characteristic

• Homology of real projective n -space

As another application of the theorem on the cellular chain complex and the homology of cell complexes we are going to compute the homology of real projective space.

First of all, recall that attaching and characteristic maps assemble to a commutative diagram

$$\begin{array}{ccc} (\coprod D^n, \coprod S^{n-1}) & \longrightarrow & (X_n, X_{n-1}) \\ & \searrow & \downarrow \\ & & (\bigvee S^n, *). \end{array}$$

We have shown that all these maps induce isomorphisms in homology. In particular,

$$(24) \quad H_n(\coprod D^n, \coprod S^{n-1}) \xrightarrow{\cong} C_n(X) = H_n(X_n, X_{n-1}).$$

We are now going to exploit this fact for the computation of $H_*(\mathbb{RP}^n)$.

Recall that the cell structure of \mathbb{RP}^n is such that

- $\text{Sk}_k(\mathbb{RP}^n) = \mathbb{RP}^k$ and
- there is exactly one k -cell in each dimension $k = 0, \dots, n$. We denote this k -cell by e^k .

Hence the cellular chain complex looks like this:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(\mathbb{RP}^n) & \xrightarrow{d_n} & \cdots & \xrightarrow{d_2} & C_1(\mathbb{RP}^n) \xrightarrow{d_1} C_0(\mathbb{RP}^n) \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}[e^n] & \xrightarrow{d_n} & \cdots & \xrightarrow{d_2} & \mathbb{Z}[e^1] \xrightarrow{d_1} \mathbb{Z}[e^0] \longrightarrow 0 \end{array}$$

In order to compute the homology of this chain complex, we need to determine the differentials d_n :

- We know that $H_0(\mathbb{R}P^n)$ is $\mathbb{Z} = \mathbb{Z}[e^0]$. That implies that the differential d_1 must be trivial.
- For $k > 1$, the differential d_k is defined as the top row in the following commutative diagram

$$\begin{array}{ccccc}
 C_k(\mathbb{R}P^n) = H_k(\mathbb{R}P^k, \mathbb{R}P^{k-1}) & \xrightarrow{\partial_k} & H_{k-1}(\mathbb{R}P^{k-1}) & \xrightarrow{j_{n-1}} & H_{k-1}(\mathbb{R}P^{k-1}, \mathbb{R}P^{k-2}) = C_{k-1}(\mathbb{R}P^n) \\
 \cong \uparrow & & \uparrow H_{k-1}(\pi) & & \cong \uparrow \\
 H_k(D^k, S^{k-1}) & \xrightarrow{\partial_k} & H_{k-1}(S^{k-1}) & \longrightarrow & H_{k-1}(S^{k-1}, *)
 \end{array}$$

The map $\pi: S^{k-1} \rightarrow \mathbb{R}P^{k-1}$ is the **attaching map** of the k -cell in $\mathbb{R}P^n$. The outer vertical maps are isomorphisms by our discussion of diagram (24). We also know that the lower differential ∂_k is an **isomorphism** by our original calculation of the homology of the sphere.

Hence, in order to understand the effect of the differential d_k , we need to understand the effect of the maps in the following commutative diagram:

$$(25) \quad \begin{array}{ccc}
 \mathbb{R}P^{k-1} & \xrightarrow{q} & \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \\
 \pi \uparrow & & \downarrow \approx \\
 S^{k-1} & \xrightarrow{g} & S^{k-1}
 \end{array}$$

where q is the quotient map. In other words, we need to calculate the **degree of the lower horizontal map g** .

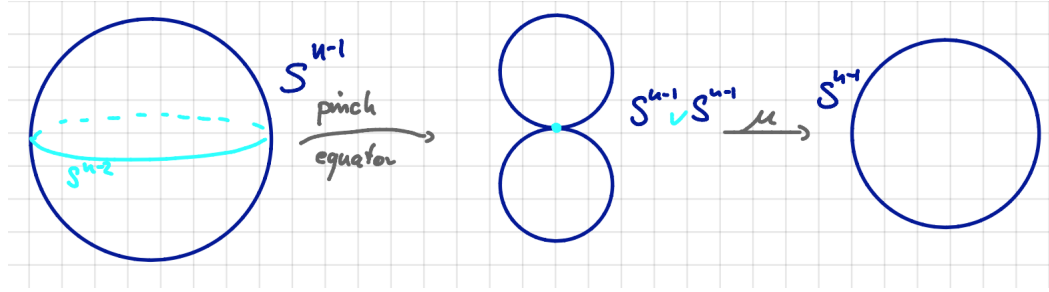
The composite

$$S^{k-1} \xrightarrow{\pi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2}$$

pinches the subspace $S^{k-2} \subset S^{k-1}$ **to a point**.

Hence the lower horizontal map g in (25) is given by

$$\begin{array}{ccccc}
 S^{k-1} & \xrightarrow{\pi} & \mathbb{R}P^{k-1} & \xrightarrow{q} & S^{k-1} \\
 & \searrow \text{pinch equator} & & \nearrow \mu & \\
 & & S^{k-1}/S^{k-2} = S_-^{k-1} \vee S_+^{k-1} & &
 \end{array}$$



To determine the effect of μ , we observe that the subspace $S^{k-1} - S^{k-2}$ consists of **two components**. Let us denote these two components by $(S^{k-1} - S^{k-2})^+$ and $(S^{k-1} - S^{k-2})^-$, respectively. The **restriction** of $q \circ \pi$ **to each component** of $S^{k-1} - S^{k-2}$ is a **homeomorphism** onto $\mathbb{RP}^{k-1} - \mathbb{RP}^{k-2}$.

Let us write $(q \circ \pi)^+$ and $(q \circ \pi)^-$ for the restrictions of $q \circ \pi$ to the subspaces $(S^{k-1} - S^{k-2})^+$ and $(S^{k-1} - S^{k-2})^-$, respectively:

$$\begin{array}{ccc}
 (S^{k-1} - S^{k-2})^+ & & \\
 \downarrow \alpha & \searrow (q \circ \pi)^+ & \\
 & \mathbb{RP}^{k-1} - \mathbb{RP}^{k-2} & \\
 & \nearrow (q \circ \pi)^- & \\
 (S^{k-1} - S^{k-2})^- & &
 \end{array}$$

By definition of \mathbb{RP}^{k-1} , both $(q \circ \pi)^+$ and $(q \circ \pi)^-$ are homeomorphisms and they **differ by precomposing with the antipodal map**.

Hence the map μ is the **identity** on one copy of S^{k-1} and the **antipodal map** α on the other copy of S^{k-1} .

Thus the effect of g on homology is given by

$$H_{k-1}(g): H_{k-1}(S^{k-1}) \rightarrow H_{k-1}(S^{k-1}), \sigma \mapsto \sigma + H_{k-1}(\alpha)(\sigma).$$

But we know what the effect of $H_{k-1}(\alpha)$ is. Namely, it is given by $H_{k-1}(\alpha) = (-1)^{k-1}$. Hence

$$H_{k-1}(g) = 1 + (-1)^k = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Summarizing, we have shown that the cellular chain complex of \mathbb{RP}^n looks like:

$$\begin{aligned} 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 & \text{ if } n \text{ is even} \\ 0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 & \text{ if } n \text{ is odd} \end{aligned}$$

where the left-hand copy of \mathbb{Z} is in dimension n and the right-hand one is in dimension 0.

And in words: in real projective space, **odd cells create new generators**, whereas **even cells create torsion** (except for the zero-cell) in the previous dimension.

Homomology of \mathbb{RP}^n

The homology of real projective n -space is given by

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n \text{ is odd} \\ \mathbb{Z}/2 & 0 < k < n \text{ and } k \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

• What homology sees and does not see

The example of \mathbb{RP}^n indicates what kind of structure of a cell complex singular homology can detect and what it cannot detect and also how we can calculate the differential in the cellular chain complex.

Let X be a cell complex. Its cell structure is determined by attaching maps

$$\coprod S_\alpha^{n-1} \rightarrow X_{n-1}.$$

Knowing these maps, up to homotopy, determines the homotopy type of the cell complex X .

However, homology does not record all of the information of the attaching maps. For, homology only sees the effect of the composite obtained by pinching

out X_{n-1} :

$$\begin{array}{ccccc} \coprod_{\alpha \in J_n} S_\alpha^{n-1} & \longrightarrow & X_{n-1} & \longrightarrow & X_{n-1}/X_{n-2} \\ & \searrow & \downarrow & \swarrow \approx & \\ & & \bigvee_{\beta \in J_{n-1}} S_\beta^{n-1} & & \end{array}$$

In other words, **homology only records what is going on modulo subskeleta**. However, we will see now that homology does a pretty good job at this recording.

Let us try to understand this picture a bit better. For each α , the left-hand diagonal map can be described as the **composite**

$$S_\alpha^{n-1} \xrightarrow{f_\alpha} X_{n-1} \xrightarrow{q_{\alpha\beta}} X_{n-1}/(X_{n-1} - e_\beta^n) =: S_\beta^{n-1}$$

where $q_{\alpha\beta}$ is the quotient map. Moreover, we identify the quotient $X_{n-1}/(X_{n-1} - e_\beta^n)$ with the boundary S_β^{n-1} of the cell e_β^n . Note that this map might be trivial from some (or all) β .

The sum of the effect of these maps in homology is actually the differential d_n in the cellular chain complex. For we have a **commutative diagram**

$$\begin{array}{ccccc} H_n(\bigvee_\alpha S_\alpha^{n-1}, *) & \xrightarrow{\partial_n} & H_{n-1}(\coprod_\alpha S_\alpha^{n-1}) & \xrightarrow{\sum_{\alpha,\beta} H_{n-1}(q_{\alpha\beta} \circ f_\alpha)} & H_n(\bigvee_\beta S_\beta^{n-1}, *) \\ \downarrow H_n(\varphi) \cong & & \downarrow & & \downarrow \cong \\ H_n(X_n, X_{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X_{n-1}) & \xrightarrow{j_{n-1}} & H_{n-1}(X_{n-1}, X_{n-2}). \\ & \searrow & & \nearrow & \\ & d_n & & & \end{array}$$

We conclude from this discussion:

Cellular differentials are sums of degrees

With the above notations, the **effect of the cellular differential** on the generator in $C_n(X)$ which corresponds to the cell e_α^n in X is given by the sum of degrees

$$d_n([e_\alpha^n]) = \sum_\beta H_{n-1}(q_{\alpha\beta} \circ f_\alpha)([e_\alpha^n]) = \sum_\beta \deg(q_{\alpha\beta} \circ f_\alpha) \cdot [e_\beta^n].$$

In other words, in order to compute the cellular differential we need to calculate the degrees of various maps.

- Euler characteristic of cell complexes

Euler characteristic of finite CW-complexes

Let X be a **finite** cell complex. Let c_k denote the number of k -cells in X . Then the **Euler characteristic** of X is defined to be the integer given by the finite sum

$$\chi(X) = \sum_k (-1)^k c_k.$$

The **main result** on $\chi(X)$ we are going to prove today is that it **only depends on the homotopy type** of X and is, in particular, **independent of the given cell structure** of X . We are going to prove this by showing that $\chi(X)$ can be computed using the singular homology of X .

Recall that we have seen an Euler number for polyhedra in the first lecture. It was defined as the number of vertices minus the number of edges plus the number of faces. This fits well with the above definition for a finite cell complex.

For, if we assume the invariance of χ for a moment, then we get $\chi(S^2) = 2$ using the **standard cell structure** on S^2 , i.e., one 0-cell and one 2-cell. This implies that Euler's polyhedra formula holds.

Corollary: Euler's polyhedra formula

For any cell structure on the 2-sphere S^2 with F 2-cells, E 1-cells and V 0-cells, we have the formula

$$F - E + V = 2.$$

As a preparation, we recall some facts about abelian groups.

Let A be an abelian group. Recall that the set of **torsion elements** is defined as

$$\text{Torsion}(A) = \{a \in A : na = 0 \text{ for some } n \neq 0\}.$$

This set is in fact a subgroup of A . A group is called **torsion-free** if $\text{Torsion}(A) = 0$. The quotient $A/\text{Torsion}(A)$ is always torsion-free.

Now we assume that A is **finitely generated**. Then $\text{Torsion}(A)$ is a finite abelian group and $A/\text{Torsion}(A)$ is a finitely generated free abelian group and

therefore isomorphic to \mathbb{Z}^r for some integer r . The number r is called the **rank** of A denoted by $\text{rank}(A)$.

In fact, by choosing generators of $A/\text{Torsion}(A)$, we can construct a homomorphism $A/\text{Torsion}(A) \rightarrow A$ which splits the projection map $A \rightarrow A/\text{Torsion}(A)$. Thus if A is finitely generated abelian, then

$$A \cong \text{Torsion}(A) \oplus \mathbb{Z}^r.$$

We are going to use the following lemma from elementary algebra without proving it:

Lemma: Ranks in exact sequences

- Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of finitely generated abelian groups. Then the ranks of these groups satisfy

$$\text{rank}(B) = \text{rank}(A) + \text{rank}(C).$$

- More generally, for a long exact sequence of finitely generated abelian groups

$$0 \rightarrow A_n \rightarrow A_{n-1} \rightarrow \dots \rightarrow A_1 \rightarrow A_0 \rightarrow 0$$

the ranks satisfy

$$0 = \sum_{i=0}^n (-1)^i \text{rank}(A_i).$$

Now we are equipped for the proof of the above mentioned result:

Theorem: Euler characteristic via homology

Let X be a **finite** cell complex. Then the Euler characteristic of X satisfies

$$\chi(X) = \sum_k (-1)^k \text{rank}(H_k(X)).$$

Proof: Let c_k be again the number of k -cells in the given finite cell structure of X . Let $C_* := C_*(X)$ denote the **cellular chain complex** of X . To simplify the notation let us denote by Z_* , B_* , and H_* the cycles, boundaries and homology, respectively, in this complex.

By their definition, they fit into **two short exact sequences**:

$$0 \rightarrow Z_k \rightarrow C_k \rightarrow B_{k-1} \rightarrow 0$$

and

$$0 \rightarrow B_k \rightarrow Z_k \rightarrow H_k \rightarrow 0,$$

By our previous study of the cellular chain complex, we know

$$c_k = \text{rank}(C_k).$$

Hence, using the above discussion, we can rewrite $\chi(X)$ as follows:

$$\begin{aligned} \chi(X) &= \sum_k (-1)^k \text{rank}(C_k) \\ &= \sum_k (-1)^k (\text{rank}(Z_k) + \text{rank}(B_{k-1})) \\ &= \sum_k (-1)^k (\text{rank}(B_k) + \text{rank}(H_k) + \text{rank}(B_{k-1})). \end{aligned}$$

When we take the sum over all k , **the summands $\text{rank}(B_k)$ and $\text{rank}(B_{k-1})$ will cancel out**. Thus we get

$$\chi(X) = \sum_k (-1)^k \text{rank}(H_k).$$

But by the theorem on the homology of the cellular chain complex, H_k is exactly the singular homology group $H_k(X)$ of X . **QED**

Note that the numbers $\text{rank}(H_k(X))$ are called the **Betti numbers** of X . They had already played an important role in mathematics, before homology groups had been systematically developed. As the theorem shows, these numbers are an interesting invariant of a space.

The description of the Euler number in the theorem now generalizes easily:

Definiton: Euler characteristic revisited

Let X be a topological space such that each $H_n(X)$ has finite rank and that there is an d such that $H_n(X) = 0$ for all $n > d$. Then the **Euler characteristic** of X is defined to be the integer given by the finite sum

$$\chi(X) = \sum_k (-1)^k \text{rank}(H_k(X)).$$

• Designing cell complexes

For example, for $m \in \mathbb{Z}$, we can easily construct a space X with $H_n(X) = \mathbb{Z}/m$ and $\tilde{H}_i(X) = 0$ for $i \neq n$. We start with S^n and attach an $n+1$ -cell to it via a map $f: S^n \rightarrow S^n$ of degree m . The cellular chain complex of this space is

$$0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow 0$$

with the copies of \mathbb{Z} in dimensions $n+1$ and n , respectively. The homology of this space is exactly what we wanted.

This procedure can easily be generalized.

Theorem: Moore spaces

Let A_* be any graded abelian group with $A_n = 0$ for $n < 0$. Then there exists a cell complex X with $\tilde{H}_*(X) = A_*$.

We are going to prove this result in the next lecture.

LECTURE 16

Designing homology groups and homology with coefficients

- Designing cell complexes

We announced last time that cell complexes enable us to design spaces with prescribed homology groups. We are going to prove this result today.

We will need a construction on spaces that we have already used in special cases.

Recall that the **wedge** $X \vee Y$ of two **pointed spaces** (X, x) and (Y, y) is defined as the quotient of $X \sqcup Y$ modulo $x \sim y$, i.e., the disjoint union with x and y identified. We can think of $X \vee Y$ glued together at the joint point $[x] = [y]$. This generalizes the wedge of spheres that we have seen before. This construction generalizes to **infinite wedges**.

If each point x_α is a deformation retract of a neighborhood U_α in X_α , then the wedge satisfies a formula for **reduced homology** that we are used to for the homology of disjoint unions:

$$\tilde{H}_*\left(\bigvee_{\alpha} X_{\alpha}\right) \cong \bigoplus_{\alpha} \tilde{H}_*(X_{\alpha}).$$

Now we can prove the following result:

Theorem: Moore spaces

Let A_* be any graded abelian group with $A_n = 0$ for $n < 0$. Then there exists a cell complex X with $\tilde{H}_*(X) = A_*$.

Proof: Let us start with just a single abelian group A . By choosing generators for A , we can define a surjective homomorphism

$$F_0 \rightarrow A$$

from a **free abelian group** F_0 . The **kernel** of this homomorphism, denoted by F_1 , is also **free**, since it is a subgroup of a free abelian group.

We write J_0 for a minimal set of generators of F_0 such that we have a surjection $F_0 \rightarrow A$ and J_1 for a minimal set of generators of F_1 .

For $n \geq 1$, we define X_n to be

$$X_n := \bigvee_{\alpha \in J_0} S_\alpha^n.$$

The n th homology of X_n is $H_n(X_n) = \mathbb{Z}[J_0]$.

Now we are going to define an attaching map

$$f: \prod_{\beta \in J_1} S_\beta^n \rightarrow X_n$$

by specifying it on each summand S_β^n .

In F_0 , we can write the generator β of F_1 as a linear combination of the generators of F_0

$$\beta = \sum_{i=1}^s n_i \alpha_i.$$

We can reproduce this relation in topology. For, let

$$S^n \rightarrow \bigvee_{i=1}^s S_{\alpha_i}^n$$

be the map obtained by pinching $s - 1$ circles on S^n to points. The effect of this map in homology is to send the generator in $H_n(S^n)$ to the s -tuple of generators in $H_n(S_{\alpha_i}^n)$:

$$H_n(S^n) \rightarrow H_n\left(\bigvee_{i=1}^s S_{\alpha_i}^n\right) = \bigoplus_{i=1}^s H_n(S_{\alpha_i}^n), \quad 1 \mapsto (1, \dots, 1).$$

For each i , we choose a map $S_{\alpha_i}^n \rightarrow S_{\alpha_i}^n$ of **degree** n_i .

The map on the summand S_β^n is now defined as the composite

$$S_\beta^n \rightarrow \bigvee_{i=1}^s S_{\alpha_i}^n \rightarrow \bigvee_{\alpha} S_\alpha^n.$$

Taking the disjoint union of all these maps as attaching maps, we get a cell complex X whose cellular chain complex looks like

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

with F_0 in dimension n and F_1 in dimension $n + 1$, and whose homology is

$$\tilde{H}_q(X) = \begin{cases} A & \text{for } q = n \\ 0 & \text{for } q \neq n. \end{cases}$$

We write $M(A, n)$ for the CW -complex produced this way and call it a **Moore space** of type A and n .

Finally, for a **graded abelian group** A_* as in the theorem, we define X to be the **wedge** of all the $M(A_n, n)$. **QED**

Moore spaces are not functorial

It is important to note that the construction of Moore spaces **cannot** be turned into a **functor** $\mathbf{Ab} \rightarrow \mathbf{hoTop}$. This might surprise at first glance. For given a homomorphism $g: A \rightarrow B$ we can construct a continuous map $\gamma: M(A, n) \rightarrow M(B, n)$ such that $H_n(\gamma) = g$.

However, this construction **depends on the various choices** we make. That means that for homomorphisms

$$A \xrightarrow{g_1} B, \text{ and } B \xrightarrow{g_2} C$$

we cannot guarantee that $\gamma_2 \circ \gamma_1$ is the same map as the one we would have constructed by starting with $g_2 \circ g_1: A \rightarrow C$ directly.

Despite this caveat, we have witnessed an important phenomenon that still motivates a lot of exciting research:

From Algebra to Topology

The proof demonstrates a common phenomenon in Algebraic Topology. Whereas our initial goal was to translate topology into algebra, now we went in the opposite direction. Starting with an algebraic structure **we modelled a space whose homology reproduces the algebra**. Since being an abelian group is not the only the algebraic structure and homology not the only invariant out there, we can imagine that the above theorem is only a first glance at the makings of a **huge mathematical industry**.

The proof also shows why **Topology is particularly well suited** for this endeavour. The gluing construction we used for **building cell complexes is unique for topological spaces**. Requiring any additional structure usually stops us from producing cell complexes. For example, **there are no cell complexes of smooth manifolds or algebraic varieties**. Nevertheless, there are some inventive procedures to remedy this defect...

• Homology with coefficients

Now it is time to move on and to develop new algebraic invariants which add to the information we get from singular homology, or possibly simplify computations.

Recall that homology produces abelian groups. As nice as abelian groups are, it would be good to have additional structure, for example as vector spaces over a field. So one might wonder if there is a **version of singular homology** with values in the **category of vector spaces** over a field, or more generally the **category of modules** over a ring.

Actually, if R is a ring (with unit and commutative), there is an obvious candidate for such a theory: We define

$$S_n(X; R) := R\text{Sing}_n(X)$$

to be the free R -module over the set $\text{Sing}_n(X)$ of n -simplices. What we have done so far, was the special case $R = \mathbb{Z}$.

Now we can use the face maps and the same formula we had before for defining a boundary operator

$$(26) \quad \partial_n: S_n(X; R) \rightarrow S_{n-1}(X; R), \quad \sum_j r_j \sigma_j \mapsto \sum_j \sum_i (-1)^i r_j (\sigma_j \circ \phi_i^n).$$

which is now a homomorphism of R -modules. The same calculations as before yield $\partial \circ \partial = 0$.

Now we can form the **homology** as usual

$$H_n(X; R) := \frac{\text{Ker}(\partial_n: S_n(X; R) \rightarrow S_{n-1}(X; R))}{\text{Im}(\partial_{n+1}: S_{n+1}(X; R) \rightarrow S_n(X; R))}.$$

For each $n \geq 0$, $H_n(X; R)$ is an R -module and is called the **singular homology of X with coefficients in R** .

More generally, if M is **any abelian group**, we can form the tensor product

$$\begin{aligned} S_n(X; M) &= S_n(X) \otimes_{\mathbb{Z}} M = \bigoplus_{\sigma \in \text{Sing}_n(X)} M \\ &= \left\{ \sum_j m_j \sigma_j : \sigma_j \in \text{Sing}_n(X), m_j \in M \right\}. \end{aligned}$$

The boundary operator is defined as before by

$$\partial_n^M : S_n(X; M) \rightarrow S_{n-1}(X; M), \partial_n^M = \partial_n \otimes 1.$$

More explicitly, ∂_n^M is given by the formula in (26) with r_j s replaced with m_j s. Since $\partial \circ \partial = 0$, we get $\partial^M \circ \partial^M = 0$.

Homology with coefficients

For a pair of spaces (X, A) , we define **singular homology of (X, A) with coefficients in M** $H_n(X, A; M)$ to be the n th homology of the chain complex

$$S_*(X, A; M) := \frac{S_*(X; M)}{S_*(A; M)}.$$

Homology with coefficients is **functorial**: That is, a map of pairs

$$f : (X, A) \rightarrow (Y, B)$$

induces, by composing simplices with f , a homomorphism

$$f_* : H_n(X, A; M) \rightarrow H_n(Y, B; M) \text{ for all } n \geq 0$$

which we denote just by f_* to keep the notation simple. Moreover, we have $(g \circ f)_* = g_* \circ f_*$.

Note if $M = R$ is a ring, this is the same definition as above, and for $M = \mathbb{Z}$ we recover $H_n(X, A; \mathbb{Z}) = H_n(X, A)$. We will often refer to these groups as **integral homology groups**.

If M is an R -module, then the groups $H_n(X, A; M)$ have the additional structure as an R -module itself.

Eilenberg-Steenrod Axioms are satisfied

Singular homology with coefficients in M satisfies the **Eilenberg-Steenrod axioms** with the only **modification**

$$H_n(\text{pt}; M) = \begin{cases} M & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

Since everything we proved for singular homology was based on these properties, we can **transfer basically all our work to homology with coefficients**. Let us point out **two crucial facts**:

- The calculations for **spheres** can be transferred and we get

$$\tilde{H}_k(S^n; M) = \begin{cases} M & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

- If X is a **cell complex**, there is a cellular chain complex

$$C_*(X; M) \text{ with } C_n(X; M) = \bigoplus_{e_\alpha^n} M$$

where the sum is taken over the n -cells of X . As for $M = \mathbb{Z}$, the n th homology of $C_*(X; M)$ is isomorphic to $H_n(X; M)$.

The **reduced homology** groups $\tilde{H}_n(X; M)$ with coefficients in M are defined as the homology groups of the augmented chain complex

$$\dots \rightarrow S_1(X; M) \rightarrow S_0(X; M) \xrightarrow{\epsilon} M \rightarrow 0$$

where ϵ is the homomorphism which sends $\sum_j m_j \sigma_j$ to $\sum_j m_j \in M$.

For a **homomorphism of groups** $\varphi: M \rightarrow N$ there is an induced morphism of chain complexes $S_*(X, A; M) \rightarrow S_*(X, A; N)$ which induces a homomorphism in homology

$$\varphi_*: H_*(X, A; M) \rightarrow H_*(X, A; N).$$

This homomorphism is compatible with f_* for maps of pairs and with long exact sequences of pairs.

For the calculations using the cellular chain complex, we need to check the following lemma:

Lemma: Degrees with coefficients

Let $f: S^n \rightarrow S^n$ be a map of **degree k** . Then $f_*: \tilde{H}_n(S^n; M) \rightarrow \tilde{H}_n(S^n; M)$ is given by **multiplication with k** , where k denotes the image of k in M .

Proof: Let $\varphi: \mathbb{Z} \rightarrow M$ be a homomorphism of groups ($0_{\mathbb{Z}} \mapsto 0_M$) which sends $1 \in \mathbb{Z}$ to an element $m \in M$. Then the assertion follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
 \mathbb{Z} & \xlongequal{\quad} & H_n(S^n; \mathbb{Z}) & \xrightarrow{f_*} & H_n(S^n; \mathbb{Z}) & \xlongequal{\quad} & \mathbb{Z} \\
 \varphi \downarrow & & \varphi_* \downarrow & & \downarrow \varphi_* & & \downarrow \varphi \\
 M & \xlongequal{\quad} & H_n(S^n; M) & \xrightarrow{f_*} & H_n(S^n; M) & \xlongequal{\quad} & M.
 \end{array}$$

That the outer diagram commutes follows from the way we compute the homology groups of S^n with coefficients \mathbb{Z} and M via the Eilenberg-Steenrod axioms. **QED**

• Why coefficients?

The coefficients that are most often used are the **fields** \mathbb{F}_p , for a prime number p , and the field \mathbb{Q} , \mathbb{R} and sometimes \mathbb{C} .

In order to get an idea of what happens when we use different coefficients, let us look at the homology of \mathbb{RP}^n for $R = \mathbb{F}_2$. We use the cellular chain complex which looks like this

$$0 \rightarrow \mathbb{F}_2 \rightarrow \mathbb{F}_2 \rightarrow \dots \rightarrow \mathbb{F}_2 \rightarrow \mathbb{F}_2 \rightarrow 0.$$

We showed that the differentials alternated between multiplication by 2 and 0. But in \mathbb{F}_2 , $2 = 0$ which means that all differentials vanish and we get

$$H_k(\mathbb{RP}^n; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{for } 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

We learn from this example that

- The calculation of \mathbb{F}_2 -homology can be particularly easy, and it might see more nontrivial groups than integral homology.
- Nevertheless, \mathbb{F}_2 -homology is often sufficient to distinguish between trivial and nontrivial spaces or maps.

The situation is quite different if we take R to be any field of **characteristic different from two**. Then the cellular chain complex of $\mathbb{R}P^n$ looks like (with the left-hand copy of R in dimension n)

$$0 \rightarrow R \xrightarrow{\cong} R \xrightarrow{0} R \xrightarrow{\cong} \dots \xrightarrow{\cong} R \xrightarrow{0} R \rightarrow 0$$

for n even, and

$$0 \rightarrow R \xrightarrow{0} R \xrightarrow{\cong} R \xrightarrow{0} \dots \xrightarrow{\cong} R \xrightarrow{0} R \rightarrow 0$$

for n odd.

Thus, for n even, we get

$$H_k(\mathbb{R}P^n; R) = \begin{cases} R & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and, for n odd,

$$H_k(\mathbb{R}P^n; R) = \begin{cases} R & \text{for } k = 0 \\ R & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

In other words, **away from 2**, real projective n -space looks for R -homology like a point if n is even and like an n -sphere if n is odd.

This teaches us already that **different coefficients can tell quite different stories**.

This notwithstanding one might wonder whether integral homology is the finest invariant and all other homologies are just coarser variations. This is not the case, and it is indeed **possible that homology with coefficients detects more than integral homology**. Let us look at an example:

Example: When \mathbb{Z}/m -homology sees more

Let $X = M(\mathbb{Z}/m, n)$ be a Moore space we constructed in the previous lecture: We start with an n -sphere S^n and form X by attaching an $n+1$ -dimensional cell to it via a map $f: S^n \rightarrow S^n$ of degree m

$$X = S^n \cup_f D^{n+1}.$$

Let

$$q: X \rightarrow X/S^n \approx S^{n+1}$$

be the quotient map. **It induces a trivial homomorphism** in reduced integral homology. For, the only nontrivial homology occurs in degrees n and $n + 1$ where we have

$$\tilde{H}_{n+1}(X; \mathbb{Z}) = 0 \xrightarrow{q_*} \tilde{H}_{n+1}(X/S^n; \mathbb{Z})$$

and

$$\tilde{H}_n(X; \mathbb{Z}) \xrightarrow{q_*} \tilde{H}_{n+1}(X/S^n; \mathbb{Z}) = \tilde{H}_n(S^{n+1}; \mathbb{Z}) = 0.$$

Hence **integral homology cannot distinguish** between the quotient map and a constant map.

However, **\mathbb{Z}/m -homology does see the difference between q and a constant map.**

For, the \mathbb{Z}/m -cellular chain complex of X is

$$0 \rightarrow \mathbb{Z}/m \xrightarrow{m} \mathbb{Z}/m \rightarrow 0$$

with copies of \mathbb{Z}/m in dimensions $n + 1$ and n . Thus, the \mathbb{Z}/m -homology of X is

$$\tilde{H}_k(X; \mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & \text{if } k = n + 1 \\ \mathbb{Z}/m & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

The long exact sequence of the pair (X, S^n) in dimension $n + 1$ then yields

$$0 = \tilde{H}_{n+1}(S^n; \mathbb{Z}/m) \rightarrow \tilde{H}_{n+1}(X; \mathbb{Z}/m) \xrightarrow{q_*} \tilde{H}_{n+1}(X/S^n; \mathbb{Z}/m).$$

Since the left-hand group is 0, exactness implies that **q_* is injective and hence nontrivial**, since both $\tilde{H}_{n+1}(X; \mathbb{Z}/m)$ and $\tilde{H}_{n+1}(X/S^n; \mathbb{Z}/m)$ are isomorphic to \mathbb{Z}/m .

This example demonstrates that homology groups with coefficients are similar, but often a bit different than integral homology groups. This raises the question how different they can be. More generally, we could ask:

Question

Given an R -module M $H_*(X; R)$, what can we deduce about $H_*(X; M)$?

For **example**, let M be the \mathbb{Z} -module \mathbb{Z}/m . One might wonder if $H_n(X; \mathbb{Z}/m)$ is just the quotient $H_n(X; \mathbb{Z})/mH_n(X; \mathbb{Z})$, since the latter is isomorphic to the tensor product $H_n(X; \mathbb{Z}) \otimes \mathbb{Z}/m$.

But we have to be **careful**. For, we do have a short exact sequence of chain complexes

$$0 \rightarrow S_*(X; \mathbb{Z}) \xrightarrow{m} S_*(X; \mathbb{Z}) \rightarrow S_*(X; \mathbb{Z}/m) \rightarrow 0.$$

Such a short exact sequence induces a long exact sequence of the respective homology groups a part of which looks like

$$H_n(X; \mathbb{Z}) \xrightarrow{m} H_n(X; \mathbb{Z}) \rightarrow H_n(X; \mathbb{Z}/m) \rightarrow H_{n-1}(X; \mathbb{Z}) \xrightarrow{m} H_{n-1}(X; \mathbb{Z}).$$

Using the exactness of this sequence yields a **short exact sequence**

(27)

$$0 \rightarrow H_n(X; \mathbb{Z})/mH_n(X; \mathbb{Z}) \rightarrow H_n(X; \mathbb{Z}/m) \rightarrow m\text{-Torsion}(H_{n-1}(X; \mathbb{Z})) \rightarrow 0$$

where $m\text{-Torsion}(H_{n-1}(X; \mathbb{Z}))$ denotes the m -torsion, i.e., the kernel of the map $H_{n-1}(X; \mathbb{Z}) \xrightarrow{m} H_{n-1}(X; \mathbb{Z})$ given by multiplication by m .

In fact, the short exact sequence (27) provides a tool to determine $H_n(X; \mathbb{Z}/m)$ when we know both $H_n(X; \mathbb{Z})$ and $H_{n-1}(X; \mathbb{Z})$. **However**, in general, we will **need a more sophisticated method** to understand the relationship of $H_*(X; \mathbb{Z}) \otimes M$ and $H_*(X; M)$.

As a first generalization, we have the following result:

Long exact sequence of coefficients

Assume we have a short exact sequence of abelian groups

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

For any pair of spaces (X, A) , there is an induced short exact sequence of chain complexes

$$0 \rightarrow S_*(X, A; M') \rightarrow S_*(X, A; M) \rightarrow S_*(X, A; M'') \rightarrow 0.$$

Such a short exact sequence induces a **long exact sequence**

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & H_{n+1}(X, A; M'') \\ & & & & & \searrow \partial & \\ & & & & & & \\ H_n(X, A; M') & \longleftarrow & H_n(X, A; M) & \longrightarrow & H_n(X, A; M'') & & \\ & & & & \nearrow \partial & & \\ & & & & & & \\ H_{n-1}(X, A; M') & \longrightarrow & \cdots & & & & \end{array}$$

LECTURE 17

Tensor products, Tor and the Universal Coefficient Theorem

Our goal for this lecture is to prove the **Universal Coefficient Theorem** for singular homology with coefficients. This will require some preparations in homological algebra. For some this will be a review. Though to keep everybody on board, this is what we have to do.

We will not treat the most general cases, but rather focus on the main ideas. Any text book in homological algebra will provide more general results.

• Tensor products

Let A and B be abelian groups. We would like to combine A and B into just one object, denoted $A \otimes B$, in such a way that having a bilinear homomorphism

$$f: A \times B \rightarrow C$$

is the same as having a homomorphism from $A \otimes B$ into C .

That f is bilinear means

$$\begin{aligned} f(a_1 + a_2, b) &= f(a_1, b) + f(a_2, b) \\ f(a, b_1 + b_2) &= f(a, b_1) + f(a, b_2). \end{aligned}$$

We can achieve this by brute force.

Tensor product

For, we can construct $A \otimes B$ as the quotient of the free abelian group generated by the set $A \times B$ modulo the subgroup generated by $\{(a + a', b) - (a, b) - (a', b)\}$ and $\{(a, b + b') - (a, b) - (a, b')\}$ for all $a, a' \in A$ and $b, b' \in B$. We denote the equivalence class of (a, b) in $A \otimes B$ by $a \otimes b$. We call $A \otimes B$ the **tensor product** of A and B .

Let us collect some immediate **observations**:

- For any $a \in A$, $b \in B$ and any integer $n \in \mathbb{Z}$, the relations imply

$$n(a \otimes b) = (na) \otimes b = a \otimes (nb).$$

- The abelian group $A \otimes B$ is generated by elements $a \otimes b$ with $a \in A$ and $b \in B$.
- Elements in the abelian group $A \otimes B$ are finite sums of equivalence classes $\sum_{i=1}^m n_i(a_i \otimes b_i)$.
- The tensor product is symmetric up to isomorphism with isomorphism given by

$$A \otimes B \xrightarrow{\cong} B \otimes A, \quad \sum_{i=1}^m n_i a_i \otimes b_i \mapsto \sum_{i=1}^m n_i b_i \otimes a_i.$$

- The tensor product is associative up to isomorphism:

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C.$$

- For homomorphisms $f: A \rightarrow A'$ and $g: B \rightarrow B'$, there is an induced homomorphism

$$f \otimes g: A \otimes B \rightarrow A' \otimes B', \quad (f \otimes g)(a \otimes b) = f(a) \otimes g(b).$$

- The tensor product has the desired universal property:

$$\text{Hom}_{\text{bilinear}}(A \times B, C) \xrightarrow{\cong} \text{Hom}_{\mathbf{Ab}}(A \otimes B, C),$$

i.e., if we have a bilinear map $A \times B \rightarrow C$, then there is a unique (up to isomorphism) dotted map which makes the diagram commutative

$$\begin{array}{ccc} A \times B & \xrightarrow{q} & A \otimes B \\ & \searrow & \vdots \\ & & C. \end{array}$$

- The universal property of the tensor product implies that we have an isomorphism

$$\left(\bigoplus_{\alpha} A_{\alpha} \right) \otimes B \cong \bigoplus_{\alpha} (A_{\alpha} \otimes B).$$

Now it is time to see **some examples**:

- For every abelian group A , we have isomorphisms

$$A \otimes \mathbb{Z} \cong A \cong \mathbb{Z} \otimes A$$

which sends $a \otimes n \mapsto na$ and inverse $a \mapsto a \otimes 1$.

- For every abelian group A and every m , we have an isomorphism

$$A \otimes \mathbb{Z}/m \cong A/mA, \quad a \otimes [n] \mapsto [an] \in A/mA.$$

- If M is an **abelian group** and X a space, we can form the tensor product

$$\begin{aligned} S_n(X; M) &:= S_n(X) \otimes M \cong \bigoplus_{\sigma \in \text{Sing}_n(X)} M \\ &= \left\{ \sum_j m_j \sigma_j : \sigma_j \in \text{Sing}_n(X), m_j \in M \right\}. \end{aligned}$$

There is a boundary operator defined by

$$\partial_n^M : S_n(X; M) \rightarrow S_{n-1}(X; M), \partial_n^M = \partial_n \otimes 1.$$

This turns $S_*(X; M)$ into a chain complex. The homology of this complex is the homology of X with coefficients in M .

Tensor products are great. Except for the following:

- **Tor functor**

Suppose we have an abelian group M and a surjective homomorphism of abelian groups

$$B \twoheadrightarrow C.$$

Then we can check by looking at the generators that

$$B \otimes M \twoheadrightarrow C \otimes M$$

is also surjective.

More generally, we can show that tensor products preserve cokernels:

Lemma: Tensor products preserve cokernels

Let M be an abelian group. Suppose we have an exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0.$$

Then taking the tensor product $- \otimes M$ yields an exact sequence

$$A \otimes M \xrightarrow{i \otimes 1} B \otimes M \xrightarrow{j \otimes 1} C \otimes M \rightarrow 0$$

where 1 denotes the identity map on M . In other words, the functor $- \otimes M$ is **right exact** and **preserves cokernels**.

Proof: We are going to show that $- \otimes M$ preserves cokernels. This is in fact equivalent to the other statements.

Let $f: B \otimes M \rightarrow Q$ be a homomorphism. We need to show that there is a unique factorization as indicated by the dotted arrow in the diagram

$$\begin{array}{ccccc} A \otimes M & \xrightarrow{i \otimes 1} & B \otimes M & \xrightarrow{j \otimes 1} & C \otimes M \longrightarrow 0 \\ & \searrow 0 & \downarrow f & \swarrow \text{dotted} & \\ & & Q & & \end{array}$$

By the universal property and the fact that $C \times M$ generates $C \otimes M$, this is equivalent to a unique factorization of the diagram of **bilinear maps**

$$\begin{array}{ccccc} A \times M & \xrightarrow{i \times 1} & B \times M & \xrightarrow{j \times 1} & C \times M \longrightarrow 0 \\ & \searrow 0 & \downarrow F & \swarrow \text{dotted} & \\ & & Q & & \end{array}$$

But now we only need to find an appropriate extension $C \rightarrow Q$ the existence of which is implied by assumption. \square

However, suppose we have an injective homomorphism

$$A \hookrightarrow B.$$

Then it is in general **not the case** that

$$A \otimes M \rightarrow B \otimes M$$

is **injective**.

For example, take the map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ given by multiplication by 2. It is clearly **injective**. But if we tensor with $\mathbb{Z}/2$, we get the map

$$\mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2$$

which is **not injective**.

Thus, **tensor products do not preserve exact sequences**, in general.

We would like to remedy this defect. And, in fact, the tensor product is **not so far from being exact**. For, if M is a **free** abelian group, then the functor $M \otimes -$ is **exact**, i.e., it preserves all exact sequences.

We can see this as follows: Assume M is the free abelian group on the set S . Then $M \otimes N = \oplus_S N$, since tensoring distributes over direct sums, as we remarked above.

To exploit this fact we make use of the following observation:

Lemma: Direct sums of exact sequences

If $M'_i \rightarrow M_i \rightarrow M''_i$ is exact for every $i \in I$, then

$$\bigoplus_i M'_i \rightarrow \bigoplus_i M_i \rightarrow \bigoplus_i M''_i$$

is exact.

Proof: The composition is zero and if $(x_i)_i$ is sent to 0 in $\bigoplus_i M''_i$, then each x_i must be sent to 0 in M''_i . Hence each x_i comes from some x'_i , and hence $(x_i)_i$ comes from $(x'_i)_i$. We just need to remember to choose $x'_i = 0$ whenever $x_i = 0$. \square

Now let A be any abelian group. As we did in the previous lecture, we **choose a free abelian group** F_0 mapping surjectively onto A

$$F_0 \twoheadrightarrow A.$$

The **kernel** F_1 of this map is also **free** abelian as a subgroup of a free abelian group. Hence we get an **exact sequence** of the form

$$0 \rightarrow F_1 \hookrightarrow F_0 \twoheadrightarrow A.$$

Free resolutions

Such an exact sequence with F_1 and F_0 free abelian groups, is called a **free resolution of A** of length two.

(Note that the fact that we can always choose such a free resolution of length two is particular to the case of abelian groups, i.e., \mathbb{Z} -modules. For R -modules over other rings, one might only be able to find projective resolutions of higher length. The fact that \mathbb{Z} is a principal ideal domain, a PID, does the trick.)

For any abelian group M , tensoring these maps with M yields an exact sequence

$$F_1 \otimes M \rightarrow F_0 \otimes M \rightarrow A \otimes M \rightarrow 0.$$

The **kernel** of the left-hand map **is not necessarily zero**, though.

This leads to the following important definition:

Definition: Tor

The **kernel** of the map $A \otimes F_1 \rightarrow A \otimes F_0$ is called $\text{Tor}(A, M)$. Hence by definition we have an **exact sequence**

$$0 \rightarrow \text{Tor}(A, M) \rightarrow A \otimes F_1 \rightarrow A \otimes F_0 \rightarrow A \otimes M \rightarrow 0.$$

This group measures how far $- \otimes M$ is from being exact.

Note that if we replace abelian groups with R -modules over other rings than \mathbb{Z} and take tensor products over R , we might have to consider higher Tor-terms. Hence we should really write $\text{Tor}_1^{\mathbb{Z}}(A, M)$ for $\text{Tor}(A, M)$. But we are going to keep things simple and focus on the idea rather than general technicalities.

It is again time to see **some examples**:

- If M is a **free** abelian group, then $\text{Tor}(A, M) = 0$ for any abelian group A . That follows from the lemma above.
- Let $M = \mathbb{Z}/m$. Then we can take $F_0 = F_1 = \mathbb{Z}$ and

$$\mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$$

as a free resolution of \mathbb{Z}/m . For an abelian group A , the sequence defining Tor looks like

$$0 \rightarrow \text{Tor}(A, \mathbb{Z}/m) \rightarrow A \otimes \mathbb{Z} \xrightarrow{1 \otimes m} A \otimes \mathbb{Z} \rightarrow A \otimes \mathbb{Z}/m \rightarrow 0.$$

Since we know $A \otimes \mathbb{Z}/m = A/mA$, we get

$$\text{Tor}(A, \mathbb{Z}/m) = \text{Ker}(m: A \rightarrow A) = m\text{-torsion in } A.$$

Hence $\text{Tor}(A, \mathbb{Z}/m)$ is the subgroup of m -torsion elements in A .

- For a concrete case, let us try to calculate $\text{Tor}(\mathbb{Z}/4, \mathbb{Z}/6)$. We use the free resolution

$$\mathbb{Z} \xrightarrow{6} \mathbb{Z} \rightarrow \mathbb{Z}/6 \rightarrow 0.$$

Tensoring with $\mathbb{Z}/4$ yields

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \rightarrow \mathbb{Z}/4 \otimes \mathbb{Z}/6 \rightarrow 0$$

where we use $6 = 2$ in $\mathbb{Z}/4$. The kernel of $\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$ is $\mathbb{Z}/2$. Thus

$$\text{Tor}(\mathbb{Z}/4, \mathbb{Z}/6) = \mathbb{Z}/2.$$

- More generally, we get

$$\text{Tor}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(n, m)$$

where $\gcd(n, m)$ denotes the greatest common divisor of n and m .

The last three examples explain the name Tor.

We should hold our breath for a moment and check a couple of things. For example, that Tor does not depend on the choice of free resolution, that it is a functor etc. So let us get to work:

Lemma: Lifting resolutions

Let $f: M \rightarrow N$ be a homomorphism and $0 \rightarrow E_1 \xrightarrow{i} E_0 \xrightarrow{p} M$ and $0 \rightarrow F_1 \xrightarrow{j} F_0 \xrightarrow{q} N$ be free resolutions. Then we can lift f to a chain map $f_*: E_* \rightarrow F_*$, i.e., to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i} & E_0 & \xrightarrow{p} & M \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\ 0 & \longrightarrow & F_1 & \xrightarrow{j} & F_0 & \xrightarrow{q} & N \longrightarrow 0. \end{array}$$

Moreover, this lift is unique up to chain homotopy, i.e., for another lift f'_* of f , there is a chain homotopy h between f_* and f'_* :

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & 0 \\ & & f'_1 \downarrow & & \downarrow f_0 & & \\ & & \searrow f_1^h & \nearrow f'_0 & & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & 0. \end{array}$$

Proof: • Since E_0 is a free abelian group, we know there is some set S of generators such that $E_0 = \mathbb{Z}S$. Now we can map the elements in S to M via the map $E_0 \xrightarrow{p} M$, and further to N via $M \xrightarrow{f} N$. Since $F_0 \xrightarrow{q} N$ is **surjective**, we can choose lifts in F_0 of the elements in $f(p(S))$. Since a homomorphism on a free abelian group is determined by the image of the generators, we can extend this process to get a homomorphism

$$E_0 \xrightarrow{f_0} F_0 \text{ such that } f \circ p = q \circ f_0.$$

Now can define f_1 to be the restriction of f_0 to the kernel of p which is E_1 by definition. This yields the desired commutative diagram.

• Now let f'_0 and f'_1 be another choice of maps which lift f . The differences $g_0 := f_0 - f'_0$ and $g_1 := f_1 - f'_1$ are then maps which lift $f - f = 0: M \rightarrow N$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i} & E_0 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow g_1 & & \downarrow g_0 & & \downarrow 0 & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{j} & F_0 & \xrightarrow{q} & N & \longrightarrow & 0. \end{array}$$

Since the diagram commutes, we get $q \circ g_0 = p \circ 0 = 0$. Therefore, the **universal property of kernels** implies that we can lift g_0 to a map $h: E_0 \rightarrow F_1$ such that $j \circ h = g_0$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i} & E_0 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow g_1 & \nearrow h & \downarrow g_0 & & \downarrow 0 & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{j} & F_0 & \xrightarrow{q} & N & \longrightarrow & 0. \end{array}$$

Moreover, since E_1 is the kernel of i , we must have $h \circ i = g_1$. Thus h is a chain homotopy between f_* and f'_* (the next map $E_1 \rightarrow 0$ being trivial). \square

With this result at hand we can finally prove:

Corollary: Tor is independent of resolutions

Tor is independent of the choice of a free resolution: For any free resolution $0 \rightarrow E_1 \xrightarrow{i} E_0 \rightarrow M$ of M , there is a unique isomorphism

$$\text{Ker}(i \otimes 1) \xrightarrow{\cong} \text{Tor}(A, M).$$

Proof: We just apply the previous result to the **identity** of M to get that, with whatever resolution we calculate Tor, there is an isomorphism between any two different ways. And this isomorphism is unique by the theorem on chain homotopies and their induced maps on homologies. \square

There are other **properties of Tor** the proof of which we are going to omit:

- Tor is **functorial**: For any homomorphisms of abelian groups $A \rightarrow A'$ and $M \rightarrow M'$, there is a homomorphism

$$\text{Tor}(A, M) \rightarrow \text{Tor}(A', M').$$

- Tor is **symmetric**, i.e., $\text{Tor}(A, M) \cong \text{Tor}(M, A)$.
- If M is **free**, then $\text{Tor}(A, M) = 0$ for any abelian group A .

- Since the direct sum of free resolutions of each A_i is a free resolution of $\bigoplus_i A_i$, we know that Tor commutes with direct sums:

$$\text{Tor}\left(\bigoplus_i A_i, M\right) \cong \bigoplus_i \text{Tor}(A_i, M),$$

- Let $T(M)$ be the subgroup of **torsion elements** of M . Then

$$\text{Tor}(A, M) \cong \text{Tor}(A, T(M))$$

for any abelian group A .

Now we can prove the main result in this story:

Theorem: Universal Coefficient Theorem

Let C_* be a chain complex of **free** abelian groups and let M be an abelian group. Then there are natural short exact sequences

$$0 \rightarrow H_n(C_*) \otimes M \rightarrow H_n(C_* \otimes M) \rightarrow \text{Tor}(H_{n-1}(C_*), M) \rightarrow 0$$

for all n . These sequences split, but the splitting is not natural.

Proof: We write Z_n for the **kernel** and B_{n-1} for the **image** of the differential $d: C_n \rightarrow C_{n-1}$. Since C_n and C_{n-1} are **free**, both Z_n and B_{n-1} are **free as well**.

Together with the differential in C_* , this yields a morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{d_n} & B_{n-1} \longrightarrow 0 \\ & & \downarrow d_n & & \downarrow d_n & & \downarrow d_{n-1} \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{d_{n-1}} & B_{n-2} \longrightarrow 0. \end{array}$$

By definition of Z_n and B_n , the restriction of the **differentials** to these groups **vanish**. This implies that (Z_*, d) and (B_*, d) are chain complexes (with trivial differentials).

Hence we get a short exact sequence of chain complexes

$$(28) \quad 0 \rightarrow Z_* \rightarrow C_* \rightarrow B_{*-1} \rightarrow 0.$$

Since all groups in these chain complexes are **free**, tensoring with M yields again a **short exact sequence of chain complexes**

$$0 \rightarrow Z_* \otimes M \rightarrow C_* \otimes M \rightarrow B_{*-1} \otimes M \rightarrow 0.$$

This can be checked as in the above lemma on direct sums of exact sequences.

Since the **differentials** in Z_* and B_* are **trivial**, the associated long exact sequence in homology looks like

$$\cdots \rightarrow B_n \otimes M \xrightarrow{\partial_n} Z_n \otimes M \rightarrow H_n(C_* \otimes M) \rightarrow B_{n-1} \otimes M \xrightarrow{\partial_{n-1}} Z_{n-1} \otimes M \rightarrow \cdots$$

The connecting homomorphism $B_n \otimes M \xrightarrow{\partial_n} Z_n \otimes M$ in this sequence is $i_n \otimes 1$, where $i_n: B_n \hookrightarrow Z_n$ denotes the inclusion and 1 denotes the identity on M . This can be easily checked using the definition of the connecting homomorphism.

A long exact sequence can always be **cut into short exact sequences** of the form

$$0 \rightarrow \operatorname{Coker}(i_n \otimes 1) \rightarrow H_n(C_* \otimes M) \rightarrow \operatorname{Ker}(i_{n-1} \otimes 1) \rightarrow 0.$$

Since the **tensor product preserves cokernels**, the cokernel on the left-hand side is just

$$\operatorname{Coker}(i_n \otimes 1) = \operatorname{Coker}(i_n) \otimes M = Z_n/B_n \otimes M = H_n(C_*) \otimes M.$$

For $\operatorname{Ker}(i_{n-1} \otimes 1)$, we observe that

$$B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \rightarrow H_{n-1}(C_*) \rightarrow 0$$

is a free resolution of $H_{n-1}(C_*)$. Hence after tensoring with M we get an **exact sequence**

$$0 \rightarrow \operatorname{Ker}(i_{n-1} \otimes 1) \rightarrow B_{n-1} \otimes M \xrightarrow{i_{n-1} \otimes 1} Z_{n-1} \otimes M \rightarrow H_{n-1}(C_*) \otimes M \rightarrow 0.$$

Thus, since Tor is **independent of the chosen free resolution**,

$$\operatorname{Ker}(i_{n-1} \otimes 1) = \operatorname{Tor}(H_{n-1}(C_*), M).$$

Finally, to obtain the asserted **splitting** we use that subgroups of free abelian groups are free. That implies that sequence (28) splits and we have

$$C_n \cong Z_n \oplus B_{n-1}.$$

Tensoring with M yields

$$C_n \otimes M \cong (Z_n \otimes M) \oplus (B_{n-1} \otimes M).$$

Now one has to work a little bit more to get that this induces a direct sum decomposition in homology. We skip this here. \square

Since the **singular chain complex** $S_*(X, A)$ is an example of a chain complex of **free** abelian groups, the theorem implies:

Corollary: UCT for singular homology

For each pair of spaces (X, A) there are split short exact sequences

$$0 \rightarrow H_n(X, A) \otimes M \rightarrow H_n(X, A; M) \rightarrow \operatorname{Tor}(H_{n-1}(X, A), M) \rightarrow 0$$

for all n , and these sequences are natural with respect to maps of pairs $(X, A) \rightarrow (Y, B)$.

One of the goals of introducing coefficients is to simplify calculations. The simplest case is often when we consider a field as coefficients. For example, the finite fields \mathbb{F}_p or the rational numbers \mathbb{Q} . The UCT tells how we can recover integral homology from these pieces. We will figure out how this works in the [exercises](#).

Since we put so much work into defining Tor , let us mention another important theorem. It tells us how the homology of the product of two spaces depends on the homology of the individual spaces. For that relation is not as straight forward as one might hope:

Künneth Theorem

For any pair of spaces X and Y and every n , there is a split short exact sequence

$$0 \rightarrow \bigoplus_{p+q=n} (H_p(X) \otimes H_q(Y)) \rightarrow H_n(X \times Y) \rightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(X), H_q(Y)) \rightarrow 0.$$

This sequence is natural in X and Y . But the splitting is not natural.

The maps $H_p(X) \rightarrow H_n(X \times Y)$ and $H_q(Y) \rightarrow H_n(X \times Y)$ arise from the cross product construction on singular chains. We will not have time to discuss this in class though.

LECTURE 18

Singular cohomology

We are going to define a **new algebraic invariant**, called singular cohomology. At first glance it might look almost the same as homology, but we will see that there is a **striking difference** between homology and cohomology. For, singular cohomology allows us to define an additional algebraic structure: **multiplication**.

As a **motivation**, we start with the following familiar situation. Recall that in **calculus**, we learn to calculate path integrals. Given a path $\gamma: [a,b] \rightarrow \mathbb{R}^2$ and a 1-form $pdx + qdy$. Then we can form the integral $\int_{\gamma} pdx + qdy$, and we learned all kinds of things about it.

In particular, we can **consider taking the integral as a map**

$$\gamma \mapsto \int_{\gamma} pdx + qdy \in \mathbb{R}.$$

Since any path γ can be reparametrized to a 1-simplex, we can think of taking the integral of a given 1-form over a path as a map

$$\text{Sing}_1(\mathbb{R}^2) \rightarrow \mathbb{R}.$$

This map captures certain geometric and topological information. It is an important example of a 1-cochain, a concept we are now going to define.

Definition: Singular cochains

Let X be a topological space and let M be an abelian group. An **n -cochain on X with values in M** is a function

$$\text{Sing}_n(X) \rightarrow M.$$

We turn the set

$$S^n(X; M) := \text{Map}(\text{Sing}_n(X), M)$$

of n -cochains into a **group** by defining $c + c'$ to be the function which sends σ to $c(\sigma) + c'(\sigma)$.

- As an **example**, let us look at the case $M = \mathbb{Z}$. Then an n -cochain on X is just a **function** which assigns to any n -simplex $\sigma: \Delta^n \rightarrow X$ a number in \mathbb{Z} .

We know that simplices in different dimensions are connected via the face maps. As for chains, the face maps induce an operator between cochains in different dimensions. But note that, for cochains, the degree will **increase instead of decrease**.

Definition: Coboundaries

The **coboundary operator**

$$\delta: S^n(X; M) \rightarrow S^{n+1}(X; M) \quad \delta(c)(\sigma) = c(\partial\sigma)$$

is defined as follows:

Given an n -cochain c and an $n+1$ -simplex $\sigma: \Delta^{n+1} \rightarrow X$. Then we define the $n+1$ -cochain $\delta(c)$ as

$$\delta^n(c)(\sigma) = c(\partial_{n+1}(\sigma)) = \sum_{i=0}^{n+1} (-1)^i c(\sigma \circ \phi_i^{n+1})$$

where ϕ_i^{n+1} is the i th face map. This defines $\delta^n(c)$ as a function on $\text{Sing}_{n+1}(X)$.

- For an **example**, let us look again at the case $m = \mathbb{Z}$. We learned that an n -cochain on X is a **function** which assigns to any n -simplex $\sigma: \Delta^n \rightarrow X$ a number in \mathbb{Z} . In order to be an **n -cocycle**, the numbers assigned to the boundary of an $n+1$ -simplex cancel out (with the sign convention).

To get more concrete, let $c \in S^1(X; \mathbb{Z})$ be a 1-cochain. Let $\sigma: \Delta^2 \rightarrow X$ be a 2-simplex. Then, for c to be a cocycle, we need that the **numbers** it assigns to the faces of σ **cancel out** in the sense that

$$c(d_0\sigma) - c(d_1\sigma) + c(d_2\sigma) = 0.$$

- Let us have another look at the **example from calculus** we started with. A **function**

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

is a **0-cochain** on \mathbb{R}^2 with values in \mathbb{R} . For it assigns to each zero-simplex, i.e., a point $x \in \mathbb{R}^2$, a real number $f(x)$.

Then the 1-cochain $\delta(f)$ is the function which assigns to a (smooth) path γ the number $f(\gamma(1)) - f(\gamma(0))$:

$$\delta(f): \gamma \mapsto f(\gamma(1)) - f(\gamma(0)).$$

By Green's Theorem, this is also the value of the integral

$$\int_{\gamma} f_x dx + f_y dy = \int_{\gamma} df$$

which is the integral of the 1-form df along γ .

Hence the cochain complex, while it looks very much like homology, also has a **natural connection to calculus**. In fact, there is some justification for saying that **cochains and cohomology are more natural** notions than chains and homology.

Back to the **general case**. The coboundary operator turns $S^*(X; M)$ into a **cochain complex**, since $\delta \circ \delta = 0$ which follows from our previous calculation. For, given an $n + 1$ -simplex σ and an $n - 1$ -cochain c , we get

$$(\delta^n \circ \delta^{n-1}(c))(\sigma) = (\delta^n c)(\partial_n(\sigma)) = c(\partial_n \circ \partial_{n+1}(\sigma)) = 0.$$

An **equivalent way** to obtain this complex, is to look at **homomorphisms** of abelian groups from $S_n(X)$ to M , i.e., we have

$$S^n(X; M) = \text{Hom}_{\mathbf{Ab}}(S_n(X), M).$$

The coboundary operator is just the homomorphism induced by the boundary operator on chains:

$$\delta = \text{Hom}(\partial, M): \text{Hom}_{\mathbf{Ab}}(S_n(X), M) \rightarrow \text{Hom}_{\mathbf{Ab}}(S_{n+1}(X), M), \quad c \mapsto c \circ \partial.$$

In other words, $\delta = \partial^*$ **equals the pullback along ∂** .

The subgroup given as the **kernel** of δ^n is denoted by

$$Z^n(X; M) = \text{Ker}(\delta: S^n(X; M) \rightarrow S^{n+1}(X; M))$$

and called the group of n -**cocycles** of X .

The **image** of δ^{n-1} is called the group of n -**coboundaries** of X and is denoted by

$$B^n(X; M) = \text{Im}(\delta^{n-1}: S^{n-1}(X; M) \rightarrow S^n(X; M)).$$

Since $\delta \circ \delta = 0$, we have

$$B^n(X; M) \subseteq Z^n(X; M).$$

In other words, every coboundary is a cocycle.

Definition: Singular cohomology

Let X be a topological space and let M be an abelian group. The **n th singular cohomology group of X** is defined as the n th cohomology group of the cochain complex $S^*(X; M)$, i.e.,

$$H^n(X; M) = \frac{Z^n(X; M)}{B^n(X; M)}.$$

Integrals over forms yield elements in cohomology with coefficients in \mathbb{R} . This is in fact the origin of cohomology theory and is connected to **de Rham cohomology**. Though as natural as de Rham cohomology is, it has the drawback that we have to stick to coefficients in \mathbb{R} .

This demonstrates why it might be smart to take the detour via singular simplices and taking maps in chains. For **we gain the flexibility** to study singular cohomology with coefficients in an arbitrary abelian group.

As a first example, let us try to understand $H^0(X; M)$.

Cohomology in dimension zero

A 0-cochain is a function

$$c: \text{Sing}_0(X) \rightarrow M.$$

Since $\text{Sing}_0(X)$ is just the **underlying set** of X , a 0-cochain corresponds to just an arbitrary, that is **not necessarily continuous**, function

$$f: X \rightarrow M.$$

Now what does it mean for such a function to be a cocycle? To figure this out we need to calculate $\delta(f)$. Since $\delta(f)$ is defined on 1-simplices, let $\sigma: \Delta^1 \rightarrow X$ be a 1-simplex on X . The effect of $\delta(f)$ is to **evaluate f on the boundary** of σ :

$$\delta(f)(\sigma) = f(\partial\sigma) = f(\sigma(e_0)) - f(\sigma(e_1)).$$

Since this expression must be 0 for every 1-simplex, we deduce that f is a **cocycle** if and only if it is **constant on the path-components** of X .

If we denote by $\pi_0(X)$ the set of path-components, then we have shown:

$$H^0(X; M) = \text{Map}(\pi_0(X), M).$$

Cohomology of a point

If X is just a point, then $\text{Sing}_n(\text{pt})$ consists just of the **constant map** for each n . Hence an n -cochain $c \in S^n(\text{pt}; M)$ is **completely determined by its value m_c** on the constant map, and therefore $\text{Hom}(\text{Sing}_n(\text{pt}), M) \cong M$ for all n .

The coboundary operator takes $c \in \text{Hom}(\text{Sing}_n(\text{pt}), M)$ to the alternating sum

$$\delta(c) = \sum_{i=0}^{n+1} (-1)^i c(\text{constant map} \circ \phi_i^n) = \sum_{i=0}^{n+1} (-1)^i m_c.$$

Hence the coboundary is **trivial if n is even** and the **identity if n is odd**. The cochain complex therefore looks like

$$0 \rightarrow M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} \dots$$

Thus the **cohomology of a point** is given by

$$H^n(\text{pt}; M) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

Now let us see what else we know about singular cohomology.

Properties of singular cohomology

Fix an abelian group M . Singular cohomology has the following properties:

- **Cohomology is contravariant**, i.e., a continuous map $f: X \rightarrow Y$ induces a homomorphism

$$f^*: S^*(Y; M) \rightarrow S^*(X; M).$$

This map works as follows: Let $c \in S^n(Y; M)$ be an n -cochain on Y . Then f^*c is the map which assigns to n -simplex $\sigma: \Delta^n \rightarrow X$, the value

$$(f^*c)(\sigma) = c(f \circ \sigma) = c(\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y).$$

Since f^* is in fact a map of cochain complexes (which is defined in analogy to maps of chain complexes), this induces a homomorphism on cohomology

$$f^*: H^*(Y; M) \rightarrow H^*(X; M).$$

This assignment is functorial, i.e., the identity map is sent to the identity homomorphism of cochains and if $g: Y \rightarrow Z$ is another map, then

$$(g \circ f)^* = f^* \circ g^*.$$

- In our **calculus example**, the **contravariance corresponds to restriction** of a form to an open subspace.

Why cohomology?

At first glance it seems like cohomology and homology are the same guys, just wrapped up in slightly different cloths and reversing the arrows. In fact, this is kind of true as we will see in the next lecture. However, there is also a striking difference which is due to the innocent looking fact that cohomology is contravariant. We are going to exploit this fact as follows:

Assume that R is a ring, and let

$$X \xrightarrow{\Delta} X \times X, x \mapsto (x, x)$$

be the **diagonal map**. It induces a homomorphism in cohomology

$$H^*(X \times X; R) \xrightarrow{\Delta^*} H^*(X; R).$$

Now we only need to construct a suitable map $H^p(X; R) \otimes H^q(X; R) \rightarrow H^{p+q}(X \times X; R)$ to get a multiplication on the direct sum $H^*(X; R) = \bigoplus_q H^q(X; R)$:

$$H^p(X; R) \otimes H^q(X; R) \rightarrow H^{p+q}(X \times X; R) \xrightarrow{\Delta^*} H^{p+q}(X; R).$$

It will still take some effort to make this idea work. Nevertheless, this gives us a first idea of how contravariance can be useful.

- **Cohomology is homotopy-invariant**, i.e., if the maps f and g are **homotopic** $f \simeq g$, then they **induce the same map in cohomology** $f^* = g^*$.

In fact, the **proof** we used for homology **dualizes to cohomology**.

For, recall that a homotopy between f and g induces a chain homotopy between h between the maps f_* and g_* on singular chain complexes. Now we use that the singular cochain complex is the value of the singular chain complex under the functor $\text{Hom}(-, M)$.

Then $\text{Hom}(h, M)$ is a homotopy between the maps of cochain complexes

$$f^* = \text{Hom}(f_*, M) \text{ and } g^* = \text{Hom}(g_*, M).$$

For the relation

$$f_* - g_* = h \circ \partial + \partial \circ h$$

implies the relation

$$f^* - g^* = \delta \circ \text{Hom}(h, M) + \text{Hom}(h, M) \circ \delta.$$

- If A is a **subspace** of X , then there are also **relative cohomology** groups. Let $i: A \hookrightarrow X$ denote the inclusion map. We consider the cochain complex

$$S^*(X, A; M) = \text{Ker}(S^*(X; M) \xrightarrow{i^*} S^*(A; M))$$

consisting of **those maps** $\text{Sing}_n(X) \rightarrow M$ **which vanish on the subset** $\text{Sing}_n(A)$.

The n th **relative cohomology** group is defined as the cohomology of this cochain complex

$$H^n(X, A; M) = H^n(S^*(X, A; M)).$$

By definition, there is a **short exact sequence**

$$0 \rightarrow S^*(X, A; M) \rightarrow S^*(X; M) \rightarrow S^*(A; M) \rightarrow 0$$

which induces a long exact sequence of the cohomology groups of the complexes in the same way as this was the case for chain complexes and homology:

$$\cdots \rightarrow H^n(X, A; M) \rightarrow H^n(X; M) \rightarrow H^n(A; M) \xrightarrow{\partial^n} H^{n+1}(X, A; M) \rightarrow \cdots$$

- There is also a **reduced version** of cohomology. Let $\epsilon: S_0(X; M) \rightarrow M$ be the **augmentation map** sending $\sum_i m_i \sigma_i$ to $\sum_i m_i \in M$. Since we know $\partial_0 \circ \epsilon = 0$, we observe that applying the functor $\text{Hom}(-, M)$ yields the **augmented singular cochain complex**

$$0 \rightarrow M \xrightarrow{\epsilon^*} S^0(X; M) \xrightarrow{\delta^0} S^1(X; M) \xrightarrow{\delta^1} \cdots$$

The **reduced cohomology** of X with coefficients in M is the cohomology of the augmented singular cochain complex.

- **Cohomology satisfies Excision**, i.e., if $Z \subset A \subset X$ with $\bar{Z} \subset A^\circ$, then the inclusion $k: (X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism

$$k^*: H^*(X, A; M) \xrightarrow{\cong} H^*(X - Z, A - Z; M).$$

- **Cohomology sends sums to products**, i.e.,

$$H^*\left(\coprod_{\alpha} X_{\alpha}; M\right) \cong \prod_{\alpha} H^*(X_{\alpha}; M).$$

- **Cohomology has Mayer-Vietoris sequences**, i.e., if $\{A, B\}$ is a **cover** of X , then, for every n , there are connecting homomorphisms d which fit into a long exact sequence

$$\cdots \xrightarrow{d} H^n(X; M) \xrightarrow{\begin{bmatrix} i_A^* \\ -i_B^* \end{bmatrix}} H^n(A; M) \oplus H^n(B; M) \xrightarrow{\begin{bmatrix} j_A^* & j_B^* \end{bmatrix}} H^n(A \cap B; M) \xrightarrow{d} H^{n+1}(X; M) \rightarrow \cdots$$

Note that the maps go in the **other direction** and the **degree** of the connecting homomorphism **increases**.

Here we used the inclusion maps

$$\begin{array}{ccc} A \cap B & \xhookrightarrow{j_A} & A \\ j_B \downarrow & & \downarrow i_A \\ B & \xhookrightarrow{i_B} & X. \end{array}$$

To **prove**, for example, the statement about **Mayer-Vietoris sequences**, let us go back to the proof in homology.

Let $\mathcal{A} = \{A, B\}$ be our cover. We used a short exact sequence of chain complexes

$$0 \rightarrow S_*(A \cap B) \rightarrow S_*(A) \oplus S_*(B) \rightarrow S_*^{\mathcal{A}}(X) \rightarrow 0$$

where $S_*^{\mathcal{A}}(X)$ denotes the \mathcal{A} -small chains.

We would like to **turn this** into an exact sequence **in cohomology**. As we have learned last time, **not all functors preserve exactness**.

And, in fact, $\text{Hom}(-, M)$ is **unfortunately no exception**. We will study the behaviour of Hom next time, but for the present purpose we observe a fact which saves our day.

For, the singular chain complexes involved in the above short exact sequence consist of **free** abelian groups in each dimension. And exactness is indeed preserved by Hom for such sequences.

More precisely, we would like to use the following fact:

Lemma: Exactness of Hom -functor on free complexes

Let M be an abelian group and let

$$0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$$

be an **exact** sequence of chain complexes of **free** abelian groups. Then the induced sequence of cochain complexes

$$0 \rightarrow \text{Hom}(C_*, M) \rightarrow \text{Hom}(B_*, M) \rightarrow \text{Hom}(A_*, M) \rightarrow 0$$

is **exact**.

Proof: By definition of exactness for sequences of complexes, it **suffices** to show the assertion for a short exact sequence of **free abelian groups**.

The key is that any short exact sequence of free abelian groups **splits**. The splitting induces a splitting on the induced sequence of Hom -groups.

More concretely, let

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

be a short exact sequence of **free abelian groups**.

Since C is **free** and p is **surjective**, there is a **dotted lift** in the solid diagram

$$\begin{array}{ccc} & B & \\ & \downarrow p & \\ C & \xleftarrow{s} & C \end{array}$$

which makes the diagram commute, i.e., $p \circ s = 1_C$.

This implies

$$s^* \circ p^* = 1_{\text{Hom}(C, M)},$$

and hence s^* is a **section of p^*** in

$$0 \rightarrow \text{Hom}(C, M) \xrightarrow{p^*} \text{Hom}(B, M) \xrightarrow{i^*} \text{Hom}(A, M) \rightarrow 0.$$

This implies that $\text{Hom}(B, M) = \text{Hom}(A, M) \oplus \text{Hom}(C, M)$ and that i^* is surjective. That the sequence is exact at $\text{Hom}(B, M)$ is now obvious as well. **QED**

Warning: Note that if $A = \mathbb{Z}[S]$ is a free abelian group, then

$$\mathrm{Hom}(A, M) = \mathrm{Hom}\left(\bigoplus_S \mathbb{Z}, M\right) = \prod_S \mathrm{Hom}(\mathbb{Z}, M)$$

which **might be an uncountable product**. This leads to the annoying fact that $\mathrm{Hom}(A, M)$ is not a free abelian group, in general.

Back to the proof of the MVS, with this fact at hand we get an induced short exact sequence of cochain complexes

$$0 \rightarrow \mathrm{Hom}(S_*^A(X), M) \rightarrow \mathrm{Hom}(S_*(A), M) \oplus \mathrm{Hom}(S_*(B), M) \rightarrow \mathrm{Hom}(S_*(A \cap B), M) \rightarrow 0$$

where we also use that Hom commutes with direct sums.

Again, such a short exact sequence of cochain complexes **induces a long exact sequence** of cohomology groups.

The **final step** of the proof is that we **need to check** that the induced map of cochain complexes

$$\mathrm{Hom}(S_*(X), M) \rightarrow \mathrm{Hom}(S_*^A(X), M)$$

induces an isomorphism in cohomology.

In fact, this follows from the Small Chain Theorem and the following fact:

Proposition: From isos in homology to isos in cohomology

Let C_* and D_* be two chain complexes of **free** abelian groups.

Assume that there is a map $C_* \xrightarrow{\varphi} D_*$ which induces an **isomorphism in homology**

$$\varphi_*: H_*(C_*) \xrightarrow{\cong} H_*(D_*).$$

Then, for any abelian group M , the map φ

$$\varphi^*: H^*(D; M) \rightarrow H^*(C; M)$$

induces an **isomorphism in cohomology** with coefficients in M as well.

Here we wrote $H^*(C; M) = H^*(\mathrm{Hom}(C_*, M))$ and $H^*(D; M) = H^*(\mathrm{Hom}(D_*, M))$ for the cohomology of the induced cochain complexes.

We are going to deduce this result from the **Universal Coefficient Theorem in cohomology** which we will prove in the next lecture. Roughly speaking, it will tell us how homology and cohomology are related.

As a first approach, we observe the following phenomenon.

The Kronecker pairing

Let M be an abelian group. For a chain complex C_* and cochain complex $C^* := \text{Hom}(C_*, M)$ there is a natural pairing given by evaluating a cochain on chains:

$$\langle -, - \rangle: C^n \otimes C_n \rightarrow M, (\varphi, a) \mapsto \langle \varphi, a \rangle := \varphi(a).$$

This is called the **Kronecker pairing**.

The boundary and coboundaries are compatible with this pairing, i.e.,

$$\langle \delta\varphi, a \rangle = \delta(\varphi)(a) = \varphi(\partial(a)) = \langle \varphi, \partial a \rangle.$$

Lemma: Kronecker pairing

The Kronecker pairing induces a well-defined pairing on the level of cohomology and homology, i.e., we get an induced pairing

$$\langle -, - \rangle: H^n(C^*) \otimes H_n(C_*) \rightarrow M.$$

Proof: Let φ be a **cocycle**, i.e., $\delta\varphi = 0$. Then we get

$$\langle \varphi, a + \partial b \rangle = \langle \varphi, a \rangle + \langle \varphi, \partial b \rangle = \langle \varphi, a \rangle + \langle \delta\varphi, b \rangle = \langle \varphi, a \rangle.$$

Thus, the map $\langle \varphi, - \rangle$ **descends to homology** if φ is a cocycle.

It remains to check that this map **vanishes** if φ is a **coboundary**. So assume $\varphi = \delta\psi$ and a is a **cycle**, i.e., $\partial a = 0$. Then we get

$$\langle \varphi, a \rangle = \langle \delta\psi, a \rangle = \langle \psi, \partial a \rangle = 0.$$

This shows that the pairing is well-defined on $H^n(C^*)$ and $H_n(C_*)$. \square

Kronecker homomorphism

Thus, applied to the integral singular chain complex and the cochain complex with coefficients in M , this pairing yields a natural homomorphism

$$\kappa: H^n(X; M) \rightarrow \text{Hom}(H_n(X), M),$$

which sends the class $[c]$ of a cocycle to the homomorphism $\kappa([c])$ defined by

$$\kappa([c]): H_n(X) \rightarrow M, [\sigma] \mapsto \langle c, \sigma \rangle = c(\sigma).$$

This leads to the important question:

From homology to cohomology?

If we know the singular homology of a space, what can we deduce about its cohomology?

More concretely, what can we say about the map κ ? Is κ injective? Is κ surjective?

LECTURE 19

Ext and the Universal Coefficient Theorem for cohomology

In the previous lecture, we introduced the singular cochain complex and defined singular cohomology. Along the way we ran into some exact sequences to which we applied the Hom-functor. In particular, we constructed the Kronecker map

$$\kappa: H^n(X; M) \rightarrow \text{Hom}(H_n(X), M).$$

Our goal for this lecture is to study the Hom-functor in more detail and to prove the **Universal Coefficient Theorem** for singular cohomology which will tell us that κ is surjective. However, κ is not injective in general, but the UCT will tell us what the kernel is.

Again, for some this will be a review of known results in homological algebra. Nevertheless, those who have not seen this before, should get a chance to catch up.

We will again focus on the main ideas.

Let M be an abelian group. We would like to understand the effect of the functor $\text{Hom}(-, M)$ on exact sequences.

Before we start, note that Hom is not symmetric in general, i.e., $\text{Hom}(A, M)$ and $\text{Hom}(M, A)$ might be very different indeed. For example,

$$\text{Hom}(\mathbb{Z}, \mathbb{Z}/n) \cong \mathbb{Z}/n, \text{ but } \text{Hom}(\mathbb{Z}/n, \mathbb{Z}) = 0.$$

Our next observation tells us that Hom is left-exact:

Lemma: Hom is left-exact

(a) Let M be an abelian group. Suppose we have an exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0.$$

Then applying $\text{Hom}(-, M)$ yields an exact sequence

$$0 \rightarrow \text{Hom}(C, M) \xrightarrow{j^*} \text{Hom}(B, M) \xrightarrow{i^*} \text{Hom}(A, M).$$

In other words, the functor $\text{Hom}(-, M)$ is **left-exact** and sends **cokernels to kernels**.

(b) Similarly, applying $\text{Hom}(M, -)$ to an exact sequence of the form

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

yields an exact sequence

$$0 \rightarrow \text{Hom}(M, A) \xrightarrow{i_*} \text{Hom}(M, B) \xrightarrow{j_*} \text{Hom}(M, C).$$

In other words, the functor $\text{Hom}(M, -)$ is **left-exact** and sends **kernels to kernels**.

Proof: (a) To show that j^* is **injective**, assume that $\gamma \in \text{Hom}(C, M)$ satisfies $j^*(\gamma) = 0$. That means

$$j^*(\gamma)(b) = (\gamma \circ j)(b) = \gamma(j(b)) = 0 \text{ for all } b \in B.$$

But j is **surjective**, and hence every element in C is of the form $j(b)$ for some $b \in B$. Hence $\gamma = 0$ is the trivial homomorphism.

The **composition** $i^* \circ j^*$ is clearly 0, since $j \circ i = 0$ by assumption. Thus $\text{Im}(j^*) \subseteq \text{Ker}(i^*)$.

Now if $\beta \in \text{Hom}(B, M)$ is in $\text{Ker}(i^*)$, then

$$0 = i^*(\beta)(a) = \beta(i(a)) \text{ for all } a \in A.$$

In other words, β is trivial on the image of i and hence factors as

$$\beta: B \rightarrow B/\text{Im}(i) \rightarrow M.$$

But $B/\text{Im}(i) \cong C$, since the initial sequence was exact. Hence β is the composition of a map $B \xrightarrow{j} C \xrightarrow{\gamma} M$ for some $\gamma \in \text{Hom}(C, M)$. Thus, $\beta \in \text{Im}(j^*)$.

(b) The proof is of course similar. To show that i_* is **injective**, let $\alpha \in \text{Hom}(M, A)$ be a map such that $i_*(\alpha) = 0$. That means

$$i_*(\alpha(m)) = i(\alpha(m)) = 0 \text{ for all } m \in M.$$

Since i is **injective**, this implies $\alpha(m) = 0$ for all $m \in M$, and hence $\alpha = 0$.

The **composition** $j_* \circ i_*$ is clearly 0, since $j \circ i = 0$ by assumption. Thus $\text{Im}(i_*) \subseteq \text{Ker}(j_*)$.

If $\beta \in \text{Hom}(M, B)$ is in $\text{Ker}(j_*)$, then

$$0 = j_*(\beta)(m) = j(\beta(m)) \text{ for all } m \in M.$$

In other words, $\beta(m) \in \text{Ker}(j)$ for all $m \in M$. Since $\text{Ker}(j) = \text{Im}(i)$, we get $\beta(m) \in \text{Im}(i)$ for all $m \in M$. Hence β factors as

$$\beta: M \xrightarrow{\alpha} A \xrightarrow{i} B$$

for some $\alpha \in \text{Hom}(M, A)$. Thus, $\beta \in \text{Im}(i_*)$. **QED**

However, suppose we have an injective homomorphism

$$A \hookrightarrow B.$$

Then it is in general **not the case** that the induced map

$$\text{Hom}(B, M) \rightarrow \text{Hom}(A, M)$$

is **surjective**.

For example, take the map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ given by multiplication by 2. It is clearly **injective**. But if we apply $\text{Hom}(-, \mathbb{Z}/2)$, we get the map

$$\text{Hom}(\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2 \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}/2)$$

which is **not surjective**.

We would like to remedy this defect. And we can already guess how this can be achieved. As we have seen in the previous lecture, $\text{Hom}(-, M)$ is **not so far from being exact**. For, if we apply $\text{Hom}(-, M)$ to a short exact sequence of **free** abelian groups, then the induced sequence is still short exact.

So let A be an abelian group and let us choose a free resolution of A as in a previous lecture

$$0 \rightarrow F_1 \hookrightarrow F_0 \twoheadrightarrow A.$$

Applying $\text{Hom}(-, M)$ to this sequence yields an exact sequence

$$0 \rightarrow \text{Hom}(A, M) \rightarrow \text{Hom}(F_0, M) \rightarrow \text{Hom}(F_1, M).$$

The right-hand map is not necessarily surjective, or in other words, the **cokernel** of the right-hand map **is not necessarily zero**.

This leads to the following important definition:

Definition: Ext

The **cokernel** of the map $\text{Hom}(F_0, M) \rightarrow \text{Hom}(F_1, M)$ is called $\text{Ext}(A, M)$. Hence by definition we have an **exact sequence**

$$0 \rightarrow \text{Hom}(A, M) \rightarrow \text{Hom}(F_0, M) \rightarrow \text{Hom}(F_1, M) \rightarrow \text{Ext}(A, M) \rightarrow 0.$$

Roughly speaking, the group $\text{Ext}(A, -)$ measures how far $\text{Hom}(A, -)$ is from being exact.

Let us calculate **some examples**:

- Let $A = \mathbb{Z}/p$. Then we can take $F_0 = F_1 = \mathbb{Z}$ and

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$$

as a free resolution of \mathbb{Z}/p . For an abelian group M , the sequence defining Ext looks like

$$0 \rightarrow \text{Hom}(\mathbb{Z}/p, M) \rightarrow \text{Hom}(\mathbb{Z}, M) \xrightarrow{p} \text{Hom}(\mathbb{Z}, M) \rightarrow \text{Ext}(\mathbb{Z}/p, M) \rightarrow 0.$$

Since $\text{Hom}(\mathbb{Z}, M) = M$, this sequence equals

$$0 \rightarrow \text{p-torsion in } M \rightarrow M \xrightarrow{p} M \rightarrow \text{Ext}(\mathbb{Z}/p, M) \rightarrow 0.$$

Thus

$$\text{Ext}(\mathbb{Z}/p, M) = \text{Coker}(M \xrightarrow{p} M) = M/pM.$$

- For a concrete case, let us calculate $\text{Ext}(\mathbb{Z}/2, \mathbb{Z}/2)$. We use the free resolution

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Applying $\text{Hom}(-, \mathbb{Z}/2)$ yields

$$0 \rightarrow \text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) \rightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}/2) \xrightarrow{2} \text{Hom}(\mathbb{Z}, \mathbb{Z}/2).$$

This sequence is isomorphic to

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2.$$

Since $2 = 0$ in $\mathbb{Z}/2$, the second map is trivial. Hence the cokernel of this map is just $\mathbb{Z}/2$. Thus

$$\text{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2.$$

- More generally, one can show

$$\text{Ext}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(n, m)$$

where $\gcd(n, m)$ denotes the greatest common divisor of n and m .

Now we should study Ext in more detail. As a first step we show that it can be viewed as a cohomology group:

Lemma: Ext and Hom as cohomology groups

Let A and M be abelian groups and $0 \rightarrow F_1 \xrightarrow{j} F_0 \rightarrow A \rightarrow 0$ be a free resolution of A . Consider the cochain complex $\text{Hom}(F_*, M)$ given by

$$0 \rightarrow \text{Hom}(F_0, M) \xrightarrow{j^*} \text{Hom}(F_1, M) \rightarrow 0$$

with $\text{Hom}(F_0, M)$ in dimension zero and $\text{Hom}(F_1, M)$ in dimension one. Then we have

$$H^0(\text{Hom}(F_*, M)) = \text{Hom}(A, M) \text{ and } H^1(\text{Hom}(F_*, M)) = \text{Ext}(A, M).$$

Proof: By definition, $\text{Ext}(A, M)$ is the cokernel of j^* . Since the differential out of $\text{Hom}(F_0, M)$ is trivial, the first cohomology is just

$$H^1(\text{Hom}(F_*, M)) = \text{Hom}(F_0, M) / \text{Im}(j^*) = \text{Coker}(j^*) = \text{Ext}(A, M).$$

For H^0 we remember that the augmented sequence

$$0 \rightarrow \text{Hom}(A, M) \rightarrow \text{Hom}(F_0, M) \xrightarrow{j^*} \text{Hom}(F_1, M)$$

is exact.

Hence $\text{Hom}(A, M)$ is isomorphic to its image in $\text{Hom}(F_0, M)$ which is, by exactness of the sequence, the kernel of j^* . But this kernel is the cohomology group of $\text{Hom}(F_*, M)$ in dimension 0:

$$H^0(\text{Hom}(F_*, M)) = \text{Ker}(j^*) = \text{Hom}(A, M)$$

□

We should check that Ext does not depend on the choice of a free resolution. To do this, we are going to apply the lemma we proved for the Tor-case which states that maps can be lifted to resolutions and any two lifts are chain homotopic in a suitable sense.

Proposition: Ext is independent of resolutions

Ext is independent of the choice of a free resolution: If $0 \rightarrow E_1 \xrightarrow{i} E_0 \rightarrow A$ and $0 \rightarrow F_1 \xrightarrow{j} F_0 \rightarrow A$ are two free resolutions of A , there is a unique

isomorphism

$$\operatorname{Coker}(\operatorname{Hom}(i, M)) \xrightarrow{\cong} \operatorname{Coker}(\operatorname{Hom}(j, M)).$$

Proof: We know from the result on lifting resolutions that we can **lift the identity map on A** to a map of resolutions

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i} & E_0 & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \parallel \\ 0 & \longrightarrow & F_1 & \xrightarrow{j} & F_0 & \longrightarrow & A \longrightarrow 0. \end{array}$$

The other way around we get a lift

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i} & E_0 & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow g_1 & & \downarrow g_0 & & \parallel \\ 0 & \longrightarrow & F_1 & \xrightarrow{j} & F_0 & \longrightarrow & A \longrightarrow 0. \end{array}$$

We write E_* for the complex $0 \rightarrow E_1 \rightarrow E_0 \rightarrow 0$ and F_* for the complex $0 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$.

Composition yields maps $f_* \circ g_*: E_* \rightarrow E_*$ and $g_* \circ f_*: F_* \rightarrow F_*$ which lift the identity map on A . But since the identity maps on E_* and F_* , respectively, also lift the identity on A , the lemma of a previous lecture implies that there is a chain homotopy h_E between $f_* \circ g_*$ and 1_{E_*} and a chain homotopy h_F between $g_* \circ f_*$ and 1_{F_*} .

Now we apply $\operatorname{Hom}(-, M)$. Then h_E induces a cochain homotopy $\operatorname{Hom}(h_E, M)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(E_0, M) & \longrightarrow & \operatorname{Hom}(E_1, M) & \longrightarrow & 0 \\ & & \downarrow 1_{\operatorname{Hom}(E_0, M)} & & \downarrow 1_{\operatorname{Hom}(E_1, M)} & & \\ & & \downarrow g_0^* \circ f_0^* & \nearrow h^* & \downarrow g_1^* \circ f_1^* & & \\ 0 & \longrightarrow & \operatorname{Hom}(E_0, M) & \longrightarrow & \operatorname{Hom}(E_1, M) & \longrightarrow & 0. \end{array}$$

between

$$\operatorname{Hom}(f_* \circ g_*, M) = \operatorname{Hom}(g_*, M) \circ \operatorname{Hom}(f_*, M) \text{ and } \operatorname{Hom}(1_{E_*}, M) = 1_{\operatorname{Hom}(E_*, M)}.$$

Whereas h_F induces a cochain homotopy $\text{Hom}(h_F, M)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(F_0, M) & \longrightarrow & \text{Hom}(F_1, M) & \longrightarrow & 0 \\ & & \downarrow 1_{\text{Hom}(F_0, M)} & & \downarrow 1_{\text{Hom}(F_1, M)} & & \\ & & f_0^* \circ g_0^* & \xleftarrow{h^*} & f_1^* \circ g_1^* & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Hom}(F_0, M) & \longrightarrow & \text{Hom}(F_1, M) & \longrightarrow & 0. \end{array}$$

between

$$\text{Hom}(g_* \circ f_*, M) = \text{Hom}(f_*, M) \circ \text{Hom}(g_*, M) \text{ and } \text{Hom}(1_{F_*}, M) = 1_{\text{Hom}(F_*, M)}.$$

Thus, the maps induced by the compositions on cohomology are equal to the respective identity maps. In other words, the induced maps f^* and g^* on cohomology are mutual inverses to each other.

Moreover, since the chain homotopy type of f_* and g_* is unique by the lemma of the lecture on Tor, they induce in fact a unique isomorphism

$$\text{Coker}(\text{Hom}(i, M)) = H^1(\text{Hom}(E_*, M)) \xrightarrow{\cong} H^1(\text{Hom}(F_*, M)) = \text{Coker}(\text{Hom}(j, M)).$$

QED

Lemma: Induced exact sequence

Let M be an abelian group and assume we have a short exact sequence of abelian groups

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0.$$

Then there is an associated long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(C, M) & \longrightarrow & \text{Hom}(B, M) & \longrightarrow & \text{Hom}(A, M) \\ & & & & \swarrow & & \\ & & \text{Ext}(C, M) & \longrightarrow & \text{Ext}(B, M) & \longrightarrow & \text{Ext}(A, M) \longrightarrow 0. \end{array}$$

Proof: Let $0 \rightarrow E_1 \rightarrow E_0 \rightarrow A \rightarrow 0$ be a free resolution of A , and $0 \rightarrow F_1 \rightarrow F_0 \rightarrow C \rightarrow 0$ be a free resolution of C . This data gives us a free resolution of B by forming direct sums:

$$0 \rightarrow E_1 \oplus F_1 \rightarrow E_0 \oplus F_0 \rightarrow B \rightarrow 0.$$

By the result of the previous lecture, we can lift the maps in the short exact sequence to maps of resolutions

$$\begin{array}{ccccccc}
 0 & \longrightarrow & E_1 & \cdots \cdots \longrightarrow & E_1 \oplus F_1 & \cdots \cdots \longrightarrow & F_1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & E_0 & \cdots \cdots \longrightarrow & E_0 \oplus F_0 & \cdots \cdots \longrightarrow & F_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The horizontal sequences are short exact, since the middle term is a direct sum of the other terms. Hence we get a short exact sequence of chain complexes

$$0 \rightarrow E_* \rightarrow E_* \oplus F_* \rightarrow F_* \rightarrow 0.$$

Since all three complexes consist of free abelian groups, applying $\text{Hom}(-, M)$ yields a short exact sequence of cochain complexes

$$0 \rightarrow \text{Hom}(F_*, M) \rightarrow \text{Hom}(E_* \oplus F_*, M) \rightarrow \text{Hom}(E_*, M) \rightarrow 0.$$

By taking cohomology of these cochain complexes, we get an induced long exact sequence of the associated cohomology groups. This is the desired exact sequence together with the identification of H^1 with Ext and H^0 with Hom of the previous lemma. \square

This lemma also gives a hint to where the name Ext comes from:

Ext and extensions

- We can think of a short exact sequence of abelian groups

$$0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$$

as an **extension of M by A** . We can then say that two extensions are **equivalent** if they fit into an isomorphism of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & M \longrightarrow 0 \\
 & & \parallel & & \downarrow \cong & & \parallel \\
 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & M \longrightarrow 0.
 \end{array}$$

- Note that we can always construct a trivial extension by taking the direct sum of A and M :

$$0 \rightarrow A \xrightarrow{(1,0)} A \oplus M \rightarrow M \rightarrow 0.$$

Recall that we say that such a sequence splits.

- The group $\text{Ext}(A, M)$ measures how far extensions of M by A can be from being from the trivial extension. For, we have

$$\text{Ext}(A, M) = 0 \iff \text{every extension of } M \text{ by } A \text{ splits.}$$

Proof: Given an extension, applying $\text{Hom}(-, M)$ yields an exact sequence

$$\text{Hom}(B, M) \rightarrow \text{Hom}(M, M) \rightarrow \text{Ext}(A, M).$$

Thus the identity map $M \xrightarrow{1} M$ lifts to a map $B \rightarrow M$ if $\text{Ext}(A, M) = 0$. But that is equivalent to that the initial short exact sequence splits. **QED**

- Now one can show in general that $\text{Ext}(A, M)$ is in bijection with the **set of all equivalence classes** of extensions of M by A .
- For example, we computed $\text{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$. The trivial element in Ext corresponds to the trivial extension

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow 0$$

whereas the non-trivial element corresponds to the extension

$$0 \rightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

We summarize some further **properties of Ext**:

- Ext is **functorial**: For any homomorphisms of abelian groups $A \rightarrow A'$ and $M \rightarrow M'$, there are homomorphisms

$$\text{Ext}(A', M) \rightarrow \text{Ext}(A, M) \text{ and } \text{Ext}(A, M) \rightarrow \text{Ext}(A, M').$$

This follows from the lemma on liftings of resolutions.

- If A is **free**, then $\text{Ext}(A, M) = 0$ for any abelian group A . This follows from the fact that $0 \rightarrow A \xrightarrow{1} A \rightarrow 0$ is a free resolution of A .
- Ext commutes with **finite** direct sums, i.e.,

$$\text{Ext}(A_1 \oplus A_2, M) \cong \text{Ext}(A_1, M) \oplus \text{Ext}(A_2, M).$$

This follows from the fact that the direct sum of free resolutions of each A_1 and A_2 is a free resolution of $A_1 \oplus A_2$.

- Let A be a finitely generated abelian group and let $T(A)$ denote its torsion subgroup. Since $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m$, the structure theorem for finitely generated abelian groups and the previous two points imply that

$$\text{Ext}(A, \mathbb{Z}) \cong T(A).$$

Now we prove the main result which connects homology and cohomology and answers the question we raised last time about the Kronecker map κ :

Theorem: Universal Coefficient Theorem

Let C_* be a chain complex of **free** abelian groups and let M be an abelian group. We write $C^* = \text{Hom}(C_*, M)$ for the induced cochain complex. Then there are natural short exact sequences

$$0 \rightarrow \text{Ext}(H_{n-1}(C_*), M) \rightarrow H^n(C^*) \xrightarrow{\kappa} \text{Hom}(H_n(C_*), M) \rightarrow 0$$

for all n . These sequences split, but the splitting is not natural.

The proof builds on the same ideas as for the UCT in homology. But let us do it anyway to get more practice.

Proof: • We write Z_n for the **kernel** and B_{n-1} for the **image** of the differential $d: C_n \rightarrow C_{n-1}$. Since C_n and C_{n-1} are **free**, both Z_n and B_{n-1} are **free as well**.

By definition of Z_n and B_n , the restriction of the **differentials** to these groups **vanish**. This implies that (Z_*, d) and (B_*, d) are chain complexes (with trivial differentials).

Hence we get a short exact sequence of chain complexes

$$(29) \quad 0 \rightarrow Z_* \rightarrow C_* \xrightarrow{d} B_{*-1} \rightarrow 0.$$

- Since all groups in these chain complexes are **free**, applying the functor $\text{Hom}(-, M)$ yields again a **short exact sequence of cochain complexes**

$$0 \rightarrow \text{Hom}(B_{*-1}, M) \rightarrow \text{Hom}(C_*, M) \rightarrow \text{Hom}(Z_*, M) \rightarrow 0.$$

This follows from the lemma we proved in the previous lecture.

- Since the **differentials** in Z_* and B_* are **trivial**, the n th cohomology of $\text{Hom}(B_{*-1}, M)$ is just $\text{Hom}(B_{n-1}, M)$, and the n th cohomology of $\text{Hom}(Z_*, M)$ is just $\text{Hom}(Z_n, M)$.

Hence the long exact sequence in cohomology associated to the short exact sequence (29) looks like

$$\cdots \rightarrow \operatorname{Hom}(Z_{n-1}, M) \xrightarrow{\partial} \operatorname{Hom}(B_{n-1}, M) \xrightarrow{d^*} H^n(\operatorname{Hom}(C_*, M)) \xrightarrow{i^*} \operatorname{Hom}(Z_n, M) \xrightarrow{\partial} \operatorname{Hom}(B_n, M) \rightarrow \cdots$$

• The **connecting homomorphism** $\operatorname{Hom}(Z_n, M) \xrightarrow{\partial} \operatorname{Hom}(B_n, M)$ in this sequence is $i_n^* = \operatorname{Hom}(i_n, M)$, where $i_n: B_n \hookrightarrow Z_n$ denotes the inclusion. For, the connecting homomorphism is defined as follows. Consider the maps

$$\begin{array}{ccc} \operatorname{Hom}(C_n, M) & \longrightarrow & \operatorname{Hom}(Z_n, M) \\ \downarrow \delta & & \\ \operatorname{Hom}(B_n, M) & \xrightarrow{\delta} & \operatorname{Hom}(C_{n+1}, M). \end{array}$$

A preimage of $\varphi \in \operatorname{Hom}(Z_n, M)$ is any map $\psi: C_n \rightarrow M$ which restricts to Z_n . Such a preimage exists since the upper horizontal map is surjective. Then ψ is mapped to $\psi \circ d \in \operatorname{Hom}(C_{n+1}, M)$ by δ . Since every boundary is a cycle, we have

$$\psi \circ d = \varphi \circ d.$$

Now it remains to find a map $\tilde{\varphi}: B_n \rightarrow M$ such that

$$\psi \circ d = \varphi \circ d = \tilde{\varphi} \circ d.$$

There is a canonical candidate for $\tilde{\varphi}$, namely the restriction of φ to B_n . This is exactly $i_n^*(\varphi)$.

• A long exact sequence can always be **cut into short exact sequences** of the form

$$0 \rightarrow \operatorname{Coker}(\operatorname{Hom}(i_{n-1}, M)) \rightarrow H_n(C^*) \rightarrow \operatorname{Ker}(\operatorname{Hom}(i_n, M)) \rightarrow 0.$$

Since the functor $\operatorname{Hom}(-, M)$ sends **cokernels to kernels**, the kernel on the right-hand side is just

$$\operatorname{Ker}(\operatorname{Hom}(i_n, M)) = \operatorname{Hom}(\operatorname{Coker}(i_n), M) = \operatorname{Hom}(Z_n/B_n, M) = \operatorname{Hom}(H_n(C_*), M).$$

For the cokernel on the left-hand side, we use that

$$B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \rightarrow H_{n-1}(C_*) \rightarrow 0$$

is a **free resolution** of $H_{n-1}(C_*)$.

Hence, after applying $\text{Hom}(-, M)$, we get an **exact sequence**

$$0 \rightarrow \text{Hom}(H_{n-1}(C_*), M) \rightarrow \text{Hom}(B_{n-1}, M) \xrightarrow{i_{n-1}^*} \text{Hom}(Z_{n-1}, M) \rightarrow \text{Coker}(\text{Hom}(i_{n-1}, M)) \rightarrow 0.$$

Thus, since $\text{Ext}(-, M)$ is **independent of the chosen free resolution**,

$$\text{Coker}(\text{Hom}(i_{n-1}, M)) = \text{Ext}(H_{n-1}(C_*), M).$$

Finally, to obtain the asserted **splitting** we use that subgroups of free abelian groups are free. That implies that sequence (29) splits and we have

$$C_n \cong Z_n \oplus B_{n-1}.$$

Applying $\text{Hom}(-, M)$ yields

$$\text{Hom}(C_n, M) \cong \text{Hom}(Z_n, M) \oplus \text{Hom}(B_{n-1}, M).$$

Now one has to work a little bit more to get that this induces a direct sum decomposition in homology.

It remains to check that the right-hand map in the theorem is in fact the previously defined map κ . We leave this as an exercise. **QED**

Now we can prove the result we claimed in the previous lecture:

Corollary: From isos in homology to isos in cohomology

Let C_* and D_* be two chain complexes of **free** abelian groups. Let M be an abelian group.

Assume that there is a map $C_* \xrightarrow{\varphi} D_*$ which induces an **isomorphism in homology**

$$\varphi_*: H_*(C_*) \xrightarrow{\cong} H_*(D_*).$$

Then this map also induces an **isomorphism in cohomology** with coefficients in M

$$\varphi^*: H^*(D^*) \xrightarrow{\cong} H^*(C^*).$$

Proof: Since the construction of the long exact sequence we used in the proof of the theorem is functorial, we see that φ induces a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Ext}(H_{n-1}(C_*), M) & \longrightarrow & H^n(C^*) & \longrightarrow & \text{Hom}(H_n(C_*), M) \longrightarrow 0 \\
 & & \downarrow (\varphi_*)^* & & \downarrow \varphi^* & & \downarrow (\varphi_*)^* \\
 0 & \longrightarrow & \text{Ext}(H_{n-1}(D_*), M) & \longrightarrow & H^n(D^*) & \longrightarrow & \text{Hom}(H_n(D_*), M) \longrightarrow 0
 \end{array}$$

The assumption that φ_* induces an isomorphism implies that the two outer vertical maps are isomorphisms. The Five-Lemma implies that the middle vertical map φ^* is an isomorphism as well. **QED**

Our previous observations about Ext and torsion subgroups together with the theorem imply:

Corollary: Computing cohomology from homology

Assume that the homology groups $H_n(C_*)$ and $H_{n-1}(C_*)$ of the chain complex are finitely generated. Let $T_n \subseteq H_n(C_*)$ and $T_{n-1} \subseteq H_{n-1}(C_*)$ denote the torsion subgroups. Then we can calculate the integral cohomology of $C^* = \text{Hom}(C_*, \mathbb{Z})$ by

$$H^n(C^*; \mathbb{Z}) \cong (H_n(C_*)/T_n) \oplus T_{n-1}.$$

Since the **singular chain complex** $S_*(X, A)$ is an example of a chain complex of **free** abelian groups, the theorem implies:

Corollary: UCT for singular cohomology

For each pair of spaces (X, A) there are split short exact sequences

$$0 \rightarrow \text{Ext}(H_{n-1}(X, A), M) \rightarrow H^n(X, A; M) \rightarrow \text{Hom}(H_n(X, A), M) \rightarrow 0$$

for all n , and these sequences are natural with respect to maps of pairs $(X, A) \rightarrow (Y, B)$.

As a final remark, we mention that there are versions of Ext for the category of R -modules for any ring. The corresponding Ext-groups $\text{Ext}_R(M, N)$ will depend on the ring R as well as on the modules M and N . Moreover, there might be non-trivial higher Ext-groups $\text{Ext}_R^i(M, N)$ for $i \geq 2$, in general.

But the theory is very similar to the case of abelian groups, i.e., \mathbb{Z} -modules, as long as R is a principal ideal domain (PID). For, then submodules of free R -modules are still free over R (which is not true in general). Hence free resolutions of length two exist, and higher Ext groups vanish also in this case.

For example, fields are examples of PIDs. However, note that, for example, $\text{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ whereas $\text{Ext}_{\mathbb{Z}/2}^1(\mathbb{Z}/2, \mathbb{Z}/2) = 0$. Hence the base rings matter.

LECTURE 20

Cup products in cohomology

We are now going to define the **additional algebraic structure** on cohomology that we promised earlier: multiplication.

There are many different ways to define a product structure in cohomology. As always, each of these ways has its advantages and disadvantages. We will take a direct path to the construction. This has the advantage to get a product right away. The price we are going to pay is that we will have to work harder for some results later. Note also that, even though we emphasized the importance of the diagonal map in a previous lecture, this will not become clear from our direct approach today. Though it matters nevertheless. :)

What we do take advantage of and which would not work for singular chains is that a cochain is by definition a map to a ring. So we can multiply images of cochains. Hence we could try to multiply cochains pointwise. We will just need to figure out the images of which points we need to multiply.

We need to assume that we work with coefficients in a ring R . We will always assume that R is commutative and that there is a neutral element 1 for multiplication (even though not all arguments require all these assumptions). Our main examples will be, of course, \mathbb{Z} , \mathbb{Z}/n , \mathbb{Q} .

Definition: Cup products

For cochains $\varphi \in S^p(X; R)$ and $\psi \in S^q(X; R)$, we define the **cup product** $\varphi \cup \psi \in S^{p+q}(X; R)$ to be the cochain whose value on the $p + q$ -simplex $\sigma: \Delta^{p+q} \rightarrow X$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[e_0, \dots, e_p]})\psi(\sigma|_{[e_p, \dots, e_{p+q}]})$$

where the product is taken in R (here it comes already quite handy that we work with coefficients in a ring).

Note: The symbol $\sigma|_{[e_0, \dots, e_p]}$ refers to the restriction of σ to the **front face** of Δ^{p+q}

$$\sigma|_{[e_0, \dots, e_p]}: \Delta^p \hookrightarrow \Delta^{p+q} \xrightarrow{\sigma} X, \quad \sigma|_{[e_0, \dots, e_p]}(t_0, \dots, t_p) = \sigma(t_0, \dots, t_p, 0, \dots, 0).$$

Similarly, the symbol $\sigma_{|[e_p, \dots, e_{p+q}]}$ refers to the restriction of σ to the **back face** of Δ^{p+q}

$$\sigma_{|[e_p, \dots, e_{p+q}]}: \Delta^q \hookrightarrow \Delta^{p+q} \xrightarrow{\sigma} X, \quad \sigma_{|[e_p, \dots, e_{p+q}]}(t_0, \dots, t_q) = \sigma(0, \dots, 0, t_0, \dots, t_q).$$

We would can think of this construction as evaluating φ on the front face of σ , evaluating ψ on the back face of σ , and then taking the product of the two results.

To make sure that this construction yields something meaningful on the level of cohomology we need to check a couple of things.

Lemma: Cup products and coboundaries

For cochains $\varphi \in S^p(X; R)$ and $\psi \in S^q(X; R)$, we have

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^p \varphi \cup \delta\psi.$$

For the next proof and the remaining lecture, recall that the notation \hat{e}_i means that the vertex e_i is omitted.

Proof: By definition, for a simplex $\sigma \in \Delta^{p+q+1} \rightarrow X$, we have

$$\begin{aligned} \delta(\varphi \cup \psi)(\sigma) &= (\varphi \cup \psi)(\partial\sigma) \\ &= (\varphi \cup \psi) \left(\sum_{i=0}^{p+q+1} (-1)^i \sigma_{|[e_0, \dots, \hat{e}_i, \dots, e_{p+q+1}]} \right) \\ &= \sum_{i=0}^{p+1} (-1)^i \varphi(\sigma_{|[e_0, \dots, \hat{e}_i, \dots, e_p]}) \psi(\sigma_{|[e_{p+1}, \dots, e_{p+q+1}]}) \\ &\quad + \sum_{i=p}^{p+q+1} (-1)^i \varphi(\sigma_{|[e_0, \dots, e_p]}) \psi(\sigma_{|[e_p, \dots, \hat{e}_i, \dots, e_{p+q+1}]}) \end{aligned}$$

where the split into the two sums is justified by the fact that the last term of the first sum is exactly (-1) -times the first term of the second sum.

Now it remains to observe that these two sums are exactly the definition of $(\delta\varphi \cup \psi)(\sigma)$ and $(-1)^p (\varphi \cup \delta\psi)(\sigma)$. \square

We would like this construction to **descend to cohomology**. Therefore, we need to check:

- Assume that φ and ψ are **cocycles**, i.e., $\delta\varphi = 0$ and $\delta\psi = 0$. Then $\varphi \cup \psi$ is a **cocycle**, since

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi \pm \varphi \cup \delta\psi = 0 \pm 0 = 0.$$

- Assume that φ is a **cocycle**, i.e., $\delta\varphi = 0$, and ψ is a **coboundary**, i.e., there is a cochain ψ' with $\psi = \delta\psi'$. Then $\varphi \cup \psi$ is a **coboundary**, since

$$\begin{aligned} \delta(\varphi \cup \psi') &= \delta\varphi \cup \psi' \pm \varphi \cup \delta\psi' \\ &= 0 \pm \varphi \cup \psi. \end{aligned}$$

In other words, $\varphi \cup \psi$ is the image of $\pm\varphi \cup \psi'$ under δ .

- Similarly, we can show that $\varphi \cup \psi$ is a **coboundary** if φ is a **coboundary** and ψ is a **cocycle**.

Thus we have shown:

Cup product in cohomology

For any p and q , the cup product defines a map on cohomology groups

$$H^p(X; R) \times H^q(X; R) \xrightarrow{\cup} H^{p+q}(X; R).$$

As we can easily check by evaluating on a simplex:

- The product is **associative**, i.e.,

$$(\varphi \cup \psi) \cup \xi = \varphi \cup (\psi \cup \xi).$$

- The product is **distributive**, i.e.,

$$\varphi \cup (\psi + \xi) = \varphi \cup \psi + \varphi \cup \xi.$$

- The 0-cocycle $\epsilon \in H^0(X; R)$ defined by taking value 1 for every 0-simplex is a **neutral element**, i.e.,

$$\epsilon \cup \varphi = \varphi = \varphi \cup \epsilon \text{ for all } \varphi \in H^p(X; R).$$

Before we address commutativity, let us first check how the cup product behaves under induced homomorphisms:

Proposition: Cup products are natural

Let $f: X \rightarrow Y$ be a continuous map and let $f^*: H^{p+q}(Y; R) \rightarrow H^{p+q}(X; R)$ be the induced homomorphism. Then

$$f^*(\varphi \cup \psi) = f^*\varphi \cup f^*\psi$$

for all $\varphi \in H^p(Y; R)$ and $\psi \in H^q(Y; R)$.

Proof: We can check this formula already on the level of cochains. For, given a simplex $\sigma: \Delta^{p+q} \rightarrow X$, we get

$$\begin{aligned} (f^*\varphi \cup f^*\psi)(\sigma) &= f^*\varphi(\sigma|_{[e_0, \dots, e_p]}) f^*\psi(\sigma|_{[e_p, \dots, e_{p+q}]}) \\ &= \varphi(f \circ \sigma|_{[e_0, \dots, e_p]}) \psi(f \circ \sigma|_{[e_p, \dots, e_{p+q}]}) \\ &= (\varphi \cup \psi)(f \circ \sigma) \\ &= f^*(\varphi \cup \psi)(\sigma). \end{aligned}$$

Now we are going to address the remaining natural property of multiplication: commutativity. It will turn out that the cup product is not exactly symmetric. This is annoying, but so is life sometimes. However, it is very close to being symmetric. For the next result, recall that we assume that R itself is commutative.

Theorem: Cup products are graded commutative

For any classes $\varphi \in H^p(X; R)$ and $\psi \in H^q(X; R)$, we have

$$\varphi \cup \psi = (-1)^{pq}(\psi \cup \varphi).$$

The proof of this result will require some efforts. Before we think about it, let us **collect some consequences** of this theorem and of the construction of the cup product in general.

- Many cup products are trivial just for degree reasons. For classes $\varphi \in H^p(X; R)$ and $\psi \in H^q(X; R)$ with $p+q$ such that $H^{p+q}(X; R) = 0$, then $\varphi \cup \psi = 0$ no matter what.
- This can happen for example if X is a finite cell complex.
- If $\varphi \in H^p(X; R)$ and p is **odd**, then

$$\varphi^2 = (-1)^{p^2} \varphi^2 = -\varphi^2.$$

Therefore, $2\varphi^2 = 0$ in $H^{2p}(X; R)$.

If R is **torsion-free** or if R is a field of **characteristic different from 2**, this implies

$$\varphi^2 = 0.$$

Proof for a special case: In order to find a strategy for the proof of the theorem, let us look at a special case. So let $[\varphi], [\psi] \in H^1(X; R)$, and let $\sigma: \Delta^2 \rightarrow X$ be a 2-simplex.

The respective cup products are then determined by their effect on a 2-simplex $\sigma: \Delta^2 \rightarrow X$:

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[e_0, e_1]})\psi(\sigma|_{[e_1, e_2]}).$$

and

$$\begin{aligned} (\psi \cup \varphi)(\sigma) &= \psi(\sigma|_{[e_0, e_1]})\varphi(\sigma|_{[e_1, e_2]}) \\ &= \varphi(\sigma|_{[e_1, e_2]})\psi(\sigma|_{[e_0, e_1]}) \end{aligned}$$

where we use that R is commutative.

Hence in order to show that these two expressions are related, we would like to reshuffle the vertices. As a first attempt we are going to reverse the order of all vertices, i.e., we replace σ with $\bar{\sigma}$ defined by

$$\sigma(e_i) = \sigma(e_{2-i}).$$

We will also use the notation

$$\sigma|_{[e_2, e_1, e_0]} = \bar{\sigma}$$

which expresses

$$\bar{\sigma}(t_0, t_1, t_2) = \sigma(t_2, t_1, t_0).$$

Recall that, a long time ago, we showed that reversing the order of vertices on a 1-simplex corresponds, at least up to boundaries, multiplying the simplex with (-1) .

Hence we should consider inserting a sign as well. So we define maps

$$\rho_1: S_1(X) \rightarrow S_1(X) \text{ and } \rho_2: S_2(X) \rightarrow S_2(X)$$

both defined by sending a simplex σ to $-\bar{\sigma}$.

Surprisingly, the comparison of the two cup products after taking pullbacks along the ρ s becomes easier. For,

$$\begin{aligned} (\rho_1^* \varphi \cup \rho_1^* \psi)(\sigma) &= \varphi(-\sigma|_{[e_1, e_0]})\psi(-\sigma|_{[e_2, e_1]}) \\ &= \varphi(\sigma|_{[e_1, e_0]})\psi(\sigma|_{[e_2, e_1]}) \end{aligned}$$

and

$$\begin{aligned} (\rho_2^*(\psi \cup \varphi))(\sigma) &= -\psi(\sigma|_{[e_2, e_1]})\varphi(\sigma|_{[e_1, e_0]}) \\ &= -\varphi(\sigma|_{[e_1, e_0]})\psi(\sigma|_{[e_2, e_1]}) \end{aligned}$$

using that R is commutative.

Hence we get

$$\rho_1^* \varphi \cup \rho_1^* \psi = -\rho_2^*(\psi \cup \varphi).$$

In other words, up to ρ_1^* and ρ_2^* we have shown the desired equality.

Now we remember that we are still on the level of cochains. The theorem is about an equality of cohomology classes. Hence all we need to show is that ρ_1^* and ρ_2^* will vanish once we pass to cohomology.

This leads to the idea to show that ρ_1 and ρ_2 are part of a chain map which is chain homotopic to the identity. So let us try to do this.

First, we want that ρ_1 and ρ_2 commute with the boundary operator:

$$\begin{aligned} (\rho_1 \circ \partial)(\sigma) &= \rho(\sigma_{[e_1, e_2]} - \sigma_{[e_0, e_2]} + \sigma_{[e_0, e_1]}) \\ &= -\sigma_{[e_2, e_1]} + \sigma_{[e_2, e_0]} - \sigma_{[e_1, e_0]} \\ &= \partial(-\sigma_{[e_2, e_1, e_0]}) \\ &= (\partial \circ \rho_2)(\sigma). \end{aligned}$$

Now we would like to construct a chain homotopy between ρ and the identity chain map.

The **idea** is to interpolate between the identity and ρ by permuting the vertices one after the other until the order is completely reversed. Then we sum up all these maps. Along the way we need to introduce some signs.

Before we can define maps, we need to recall the prism operator we used to construct a chain homotopy which showed that singular homology is homotopy invariant.

These were maps

$$p_i^n: \Delta^{n+1} \rightarrow \Delta^n \times [0, 1]$$

determined by

$$p_i^n(e_k) = \begin{cases} (e_k, 0) & \text{if } 0 \leq k \leq i \\ (e_{k-1}, 1) & \text{if } k > i. \end{cases}$$

Let us write $e_k^0 := (e_k, 0)$ and $e_k^1 := (e_k, 1)$. Given an n -simplex σ , we would like to compose it with p_i^n and also permute vertices.

Consider the **permutation of simplices**

$$\Delta^{n+1} \xrightarrow{s_i} \Delta^{n+1}, (e_0, \dots, e_{n+1}) \mapsto (e_0, \dots, e_i, e_{n+1}, \dots, e_{i+1}).$$

To simplify the notation, we are going to write

$$\sigma_{|[e_0, \dots, e_i, e_n, \dots, e_i]}: \Delta^{n+1} \rightarrow X$$

for the $n + 1$ -simplex defined by the composition of s_i with

$$\Delta^{n+1} \xrightarrow{p_i^n} \Delta^n \times [0, 1] \xrightarrow{\text{pr}} \Delta^n \xrightarrow{\sigma} X.$$

Now we **define three maps**

$$h_0: S_0(X) \rightarrow S_1(X), \sigma \mapsto \sigma_{|[e_0, e_0]},$$

for $n = 0$,

$$h_1: S_1(X) \rightarrow S_2(X), \sigma \mapsto -\sigma_{|[e_0, e_1, e_0]} - \sigma_{|[e_0, e_1, e_1]},$$

for $n = 1$, and

$$h_2: S_2(X) \rightarrow S_3(X), \sigma \mapsto -\sigma_{|[e_0, e_2, e_1, e_0]} + \sigma_{|[e_0, e_1, e_2, e_1]} + \sigma_{|[e_0, e_1, e_2, e_2]}$$

for $n = 2$.

(You will see that it does not matter so much how these maps are defined. It is just important that we have some consistent way of moving from $S_n(X)$ to $S_{n+1}(X)$.)

For a 1-simplex $\sigma: \Delta^1 \rightarrow X$, we compute

$$\begin{aligned} (\partial \circ h_1)(\sigma) &= \partial(-\sigma_{|[e_0, e_1, e_0]} - \sigma_{|[e_0, e_1, e_1]}) \\ &= -(\sigma_{|[e_1, e_0]} - \sigma_{|[e_0, e_0]} + \sigma_{|[e_0, e_1]}) \\ &\quad - (\sigma_{|[e_1, e_1]} - \sigma_{|[e_0, e_1]} + \sigma_{|[e_0, e_1]}) \end{aligned}$$

and

$$\begin{aligned} (h_0 \circ \partial)(\sigma) &= h_0(\sigma_{|[e_1]} - \sigma_{|[e_0]}) \\ &= \sigma_{|[e_1, e_1]} - \sigma_{|[e_0, e_0]}. \end{aligned}$$

Taking these terms together we get

$$\begin{aligned} (\partial \circ h_1)(\sigma) + (h_0 \circ \partial)(\sigma) &= -\sigma_{|[e_1, e_0]} + \sigma_{|[e_0, e_0]} - \sigma_{|[e_0, e_1]} \\ &\quad - \sigma_{|[e_1, e_1]} + \sigma_{|[e_0, e_1]} - \sigma_{|[e_0, e_1]} \\ &\quad + \sigma_{|[e_1, e_1]} - \sigma_{|[e_0, e_0]} \\ &= -\sigma_{|[e_1, e_0]} - \sigma_{|[e_0, e_1]} \\ &= \rho(\sigma) - \sigma. \end{aligned}$$

Thus, we have shown the homotopy relation

$$\partial \circ h_1 + h_0 \circ \partial = \rho_1 - \text{id}.$$

Similarly, for a 2-simplex $\sigma: \Delta^2 \rightarrow X$, we calculate

$$\begin{aligned}
 (\partial \circ h_2)(\sigma) &= \partial(-\sigma|_{[e_0, e_2, e_1, e_0]} - \sigma|_{[e_0, e_1, e_2, e_1]} + \sigma|_{[e_0, e_1, e_2, e_2]}) \\
 &= -(\sigma|_{[e_2, e_1, e_0]} - \sigma|_{[e_0, e_1, e_0]} + \sigma|_{[e_0, e_2, e_0]} - \sigma|_{[e_0, e_2, e_1]}) \\
 &\quad + (\sigma|_{[e_1, e_2, e_1]} - \sigma|_{[e_0, e_2, e_1]} + \sigma|_{[e_0, e_1, e_1]} - \sigma|_{[e_0, e_1, e_2]}) \\
 &\quad + (\sigma|_{[e_1, e_2, e_2]} - \sigma|_{[e_0, e_2, e_2]} + \sigma|_{[e_0, e_1, e_2]} - \sigma|_{[e_0, e_1, e_2]}) \\
 &= -\sigma|_{[e_2, e_1, e_0]} + \sigma|_{[e_0, e_1, e_0]} - \sigma|_{[e_0, e_2, e_0]} + \sigma|_{[e_1, e_2, e_1]} + \sigma|_{[e_0, e_1, e_1]} \\
 &\quad + \sigma|_{[e_1, e_2, e_2]} - \sigma|_{[e_0, e_2, e_2]} - \sigma|_{[e_0, e_1, e_2]}
 \end{aligned}$$

and

$$\begin{aligned}
 (h_1 \circ \partial)(\sigma) &= h_1(\sigma|_{[e_1, e_2]} - \sigma|_{[e_0, e_2]} + \sigma|_{[e_0, e_1]}) \\
 &= -\sigma|_{[e_1, e_2, e_1]} - \sigma|_{[e_1, e_2, e_2]} + \sigma|_{[e_0, e_2, e_0]} + \sigma|_{[e_0, e_2, e_2]} - \sigma|_{[e_0, e_1, e_0]} - \sigma|_{[e_0, e_1, e_1]}.
 \end{aligned}$$

This gives

$$(\partial \circ h_2 + h_1 \circ \partial)(\sigma) = \rho_2(\sigma) - \sigma.$$

Thus, we have again shown the homotopy relation

$$\partial \circ h_2 + h_1 \circ \partial = \rho_2 - \text{id}.$$

This indicates that ρ_1 and ρ_2 are part of a chain map which is chain homotopic to the identity.

To prove the general case, we adapt this strategy we developed for $n = 1$.

Proof of the theorem: When we evaluate the two cup products on a simplex $\sigma: \Delta^{p+q} \rightarrow X$, they differ only by a permutation of the vertices of σ . The idea of the proof consists of

- choosing a nice permutation which simplifies notation and computations,
- and then to construct a chain homotopy between the resulting cup product and the identity.

Now let us get to work:

- For an n -simplex σ , let $\bar{\sigma}$ be the n -simplex obtained by composing it first with the linear transformation which **reverses the order** of the vertices.

In other words,

$$\bar{\sigma}(e_i) = \sigma(e_{n-i}) \text{ for all } i = 0, \dots, n.$$

or

$$\bar{\sigma}(t_0, \dots, t_n) = \sigma(t_n, \dots, t_0).$$

We will also use the notation

$$\sigma|_{[e_n, \dots, e_0]} = \bar{\sigma}.$$

For this will make it easier to combine it with the restriction to the $n - 1$ -dimensional faces of Δ^n .

• Since the reversal of the vertices is the product of $n + (n - 1) + \dots + 1 = n(n + 1)/2$ many transpositions, our test case motivates the definition of the homomorphism

$$\rho_n: S_n(X) \rightarrow S_n(X), \rho_n(\sigma) = (-1)^{\frac{n(n+1)}{2}} \bar{\sigma}.$$

To simplify the notation we will write $\epsilon_n := (-1)^{\frac{n(n+1)}{2}}$.

• **We claim** that ρ is a **map of chain complexes** which is chain **homotopic to the identity** map. Assuming that the claim is true we can finish the proof of the theorem as follows.

For $\sigma: \Delta^{p+q} \rightarrow X$, we can then calculate

$$\begin{aligned} (\rho^* \varphi \cup \rho^* \psi)(\sigma) &= \varphi(\epsilon_p \sigma|_{[e_p, \dots, e_0]}) \psi(\epsilon_q \sigma|_{[e_{p+q}, \dots, e_p]}) \\ &= \epsilon_p \epsilon_q \varphi(\sigma|_{[e_p, \dots, e_0]}) \psi(\sigma|_{[e_{p+q}, \dots, e_p]}) \end{aligned}$$

and

$$(\rho^*(\psi \cup \varphi))(\sigma) = \epsilon_{p+q} \psi(\sigma|_{[e_{p+q}, \dots, e_p]}) \varphi(\sigma|_{[e_p, \dots, e_0]}).$$

Now we observe

$$\begin{aligned} \frac{(p+q)(p+q+1)}{2} &= \frac{p^2 + 2pq + q^2 + p + q}{2} \\ &= \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + \frac{2pq}{2} \\ &= \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + pq. \end{aligned}$$

Thus

$$\epsilon_{p+q} = (-1)^{pq} \epsilon_p \epsilon_q.$$

We conclude from these two computations

$$\rho^* \varphi \cup \rho^* \psi = (-1)^{pq} \rho^* (\psi \cup \varphi).$$

Now we use that ρ is chain homotopic to the identity. That implies that when we pass to cohomology classes, ρ^* is the identity and we obtain the desired identity

$$\varphi \cup \psi = (-1)^{pq} (\psi \cup \varphi).$$

Now we are going to prove the claims we made:

- ρ is a chain map.

We need to show that $\partial \circ \rho = \rho \circ \partial$. For an n -simplex σ we calculate the effects of the two maps:

$$\begin{aligned} (\rho \circ \partial)(\sigma) &= \rho \left(\sum_{i=0}^n (-1)^i \sigma_{[e_0, \dots, \hat{e}_i, \dots, e_n]} \right) \\ &= \epsilon_{n-1} \sum_{i=0}^n (-1)^i \sigma_{[e_n, \dots, \hat{e}_i, \dots, e_0]} \\ &= \epsilon_{n-1} \sum_{i=0}^n (-1)^{n-i} \sigma_{[e_n, \dots, \hat{e}_{n-i}, \dots, e_0]} \text{ by changing the order of summation} \\ &= \epsilon_{n-1} \sum_{i=0}^n (-1)^{i-n} \sigma_{[e_n, \dots, \hat{e}_{n-i}, \dots, e_0]} \text{ using } (-1)^j = (-1)^{-j} \\ &= \epsilon_{n-1} (-1)^n \sum_{i=0}^n (-1)^i \sigma_{[e_n, \dots, \hat{e}_{n-i}, \dots, e_0]} \text{ again using } (-1)^n = (-1)^{-n} \\ &= \epsilon_n \sum_{i=0}^n (-1)^i \sigma_{[e_n, \dots, \hat{e}_{n-i}, \dots, e_0]} \\ &= \partial(\epsilon_n \sigma_{[e_n, \dots, e_0]}) \\ &= (\partial \circ \rho)(\sigma) \end{aligned}$$

where we used the identity $\epsilon_n = (-1)^n \epsilon_{n-1}$.

- There is a chain homotopy between ρ and the identity.

We are going to use again the notation we introduced for the special case above. The idea for the chain homotopy is to interpolate between ρ which reverses the order of all vertices and the identity by, step by step, reversing the order up

to some vertex while the others remain fixed. Then we throw in some signs to make things work.

We define homomorphisms h_n for each n by

$$\begin{aligned} h_n: S_n(X) &\rightarrow S_{n+1}(X) \\ \sigma &\mapsto \sum_{i=0} (-1)^i \epsilon_{n-i} \sigma|_{[e_0, \dots, e_i, e_n, \dots, e_i]}. \end{aligned}$$

Now we can show by calculating $\partial \circ h_n$ and $h_{n-1} \circ \partial$ that h is a chain homotopy, i.e., we have

$$\partial \circ h_n + h_{n-1} \circ \partial = \rho - \text{id}.$$

We have

$$\begin{aligned} (\partial \circ h_n)(\sigma) &= \partial \left(\sum_{i=0} (-1)^i \epsilon_{n-i} \sigma|_{[e_0, \dots, e_i, e_n, \dots, e_i]} \right) \\ &= \sum_{j \leq i} (-1)^i (-1)^j \epsilon_{n-i} \sigma|_{[e_0, \dots, \hat{e}_j, \dots, e_i, e_n, \dots, e_i]} \\ &\quad + \sum_{j \geq i} (-1)^i (-1)^{i+1+n-j} \epsilon_{n-i} \sigma|_{[e_0, \dots, e_i, e_n, \dots, \hat{e}_j, \dots, e_i]}. \end{aligned}$$

The overlap of the summation indices is necessary. For, at $j = i$, only the two sums together yield all the summands we need:

$$\begin{aligned} &\epsilon_n \sigma|_{[e_n, \dots, e_0]} + \sum_{i > 0} \epsilon_{n-i} \sigma|_{[e_0, \dots, e_{i-1}, e_n, \dots, e_i]} \\ &+ \sum_{i < n} (-1)^{n+i+1} \epsilon_{n-i} \sigma|_{[e_0, \dots, e_i, e_n, \dots, e_{i+1}]} - \sigma|_{[e_0, \dots, e_n]}. \end{aligned}$$

Now we observe that the two sums in the last expression cancel out, since if we replace i by $i - 1$ in the second sum turns the sign into

$$(-1)^{n+i} \epsilon_{n-i+1} = -\epsilon_{n-i}.$$

Hence, for $j = i$, what remains is exactly

$$\epsilon_n \sigma|_{[e_n, \dots, e_0]} - \sigma|_{[e_0, \dots, e_n]} = \rho(\sigma) - \sigma.$$

Hence it suffices to show that the terms with $j \neq i$ in $(\partial \circ h_n)(\sigma)$ cancel out with $(h_{n-1} \circ \partial)(\sigma)$. So we calculate

$$\begin{aligned}
(h_{n-1} \circ \partial)(\sigma) &= h_{n-1} \left(\sum_{j=0} (-1)^j \sigma_{[e_0, \dots, \hat{e}_j, \dots, e_n]} \right) \\
&= \sum_{j < i} (-1)^{i-1} (-1)^j \epsilon_{n-i} \sigma_{[e_0, \dots, \hat{e}_j, \dots, e_i, e_n, \dots, e_i]} \\
&\quad + \sum_{j > i} (-1)^i (-1)^j \epsilon_{n-i-1} \sigma_{[e_0, \dots, e_i, e_n, \dots, \hat{e}_j, \dots, e_i]}.
\end{aligned}$$

Since $\epsilon_{n-i} = (-1)^{n-i} \epsilon_{n-i-1}$, the two sums cancel with the two corresponding sums in $(\partial \circ h_n)(\sigma)$. Hence h is a chain homotopy between ρ and the identity. \square

LECTURE 21

Applications of cup products in cohomology

We are going to see some examples where we calculate or apply multiplicative structures on cohomology. But we start with a couple of facts we forgot to mention last time.

Relative cup products

Let (X, A) be a pair of spaces. The formula which specifies the cup product by its effect on a simplex

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[e_0, \dots, e_p]})\psi(\sigma|_{[e_p, \dots, e_{p+q}]})$$

extends to relative cohomology.

For, if $\sigma: \Delta^{p+q} \rightarrow X$ has image in A , then so does any restriction of σ . Thus, if either φ or ψ vanishes on chains with image in A , then so does $\varphi \cup \psi$.

Hence we get relative cup product maps

$$\begin{aligned} H^p(X; R) \times H^q(X, A; R) &\rightarrow H^{p+q}(X, A; R) \\ H^p(X, A; R) \times H^q(X; R) &\rightarrow H^{p+q}(X, A; R) \\ H^p(X, A; R) \times H^q(X, A; R) &\rightarrow H^{p+q}(X, A; R). \end{aligned}$$

More generally, assume we have two open subsets A and B of X . Then the formula for $\varphi \cup \psi$ on cochains implies that cup product yields a map

$$S^p(X, A; R) \times S^q(X, B; R) \rightarrow S^{p+q}(X, A + B; R)$$

where $S^n(X, A + B; R)$ denotes the subgroup of $S^n(X; R)$ of cochains which vanish on sums of chains in A and chains in B .

The natural inclusion

$$S^n(X, A \cup B; R) \hookrightarrow S^n(X, A + B; R)$$

induces an isomorphism in cohomology. For we have a map of long exact cohomology sequences

$$\begin{array}{ccccccccc}
 H^n(A \cup B) & \longrightarrow & H^n(X) & \longrightarrow & H^n(X, A \cup B) & \longrightarrow & H^{n+1}(A \cup B) & \longrightarrow & H^{n+1}(X) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^n(A + B) & \longrightarrow & H^n(X) & \longrightarrow & H^n(X, A + B) & \longrightarrow & H^{n+1}(A + B) & \longrightarrow & H^{n+1}(X)
 \end{array}$$

where we omit the coefficients. The small chain theorem and our results on cohomology of free chain complexes imply that $H^n(A \cup B; R) \xrightarrow{\cong} H^n(A + B; R)$ is an isomorphism for every n . Thus, the Five-Lemma implies that

$$H^n(X, A \cup B; R) \xrightarrow{\cong} H^n(X, A + B; R)$$

is an isomorphism as well.

Thus composition with this isomorphism gives a cup product map

$$H^p(X, A; R) \times H^q(X, B; R) \rightarrow H^{p+q}(X, A \cup B; R).$$

Now one can check that all the formulae we proved for the cup product also hold for the relative cup products.

Cohomology ring

All we are going to say now also works for relative cohomology. But to keep things simple, we just describe the absolute case.

We will now often drop the symbol \cup to denote the cup product and just write

$$\alpha\beta = \alpha \cup \beta.$$

The cohomology ring of a space X is defined as

$$H^*(X; R) = \bigoplus_n H^n(X; R)$$

as the direct sum of all cohomology groups. Note that, while the symbol $*$ previously often indicated that something holds for an arbitrary degree, we now use it to denote the direct sum over all degrees.

The product of two sums is defined as

$$\left(\sum_i \alpha_i\right)\left(\sum_j \beta_j\right) = \sum_{i,j} \alpha_i \beta_j.$$

This turns $H^*(X; R)$ into a ring with unit, i.e., multiplication is associative, there is a multiplicatively neutral element 1, and addition and multiplication satisfy the distributive law.

We consider the cohomological degree n in $H^n(X; R)$ as a grading of $H^*(X; R)$. If an element α is in $H^p(X; R)$ we call p the degree of α and denote it also by $|\alpha|$.

Since multiplication respects this grading in the sense that it defines a map

$$H^p(X; R) \times H^q(X; R) \rightarrow H^{p+q}(X, A; R),$$

we call $H^*(X; R)$ a graded ring.

Moreover, as we have shown with a lot of effort last time, the multiplication is commutative up to a sign which depends on the grading:

$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha.$$

Hence $H^*(X; R)$ a **graded commutative ring**.

Moreover, there is an obvious scalar multiplication by elements in R which turns $H^*(X; R)$ into a graded R -algebra.

Finally, if $f: X \rightarrow Y$ is a continuous map, then the induced map on cohomology

$$f^*: H^*(Y; R) \rightarrow H^*(X; R)$$

is a **homomorphism of graded R -algebras**.

Now we should determine some ring structures and see what they can tell us.

As a first, though disappointing, **example**, let us note that the product in the cohomology of a **sphere** S^n (with $n \geq 1$) is boring, since $H^0(S^n; R)$ is just R and the product on $H^n(S^n; R)$ is trivial for reasons of degrees:

$$H^n(S^n; R) \times H^n(S^n; R) \rightarrow H^{2n}(S^n; R) = 0.$$

So let us move on to more interesting cases.

Cohomology ring of the torus

Even though the cohomology ring of S^1 was boring, the cohomology ring of the product $T = S^1 \times S^1$, i.e., of the torus, is not. Let us assume $R = \mathbb{Z}$.

We computed the homology of T using its structure as a cell complex with one 0-cell, two 1-cells, and one 2-cell.

The cellular chain complex has the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \rightarrow 0$$

where $d_1(a, b) = a + b$ and $d_2(s) = (s, -s)$ (the attaching map of the 2-cell to the two 1-cells was $aba^{-1}b^{-1}$). This yields the homology of T .

We can then apply the UCT to deduce that the singular cohomology of T is given by

$$H^i(T; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \end{cases}$$

and $H^i(T; \mathbb{Z}) = 0$ for $i > 2$.

Let α and β be generators of $H^1(T; \mathbb{Z})$. We could obtain them for example as the dual of the basis $\{a, b\}$ of $H_1(T; \mathbb{Z})$ and the isomorphism of the UCT:

$$H^1(T; \mathbb{Z}) = \text{Hom}(H_1(T; \mathbb{Z}), \mathbb{Z}).$$

Being a dual basis means, in particular,

$$\alpha(a) = \langle \alpha, a \rangle = 1, \quad \alpha(b) = \langle \alpha, b \rangle = 0, \quad \beta(a) = \langle \beta, a \rangle = 0, \quad \beta(b) = \langle \beta, b \rangle = 1$$

where the funny brackets denote the Kronecker pairing we had defined earlier.

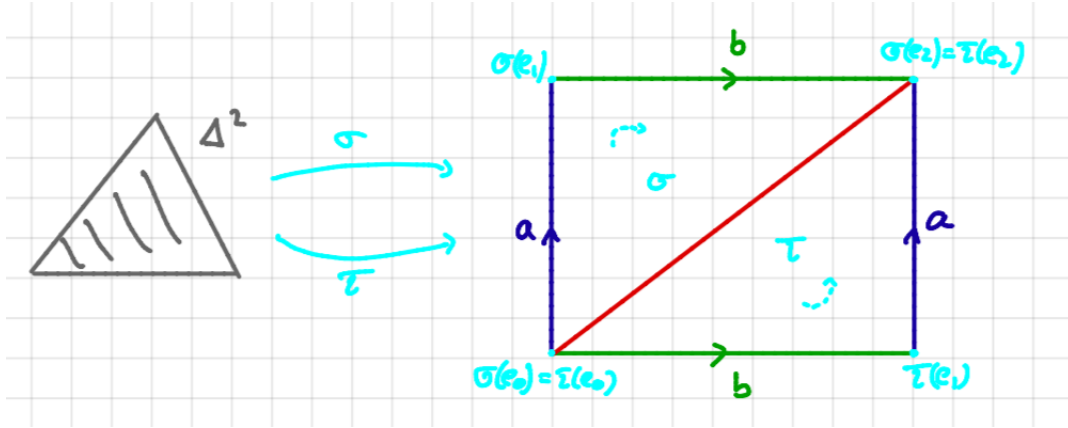
Since multiplication is graded commutative, we have

$$2\alpha^2 = 0 = 2\beta^2.$$

Since \mathbb{Z} is torsion-free, this implies

$$\alpha^2 = 0 = \beta^2.$$

Now we would like to understand the product $\alpha\beta$. Therefore, we need to evaluate it on a generator of $H_2(T; \mathbb{Z})$. Such a generator is given by the 2-chain $\sigma - \tau$, where σ and τ are the 2-simplices indicated in the picture (that this is a generator needs to be checked; we just accept this for the moment):



It is a cycle, since

$$\partial(\sigma - \tau) = \partial(\sigma) - \partial(\tau) = b - d + a - (a - d + b) = 0$$

where d denotes the diagonal.

Now we can calculate

$$\begin{aligned} (\alpha \cup \beta)(\sigma - \tau) &= \alpha(\sigma|_{[e_0, e_1]})\beta(\sigma|_{[e_1, e_2]}) - \alpha(\tau|_{[e_0, e_1]})\beta(\tau|_{[e_1, e_2]}) \\ &= \alpha(a)\beta(b) - \alpha(b)\beta(a) \\ &= 1 - 0 = 1. \end{aligned}$$

Thus, since $H^2(T; \mathbb{Z}) = \text{Hom}(H_2(T; \mathbb{Z}), \mathbb{Z})$ by the UCT, we see that $\alpha\beta$ is a generator of $H^2(T; \mathbb{Z})$.

Hence we can conclude that the cohomology ring of the torus is the ring with generators α and β and relations

$$H^*(T; \mathbb{Z}) = \mathbb{Z}\{\alpha, \beta\} / \langle \alpha^2 = 0 = \beta^2, \alpha\beta = -\beta\alpha \rangle.$$

Another way to formulate this is to say that $H^*(T; \mathbb{Z})$ is the exterior algebra over \mathbb{Z} with generators α and β :

$$H^*(T; \mathbb{Z}) = \Lambda_{\mathbb{Z}}[\alpha, \beta].$$

In general, the exterior algebra $\Lambda_R[\alpha_1, \dots, \alpha_n]$ over a commutative ring R with unit is defined as the free R -module with generators $\alpha_{i_1} \cdots \alpha_{i_k}$ for $i_1 < \dots < i_k$ with associative and distributive multiplication defined by the rules

$$\alpha_i \alpha_j = -\alpha_j \alpha_i \text{ if } i \neq j, \text{ and } \alpha_i^2 = 0.$$

Setting $\Lambda^0 = R$, $\Lambda_R[\alpha_1, \dots, \alpha_n]$ becomes a graded commutative ring with odd degrees for the α_i s and unit $1 \in R$.

For the n -torus $T^n = S^1 \times \cdots \times S^1$, defined as the n -fold product of S^1 , we then get

$$H^*(T^n; \mathbb{Z}) = \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_n].$$

Cohomology of projective spaces

The cohomology rings of projective spaces are truncated polynomial algebras:

Cohomology rings of \mathbb{RP}^n and \mathbb{CP}^n

- For every $n \geq 1$ and \mathbb{F}_2 -coefficients, we have an isomorphism of graded rings

$$H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1}), \text{ and } H^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$$

with $|x| = 1$.

- For every $n \geq 1$ and integral coefficients, we have an isomorphism of graded rings

$$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}), \text{ and } H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[y]$$

with $|y| = 2$.

The proof of this result requires some efforts. We will postpone its proof and rather see some consequences of it.

Cup products detect more

Consider the wedge of spheres $S^2 \vee S^4$. We know that its homology is given by

$$\tilde{H}_*(S^2 \vee S^4; \mathbb{Z}) = \tilde{H}_*(S^2; \mathbb{Z}) \oplus \tilde{H}_*(S^4; \mathbb{Z}).$$

In other words,

$$H_i(S^2 \vee S^4; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, 4 \\ 0 & \text{else.} \end{cases}$$

Hence the homologies of \mathbb{CP}^2 and $S^2 \vee S^4$ are the same. Since all the groups are free, this also implies that the cohomology groups of the two spaces are the same. Thus, neither homology nor cohomology groups can distinguish between these two spaces.

The cup product, however, can.

For, we know that the square of a generator in $H^n(S^n; \mathbb{Z})$ is zero, since $H^{2n}(S^n; \mathbb{Z}) = 0$. Thus

$$H^*(S^n; \mathbb{Z}) = \mathbb{Z}[t]/(t^2 = 0) \text{ with } |t| = n,$$

and hence we have a generator $s \in H^2(S^2 \vee S^4; \mathbb{Z})$ with $s^2 = 0$ and a generator $t \in H^2(S^2 \vee S^4; \mathbb{Z})$ with $t^2 = 0$.

If there was an isomorphism of graded \mathbb{Z} -algebras

$$\tilde{H}^*(\mathbb{CP}^2; \mathbb{Z}) \cong \tilde{H}^*(S^2 \vee S^4; \mathbb{Z})$$

it would have to send the generator $y \in H^2(\mathbb{CP}^2; \mathbb{Z})$ to the generator $s \in H^2(S^2 \vee S^4; \mathbb{Z})$. But $y^2 \neq 0$ in $H^4(\mathbb{CP}^2; \mathbb{Z})$, whereas $s^2 = 0$ in $H^4(S^2 \vee S^4; \mathbb{Z})$.

Thus, such an isomorphism of graded rings cannot exist.

Thus, the cup product structures show that there does not exist a homotopy equivalence between \mathbb{CP}^2 and $S^2 \vee S^4$, something our previous invariants could not prove.

Hopf maps

As an important application of what we just learned, we consider the following situation.

Many problems can be reduced to checking whether a map is **null-homotopic**, i.e., **homotopic to a constant map**, or not.

Given a map $f: X \rightarrow Y$, we can form the mapping cone $C_f = CX \cup_f Y$ (which we introduced in the exercises). It is the pushout of the diagram

$$\begin{array}{ccc} X \times \{1\} & \longrightarrow & CX \\ f \downarrow & & \downarrow \\ Y & \xrightarrow{i} & C_f. \end{array}$$

If f is homotopic to a constant map, then the diagram is equivalent to the diagram

$$\begin{array}{ccc} \text{pt} & \longrightarrow & CX/(X \times \{1\}) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{i} & SX \vee Y \end{array}$$

where we use that $CX/(X \times \{1\})$ is the suspension SX of X .

Thus, if f is **null-homotopic**, then there is a homotopy equivalence

$$C_f \xrightarrow{\cong} SX \vee Y.$$

Let us look at an **example**. Let

$$\eta: S^3 \rightarrow \mathbb{CP}^1 \approx S^2, x \mapsto [\mathbb{C}x] = \{\lambda x \in \mathbb{C}^2 : \lambda \in \mathbb{C}\}$$

be the **complex Hopf map** which sends a point $x \in S^3 \subset \mathbb{C}^2$ to the complex line in \mathbb{C}^2 which passes through x .

This is exactly the map which attaches the 4-cell to $\mathbb{CP}^1 \approx S^2$ in the cell structure of \mathbb{CP}^2 . The mapping cone C_η of η is \mathbb{CP}^2 , since the cone of S^3 is just D^4 :

$$CS^3 = (S^3 \times [0,1]) / (S^3 \times \{0\}) \approx D^4$$

and hence

$$C_\eta = CS^3 \cup_\eta S^2 \approx D^4 \cup_\eta S^2 \approx D^4 \cup_\eta \mathbb{CP}^1 \approx \mathbb{CP}^2.$$

Now we use that we showed in the exercises that the suspension of S^3 is homeomorphic to S^4 . Thus, if η was null-homotopic, then the argument above would imply

$$\mathbb{CP}^2 \approx C_\eta \xrightarrow{\cong} S^2 \vee S^4.$$

But we just showed that such a homotopy equivalence cannot exist. Thus, η is **not null-homotopic**.

More Hopf maps

Note that there is also a **quaternionic Hopf map**

$$\nu: S^7 \rightarrow S^4,$$

and an **octonionic Hopf map**

$$\sigma: S^{15} \rightarrow S^8.$$

They are constructed in the same way as η by replacing \mathbb{C} with the quaternions \mathbb{H} and the octonions \mathbb{O} , respectively. There are corresponding projective spaces \mathbb{HP}^n and \mathbb{OP}^n with $\mathbb{HP}^1 \approx S^4$ and $\mathbb{OP}^1 \approx S^8$, and polynomial rings as cohomology rings:

$$H^*(\mathbb{HP}^2; \mathbb{Z}) = \mathbb{Z}[z]/(z^3), |z| = 4, \text{ and } H^*(\mathbb{OP}^2; \mathbb{Z}) = \mathbb{Z}[w]/(w^3), |w| = 8.$$

The homotopy classes of η , ν and σ

$$[\eta] \in \pi_3(S^2), [\nu] \in \pi_7(S^4), [\sigma] \in \pi_{15}(S^8)$$

play a crucial role in the stable homotopy category.

Is there a multiplication on \mathbb{R}^n ?

For the next application, we are going to assume one more result, namely that the cohomology ring of the product of $\mathbb{R}P^n \times \mathbb{R}P^n$ is given

$$H^*(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{F}_2[\alpha_1, \alpha_2]/(\alpha_1^{n+1}, \alpha_2^{n+1}).$$

This implies the following algebraic fact:

Theorem: Multiplication on \mathbb{R}^n

Assume there is a \mathbb{R} -bilinear map

$$\mu: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

such that $\mu(x, y) = 0$ implies $x = 0$ or $y = 0$.

Then n must be a power of 2.

In fact, n must be 1, 2, 4 or 8. In all these dimensions we have such multiplications by identifying

$$\mathbb{R}^2 \cong \mathbb{C}, \mathbb{R}^4 \cong \mathbb{H}, \mathbb{R}^8 \cong \mathbb{O}.$$

But to show that there are no other such algebra structures on \mathbb{R}^n is a much harder task. The only known proofs of this fact are using algebraic topology! In fact, for showing this we need to study the famous **Hopf Invariant One-Problem**. This is beyond the scope of this lecture. So let us be modest and just prove the result stated above.

Proof: • Since μ is linear in both variables, it induces a continuous map

$$\bar{\mu}: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{n-1}.$$

Then $\bar{\mu}$ induces a homomorphism of cohomology rings which has the form

$$\bar{\mu}^*: \mathbb{F}_2[\alpha]/(\alpha^n) \rightarrow \mathbb{F}_2[\alpha_1, \alpha_2]/(\alpha_1^n, \alpha_2^n).$$

• Since μ does not have a zero-divisor, the restriction of μ to $\mathbb{R}^n \times \{a\}$ for any $a \in \mathbb{R}^n$ is an isomorphism. Hence the restriction of $\bar{\mu}$ to $\mathbb{R}P^{n-1} \times \{y\}$ for any point $y \in \mathbb{R}P^{n-1}$ is a homeomorphism.

This implies that the composite

$$\mathbb{RP}^{n-1} \rightarrow \mathbb{RP}^{n-1} \times \{y\} \hookrightarrow \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \xrightarrow{\bar{\mu}} \mathbb{RP}^{n-1}$$

is a homeomorphism as well. Hence the induced homomorphism of cohomology rings must send α to α .

Repeating this argument for $\{y\} \times \mathbb{RP}^{n-1}$, we see that the image of α under $\bar{\mu}^*$ must be

$$\bar{\mu}^*(\alpha) = \alpha_1 + \alpha_2.$$

Since both rings are polynomial algebras, $\bar{\mu}^*$ is completely determined by this identity.

- Since $\alpha^n = 0$, we must have $\bar{\mu}^*(\alpha)^n = 0$, i.e.,

$$(\alpha_1 + \alpha_2)^n = \sum_k \binom{n}{k} \alpha_1^k \alpha_2^{n-k} = 0.$$

The sum on the right-hand side can only be zero if all the coefficients of the monomials $\alpha_1^k \alpha_2^{n-k}$ vanish for $0 < k < n$. Since we are working over \mathbb{F}_2 , this means that all the numbers $\binom{n}{k}$ for $0 < k < n$ must be even.

To prove this fact is equivalent to proving the following claim about the polynomial ring $\mathbb{F}_2[x]$:

- **Claim:** In $\mathbb{F}_2[x]$, we have

$$(1+x)^n = 1+x^n \iff n \text{ is a power of } 2.$$

First, if n is a power of 2, then the equation $(a+b)^2 = a^2 + b^2$ modulo 2 shows the if part:

$$(1+x)^{2^r} = (1+x^2)^{2^{r-1}} = (1+x^{2^2})^{2^{r-2}} = \cdots = 1+x^{2^r} \text{ in } \mathbb{F}_2[x].$$

For the other direction, write n as

$$n = 2^r m \text{ with } m \text{ odd and } m > 1.$$

Then

$$(1+x)^n = (1+x)^{2^r m} = (1+x^{2^r})^m = 1 + mx^{2^r} + \cdots + x^n \neq 1 + x^n \text{ in } \mathbb{F}_2[x]$$

since m is **odd**. □

LECTURE 22

Poincaré duality and intersection form

We are going to meet an important class of topological spaces and study one of their fundamental cohomological properties. This lecture will be short of proofs, but rather aims to see an important theorem and structures at work.

Manifolds

We start with defining an important class of spaces.

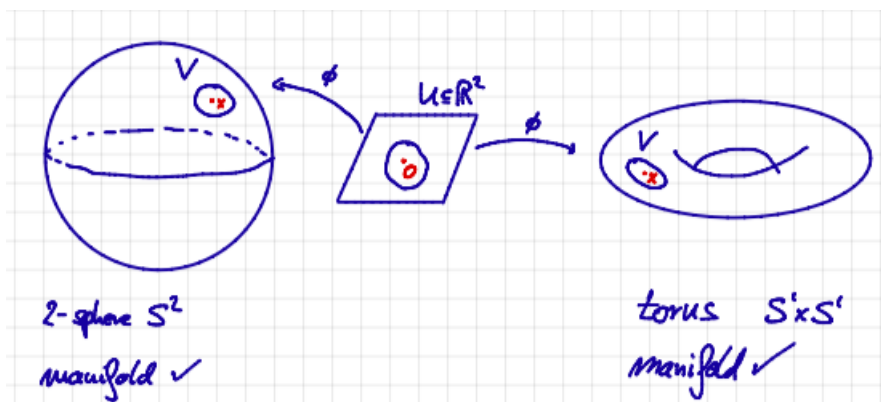
Definition: Topological manifolds

A n -dimensional topological manifold is a Hausdorff space in which each point has an open neighborhood which is homeomorphic to \mathbb{R}^n .

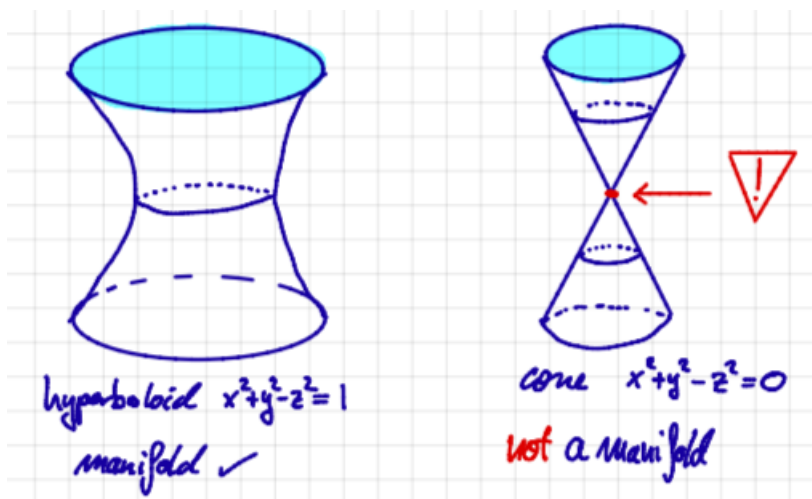
In this lecture, the word **manifold** will always mean a topological manifold.

You know many examples of manifolds, most notably \mathbb{R}^n itself, any open subset of \mathbb{R}^n , n -spheres S^n , tori, Klein bottle, projective spaces. Even though the definition does not refer to this information, any manifold M can be embedded in some \mathbb{R}^N for some large N (which depends on M).

Though it is a crucial point that N and n can and usually are different. For example, S^2 is a subset of \mathbb{R}^3 , but each point on S^2 has a neighborhood which looks like a plane, i.e., is homeomorphic to \mathbb{R}^2 .



There are many reasons why manifolds are important. One of them is that we understand and can study them locally, while they can be very complicated globally.



Poincaré duality

In this lecture, all homology and cohomology groups will be with \mathbb{F}_2 -coefficients. Recall that there is a pairing

$$H^k(X; \mathbb{F}_2) \otimes H_k(X; \mathbb{F}_2) \xrightarrow{\langle -, - \rangle} \mathbb{F}_2$$

defined by evaluating a cocycle φ on a cycle σ which is an element in \mathbb{F}_2 .

We are going to study the consequences of the following famous fact:

Theorem: Poincaré duality mod 2

Let M be a **compact topological manifold** of dimension n . Then there exists a unique class $[M] \in H_n(M; \mathbb{F}_2)$, called the **fundamental class of M** , such that, for every $p \geq 0$, the **pairing**

$$H^p(M; \mathbb{F}_2) \otimes H^{n-p}(M; \mathbb{F}_2) \xrightarrow{\cup} H^n(M; \mathbb{F}_2) \xrightarrow{\langle -, [M] \rangle} \mathbb{F}_2$$

is **perfect**.

That the pairing is perfect means that the **adjoint map**

$$H^p(X; \mathbb{F}_2) \xrightarrow{\langle a \cup -, [M] \rangle} \text{Hom}(H^{n-p}(X; \mathbb{F}_2), \mathbb{F}_2), \quad a \mapsto \langle a \cup -, [M] \rangle$$

is an **isomorphism**.

Here are some **first consequences** of this theorem:

- Since cohomology vanishes in negative dimensions, we must have $H^p(X; \mathbb{F}_2) = 0$ for $p > n$ as well.
- Since M is assumed to be compact, we know that $\pi_0(M)$, the set of connected components of M , is **finite**. Moreover, we once showed that $H^0(M; \mathbb{F}_2)$ equals $\text{Map}(\pi_0(M), \mathbb{F}_2)$. Hence we get

$$H^n(M; \mathbb{F}_2) = \text{Hom}(H^0(M; \mathbb{F}_2), \mathbb{F}_2) = \text{Hom}(\text{Map}(\pi_0(M), \mathbb{F}_2), \mathbb{F}_2) = \mathbb{F}_2[\pi_0(M)].$$

- A vector space admitting a perfect pairing is finite-dimensional. Hence $H^p(M; \mathbb{F}_2)$ is **finite-dimensional** for all p .

There is a version of the **Universal Coefficient Theorem** with \mathbb{F}_2 -coefficients. Since \mathbb{F}_2 is a field, it implies that there is an isomorphism

$$\text{Hom}(H^{n-p}(M; \mathbb{F}_2), \mathbb{F}_2) \cong H_{n-p}(M; \mathbb{F}_2).$$

(Note that we formulated the UCT with the roles of homology and cohomology reversed. But, since the map arose from the Kronecker pairing, we can also produce the claimed version of the UCT. As mentioned in the intro to this lecture, we rush through some points for the sake of telling a good story.)

Composition with the above pairing yields an isomorphism

$$H^p(X; \mathbb{F}_2) \xrightarrow{\cong} \text{Hom}(H^{n-p}(M; \mathbb{F}_2), \mathbb{F}_2) \xleftarrow{\cong} H_{n-p}(M; \mathbb{F}_2).$$

\cong

Definition: Poincaré duals

Homology and cohomology corresponding to each other under the dotted isomorphism are said to be **Poincaré dual** to each other.

Intersection pairing

Combining this isomorphism for different dimensions, we can write the cup product pairing in cohomology as a pairing in homology (where we drop the coefficients which are still \mathbb{F}_2)

$$\begin{array}{ccc} H_p(M) \otimes H_q(M) & \xrightarrow{\cap} & H_{p+q-n}(M) \\ \cong \downarrow & & \downarrow \cong \\ H^{n-p}(M) \otimes H^{n-q}(M) & \xrightarrow{\cup} & H^{2n-p-q}(M). \end{array}$$

The top map is called the **intersection pairing** in homology.

Here is how we should think about it:

- Let $\alpha \in H_p(M)$ and $\beta \in H_q(M)$ be homology classes.
- Represent them, if possible, as the image of fundamental classes of submanifolds of M . That means that there are submanifolds Y and Z in M of dimensions p and q , respectively, such that

$$\alpha = i_*[Y] \text{ and } \beta = j_*[Z]$$

where $i_*: H_p(Y) \rightarrow H_p(M)$ and $j_*: H_q(Z) \rightarrow H_q(M)$ are the homomorphisms induced by the inclusions $i: Y \hookrightarrow M$ and $j: Z \hookrightarrow M$.

- Move them a bit if necessary to make them **intersect transversally**.
- Then their intersection is a submanifold of dimension $p + q - n$ and its image will represent the homology class $\alpha \cap \beta$.

Let us look at an example:

Example: Intersection on a torus

Let $M = T^2 = S^1 \times S^1$ be the two-dimensional torus. We know

$$H^1(M) = \mathbb{F}_2\langle a, b \rangle$$

with $a^2 = 0 = b^2$, and $H^2(M)$ is generated by $ab = ba$.

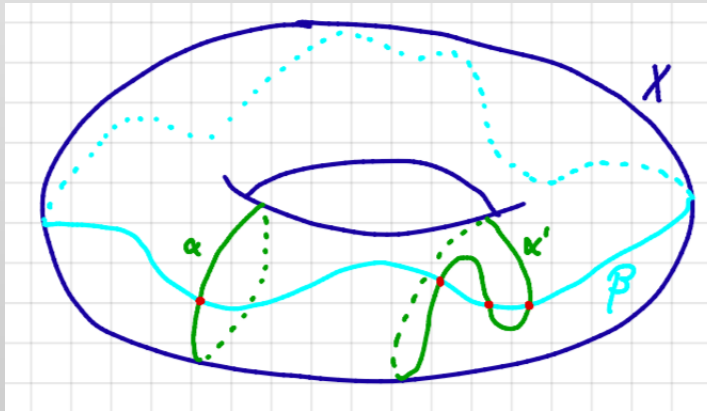
The **Poincaré duals** α and β of a and b are represented by cycles which wrap around one or the other factor circle of M .

The cycles α and β can be made to intersect in a single point. This reflects the equation

$$\langle a \cup b, [M] \rangle = 1.$$

But this equation also tells us that α and β can only be moved in such a way that they intersect in an **odd** number of points.

The fact that $a^2 = 0$ reflects the fact that its Poincaré dual α can be moved so as not to intersect itself.



Intersection form

Let us look at a particular case of Poincaré duality. Let us assume that M is **even-dimensional**, say of dimension $n = 2p$. Then Poincaré duality implies that we have a **symmetric bilinear form** on the \mathbb{F}_2 -vector space $H^p(M)$:

$$H^p(M) \otimes_{\mathbb{F}_2} H^p(M) \rightarrow H^{2p}(M) \cong \mathbb{F}_2.$$

As we just observed, this can be interpreted as a bilinear form on homology $H_p(M)$. Evaluating this form can be viewed as describing (modulo 2) the number of points where two p -cycles intersect, after they have been moved in general position, i.e., a position where they intersect transversally.

Definition: Intersection form

This form $H_p(M) \otimes H_p(M) \rightarrow \mathbb{F}_2$ is called **intersection form** and will be denoted

$$\alpha \cdot \beta := \langle a \cup b, [M] \rangle$$

where a and b are Poincaré dual to α and β , respectively.

Let us consider two examples:

- For the **sphere** S^2 , the first homology is **trivial**, and so is the intersection form on S^2 .
- In the **example of the torus**, the intersection form can be described in terms of the basis α and β by the matrix (since any such form looks like $(v, w) \mapsto v^T H w$)

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Such a form is called **hyperbolic**.

Apparently, it would good to know a bit more about such forms. We are going to review what we need to know about them now and then get back to the application in topology in the next lecture.

A digression on symmetric bilinear forms

We need to have a brief look at such forms.

So let V be a finite-dimensional vector space over \mathbb{F}_2 together with a nondegenerate symmetric bilinear form. Such a form restricts to any subspace W of V , but the restricted form may be degenerate. But any subspace has an orthogonal complement

$$W^\perp = \{v \in V : v \cdot w = 0 \text{ for all } w \in W\}.$$

Then we have the following lemma:

Lemma

The restriction of a nondegenerate symmetric bilinear form on V to a subspace W is nondegenerate if and only if $W \cap W^\perp = 0$.

In this case, the restriction to W^\perp is also nondegenerate and the splitting

$$V \cong W \oplus W^\perp$$

respects the forms.

We can use this lemma to inductively decompose all finite-dimensional symmetric bilinear forms:

- If there is a vector $v \in V$ with $v \cdot v = 1$, then it generates a nondegenerate subspace, i.e., a subspace on which the restriction of the form is nondegenerate, and

$$V = \langle v \rangle \oplus \langle v \rangle^\perp$$

where $\langle v \rangle$ denotes the subspace generated by v .

- Continue to split off one-dimensional subspaces until we reach a nondegenerate symmetric bilinear form such that $v \cdot v = 0$ for all vectors.
- Unless we ended up with zero space, we can pick a nonzero vector v . Since the form is nondegenerate, there must be a vector w such that $v \cdot w = 1$.
- The two vectors v and w generate a hyperbolic subspace, i.e., one on which the form is represented by the matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- Split this space off, and continue the process.

This procedure shows:

Proposition: Classification of nondegenerate forms

Any finite-dimensional nondegenerate symmetric bilinear form over \mathbb{F}_2 splits as an orthogonal sum of forms with matrices

$$I = (1) \text{ and } H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We are going to continue the study of forms and get back to topology in the next lecture.

LECTURE 23

Classification of surfaces

We will first continue the study of bilinear forms, and then use this knowledge to classify all compact connected surfaces, i.e., compact connected two-dimensional manifolds. Then we are going to contemplate a bit more on Poincaré duality. In this lecture, all vector spaces, homology and cohomology groups will be **over \mathbb{F}_2** .

The monoid of nondegenerate symmetric bilinear forms

Last time we showed:

Proposition: Classification of nondegenerate forms

Any finite-dimensional nondegenerate symmetric bilinear form over \mathbb{F}_2 splits as an orthogonal sum of forms with matrices

$$I = (1) \text{ and } H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now let **Bil** be the set of isomorphism classes of nondegenerate symmetric bilinear forms over \mathbb{F}_2 . This is a **commutative monoid** under the operation of taking orthogonal direct sums (that means it is like a group except that there no inverses).

Since any such form corresponds to a matrix, we can identify **Bil** also with the **set of invertible symmetric matrices modulo the equivalence relation of similarity**:

- Two matrices A and B are called **similar**, denoted $A \sim B$, if $B = PAP^T$ for some invertible matrix P .
- Every form corresponds to a matrix A determined by $v \cdot w = v^T Aw$.
- Assume we have given two vector spaces V_1 and V_2 with nondegenerate symmetric bilinear forms which are represented by matrices A_1 and A_2 ,

respectively. Then there is an isomorphism $\varphi: V \xrightarrow{\cong} W$ such that

$$\varphi(v \cdot_V w) = \varphi(v) \cdot_W \varphi(w),$$

if and only if A_1 and A_2 are similar.

Hence, in order to understand **Bil** we can aim to understand invertible matrices modulo similarity. Here is a crucial fact:

Lemma

Over \mathbb{F}_2 we have the similarity

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof: The assertion is equivalent to saying there is an invertible matrix P such that

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = PP^T.$$

This is the case for $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ where we need to remember that we work over \mathbb{F}_2 . **QED**

Since neither nI nor mH are similar to other matrices, $I + H = 3I$ is the only relation. As a consequence we get:

Bilinear forms via generators and relations

The commutative monoid **Bil** is generated by I and H modulo the relation

$$I + H = 3I.$$

Now we are going to apply this knowledge to the intersection form.

Intersection form

Let us look at a particular case of Poincaré duality. Let us assume that M is **even-dimensional**, say of dimension $n = 2p$. Then Poincaré duality defines a

symmetric bilinear form on the \mathbb{F}_2 -vector space $H^p(M)$:

$$H^p(M) \otimes_{\mathbb{F}_2} H^p(M) \rightarrow H^p(M).$$

As we observed last time, this can be interpreted as a bilinear form on homology $H_p(M)$. Recall that evaluating this form can be viewed as describing (modulo 2) the number of points where two p -cycles intersect, after they have been moved in general position, i.e., a position where they intersect transversally.

Intersection form

For a compact manifold of dimension $n = 2p$, the intersection pairing

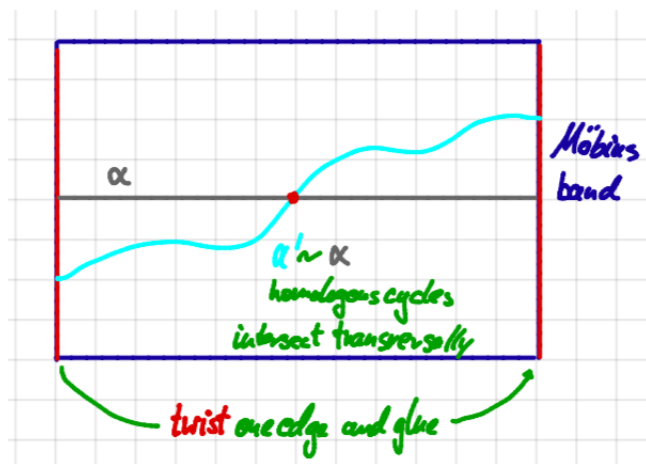
$$H_p(M; \mathbb{F}_2) \otimes H_p(M; \mathbb{F}_2) \rightarrow \mathbb{F}_2, \alpha \cdot \beta := \langle a \cup b, [M] \rangle$$

defines a nondegenerate symmetric bilinear form on $H_p(M; \mathbb{F}_2)$, called the **intersection form**. Here a and b are Poincaré dual to α and β , respectively.

- We have seen the **example of the torus** for which the intersection form is **hyperbolic**, i.e., can be described in terms of the basis α and β by the matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- For another **example**, take $M = \mathbb{RP}^2$. We know $H_1(\mathbb{RP}^2) = \mathbb{F}_2$. Moreover, \mathbb{RP}^2 can be viewed as a **Möbius band with a disk glued along the boundary**. On the Möbius band, there is a nontrivial intersection. Hence the **intersection form is nontrivial** and therefore given by I according to our classification, since on a one-dimensional space there are only two options. As a consequence we see that in whatever way we try to move the boundary of the Möbius band in \mathbb{RP}^2 , it will always **intersect itself in an odd number of points**.



Note that the **open Möbius band** itself is a two-dimensional manifold, but it is **not compact**. While the **closed** Möbius is compact, it is not a manifold according to the definition we stated last time. Though it is a **manifold with boundary**. The story is different if we allow boundaries.

Connected sums

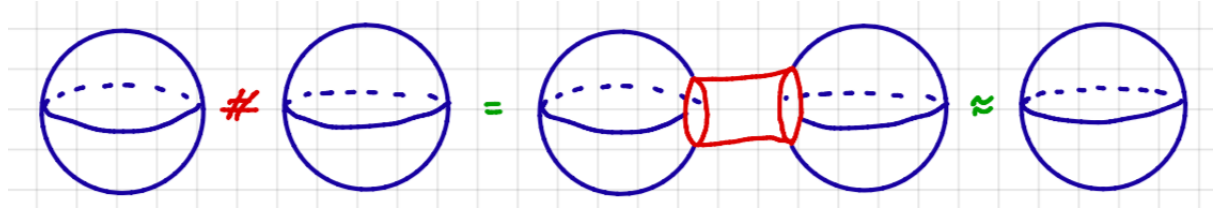
There is an interesting **geometric operation** on manifolds which **produces new ones out of old**:

Given two compact connected manifolds M_1 and M_2 both of dimension n . Then we can

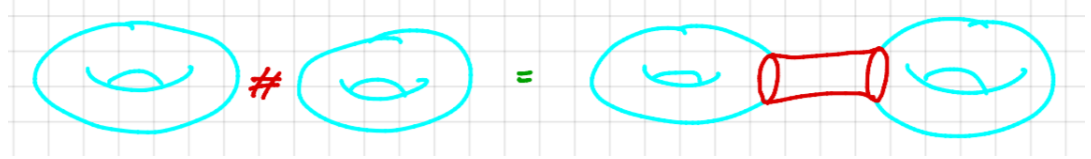
- cut out a small open n -dimensional disk D^n of each one, and
- sew them together along the resulting boundary spheres S^{n-1} , i.e., identify the boundaries via a homeomorphism.
- The resulting space is called the **connected sum** of M_1 and M_2 and is denoted by $M_1 \# M_2$. Note $M_1 \# M_2$ is a **connected compact n -dimensional manifold**.

Let us see two **examples**:

- There is not much happening if we take $S^2 \# S^2$ as it is homeomorphic to S^2 :



- But we get a new surface for $T^2 \# T^2$:



Lemma: Homology of connected sums

There is an isomorphism

$$H_i(M_1 \# M_2) \cong H_i(M_1) \oplus H_i(M_2) \text{ for all } 0 < i < n.$$

Proof: We start with the pair $(M_1 \# M_2, S^{n-1})$. Since M_1 and M_2 are manifolds of dimension n , there is an open neighborhood around S^{n-1} in $M_1 \# M_2$ which retracts onto S^{n-1} . Thus, by a result we showed some time ago when we discussed cell complexes and wedge sums, we know

$$H_*(M_1 \# M_2, S^{n-1}) \cong H_*((M_1 \# M_2)/S^{n-1}, \text{pt}) \cong \tilde{H}_*(M_1 \vee M_2).$$

Now we consider the long exact sequence of the pair $(M_1 \# M_2, S^{n-1})$:

$$\cdots \rightarrow \tilde{H}_i(S^{n-1}) \rightarrow H_i(M_1 \# M_2) \rightarrow H_i(M_1 \# M_2, S^{n-1}) \rightarrow \tilde{H}_{i-1}(S^{n-1}) \rightarrow \cdots$$

Since only $\tilde{H}_{n-1}(S^{n-1})$ is nonzero, we deduce

$$\tilde{H}_i(M_1 \# M_2) \cong H_i(M_1 \# M_2, S^{n-1}) \cong \tilde{H}_*(M_1 \vee M_2) \text{ for all } i < n - 1.$$

Hence, for $0 < i < n - 1$, we have

$$H_i(M_1 \# M_2) \cong H_i(M_1) \oplus H_i(M_2).$$

The remaining part of the long exact sequence is then

$$0 \rightarrow H_n(M_1 \# M_2) \rightarrow H_n(M_1 \vee M_2) \rightarrow H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(M_1 \# M_2) \rightarrow H_{n-1}(M_1 \vee M_2) \rightarrow 0$$

where the zeros on both ends are explained by the vanishing of the corresponding homologies of S^{n-1} .

Since fundamental classes are natural, the map

$$(30) \quad H_n(M_1) \oplus H_n(M_2) \xrightarrow{\cong} H_n(M_1 \vee M_2) \rightarrow H_{n-1}(S^{n-1})$$

sends the fundamental classes of both M_1 and M_2 to the fundamental class of S^{n-1} . Thus, this map is surjective and we deduce from the exactness of the sequence that

$$H_{n-1}(M_1 \# M_2) \cong H_{n-1}(M_1) \oplus H_{n-1}(M_2).$$

We also see that $H_n(M_1 \# M_2)$ is the kernel of the map in (30).

QED

Lemma: Connected sums and intersection forms

Assume both M_1 and M_2 are of dimension $n = 2p$. Then the isomorphism

$$H_p(M_1 \# M_2) \xrightarrow{\cong} H_p(M_1) \oplus H_p(M_2)$$

is **compatible with the intersection form**.

Proof: Fundamental classes are natural in the sense that the homomorphism

$$H_n(M_1 \# M_2) \xrightarrow{\cong} H_n(M_1 \vee M_2) \xrightarrow{\cong} H_n(M_1) \oplus H_n(M_2), [M_1 \# M_2] \mapsto [M_1] + [M_2]$$

sends the fundamental class of $[M_1 \# M_2]$ to the sum of the fundamental classes of M_1 and M_2 .

Moreover, the cup product is natural so that we get a commutative diagram

$$\begin{array}{ccccc} H^p(M_1 \# M_2) \otimes H^p(M_1 \# M_2) & \xrightarrow{\cup} & H^{2p}(M_1 \# M_2) & \xrightarrow{\langle -, [M_1 \# M_2] \rangle} & \mathbb{F}_2 \\ \downarrow & & \downarrow & & \downarrow \\ H^p(M_1) \otimes H^p(M_1) \oplus H^p(M_2) \otimes H^p(M_2) & \xrightarrow{\cup} & H^{2p}(M_1) \oplus H^{2p}(M_2) & \xrightarrow{\langle -, [M_1] \rangle + \langle -, [M_2] \rangle} & \mathbb{F}_2. \end{array}$$

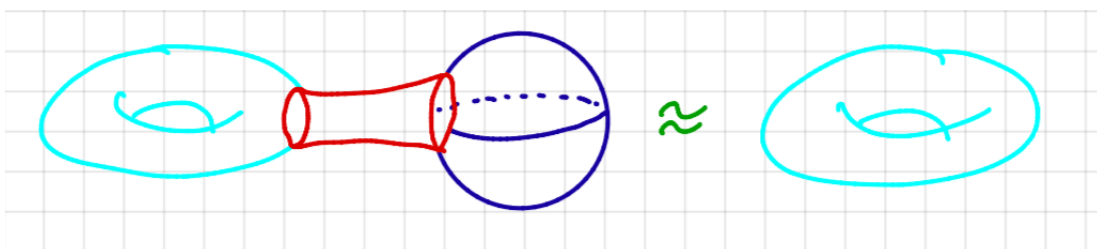
Now it remains to translate this into the intersection pairing in homology which proves the claim. **QED**

Classification of surfaces

Motivated by the examples of the torus and real projective plane we are going to focus now on **the case $n = 2$** , i.e., two-dimensional manifolds which we are

going to call **surfaces**. In fact, we are going to study **compact surfaces**. In this case we have an intersection form on $H_1(M)$.

We write **Surf** for the set of **homeomorphism classes of compact connected surfaces**. The connected sum operation provides it with the structure of a **commutative monoid**. The **neutral element** being S^2 , since $S^2 \# \Sigma \approx \Sigma$ for any surface Σ .



There is the following important result:

Theorem: Classification of surfaces

Associating the intersection form to a surface defines an **isomorphism** of commutative monoids

$$\mathbf{Surf} \xrightarrow{\cong} \mathbf{Bil}.$$

This theorem is **great** because it gives us a **complete algebraic classification of a class of geometric objects**. This is one reason why algebraic topology is so useful.

Actually, we are not finished with proving the theorem yet. Our examples show us that T^2 corresponds to H and \mathbb{RP}^2 corresponds to I . And S^2 is sent to the neutral element.

It remains to **show the relation** (and that this is the only relation)

$$(31) \quad T^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2.$$

One way to do this is to **triangulate the surfaces** involved. This requires too much geometric thinking for us today.

Instead, we make the following **observation**. We have not defined an orientation, but assuming we know what that means it is a surprising fact that

even though we never assumed anything about orientations and worked with F_2 -coefficients, the theorem tells us what the **orientable surfaces** look like.

For, the **orientable surfaces** correspond to the forms gH where g is the **genus** of the surface. This follows from the facts that T^2 is **orientable** whereas \mathbb{RP}^2 is **not**.

In other words, any compact connected **orientable** surface Σ of genus g is homeomorphic to the connected sum of g tori

$$\Sigma \cong T^2 \# \cdots \# T^2.$$

Since S^2 is also an orientable surface, we allow $g = 0$ for this case too.

The real projective plane is not orientable. Therefore, any surface which is homeomorphic to a connected sum of at least one copy of \mathbb{RP}^2 is **not orientable**. In fact, the classification theorem and the relation (31) tell us that such a surface is actually homeomorphic to a connected sum of copies of just \mathbb{RP}^2 s.

Quadratic refinement and the Kervaire invariant

Recall that away from characteristic 2 there is a bijection between quadratic forms and symmetric bilinear forms. However, since we are working over F_2 , we can ask whether there is a quadratic refinement q of the intersection form such that

$$q(x + y) = q(x) + q(y) + x \cdot y.$$

For such a refinement to exist requires $x \cdot x = 0$ for all $x \in H_1(M; \mathbb{F}_2)$, since

$$0 = q(2x) = q(x) + q(x) + x \cdot x = x \cdot x.$$

Hence we can only expect such a refinement on a sum of tori, i.e., on an orientable surface.

The existence of a quadratic refinement is an additional structure associated with the intersection form. Geometrically, it corresponds to a trivialization of the normal bundle of an embedding into an \mathbb{R}^N for some N sufficiently large. Such a trivialization is called a **framing**. There is an invariant for quadratic forms in characteristic two, called the **Arf invariant**. In the case of a surface, or more generally a manifold of dimension $4k + 2$ (the only dimension where interesting things happen for this invariant), this invariant is called the **Kervaire invariant**. This invariant is a measure for if we can do certain **surgery** manoeuvres on a manifold or not. Kervaire and Milnor used this invariant to study the differentiable structures on spheres.

But there were certain dimensions they could not completely explain. To settle the missing dimensions remained an open problem for about 60 years

until Mike [Hill](#), Mike [Hopkins](#), and Douglas [Ravenel](#) finally solved the mystery (almost completely as there is one dimension left, it is 126) in a groundbreaking work in 2009 (published in 2016) using highly sophisticated methods in [equivariant stable homotopy theory](#).

LECTURE 24

More on Poincaré duality

We continue our discussion of Poincaré duality. First we see two applications of the theorem with \mathbb{F}_2 coefficients. Then we will discuss what we need to do for other coefficients. This will lead to an important concept, orientations of manifolds, and an important algebraic structure, the cap product.

Dualities reflect fundamental properties

Poincaré duality is extremely interesting, since it reflects a **deep symmetry** in the homology and cohomology groups on manifolds. For, the cohomology in dimension p determines the homology in dimension $n-p$. This symmetry has many consequences which make the study of manifolds particularly interesting.

Duality theorems arise in many areas of mathematics and always reflect deep and interesting structures.

We start with an application of Poincaré duality modulo 2.

Applications of Poincaré duality with \mathbb{F}_2 -coefficients

Recall the important theorem:

Theorem: Poincaré duality mod 2

Let M be a connected **compact manifold** of dimension n . Then there exists a unique class $[M] \in H_n(M; \mathbb{F}_2)$, called the **fundamental class of M** , such that, for every $p \geq 0$, the **pairing**

$$H^p(M; \mathbb{F}_2) \otimes H^{n-p}(M; \mathbb{F}_2) \xrightarrow{\cup} H^n(M; \mathbb{F}_2) \xrightarrow{\langle -, [M] \rangle} \mathbb{F}_2$$

is **perfect**.

Since real projective space is a compact connected n -dimensional manifold, Poincaré duality applies. And, in fact, we can use this result to deduce the algebra structure on the cohomology of real projective space:

Corollary: Cohomology of \mathbb{RP}^n

Let x be the nonzero element in $H^1(\mathbb{RP}^n; \mathbb{F}_2)$. Then x^k is the nonzero element of $H^k(\mathbb{RP}^n; \mathbb{F}_2)$ for $k = 2, \dots, n$.

Thus $H^*(\mathbb{RP}^n; \mathbb{F}_2)$ is the **truncated polynomial algebra**

$$H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$$

generated by x in degree 1 and truncated by setting $x^{n+1} = 0$.

Moreover, $H^*(\mathbb{RP}^\infty; \mathbb{F}_2)$ is a polynomial algebra

$$H^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$$

generated by x in degree 1.

Proof: The proof is **by induction** on n .

By the construction of the cell structure on \mathbb{RP}^n , we know that the inclusion $j_k: \mathbb{RP}^k \hookrightarrow \mathbb{RP}^{k+1}$ is a **map of cell complexes** which induces an isomorphism

$$H^i(\mathbb{RP}^k; \mathbb{F}_2) \xrightarrow{\cong} H^i(\mathbb{RP}^{k+1}; \mathbb{F}_2) \text{ for all } i = 0, \dots, k,$$

which sends the nonzero element $x \in H^1(\mathbb{RP}^k; \mathbb{F}_2)$ to the nonzero element in $H^1(\mathbb{RP}^{k+1}; \mathbb{F}_2)$ which we therefore also denote by x .

Hence, assuming x^k is the nonzero element in $H^k(\mathbb{RP}^k; \mathbb{F}_2)$, it suffices to show that $x \cup x^k$ is nonzero in $H^{k+1}(\mathbb{RP}^{k+1}; \mathbb{F}_2)$.

By **Poincaré duality**, the pairing

$$H^1(\mathbb{RP}^{k+1}; \mathbb{F}_2) \otimes H^k(\mathbb{RP}^{k+1}; \mathbb{F}_2) \xrightarrow{\cup} H^{k+1}(\mathbb{RP}^{k+1}; \mathbb{F}_2) \xrightarrow{\langle -, [\mathbb{RP}^{k+1}] \rangle} \mathbb{F}_2$$

is perfect. Since x and x^k are nonzero by assumption, this implies $x \cup x^k = x^{k+1}$ is nonzero as well.

For \mathbb{RP}^n , we know that $H^i(\mathbb{RP}^n; \mathbb{F}_2) = 0$ for $i > n$, since there are no cells in dimensions bigger than n . Thus $x^{n+1} = 0$.

For \mathbb{RP}^∞ we just continue the induction process. □

As an application of this calculation, we are going to prove another famous theorem, the **Borsuk-Ulam Theorem**.

Lemma

Let $f: \mathbb{RP}^m \rightarrow \mathbb{RP}^n$ be a continuous map which induces a nontrivial map

$$f_* \neq 0: H_1(\mathbb{RP}^m; \mathbb{F}_2) \rightarrow H_1(\mathbb{RP}^n; \mathbb{F}_2)$$

Then $m \leq n$.

Proof: Since $H^1(X; \mathbb{F}_2) \cong \text{Hom}(H_1(X; \mathbb{F}_2))$, the assumption implies that the induced map in cohomology

$$f^*: H^1(\mathbb{RP}^n; \mathbb{F}_2) \rightarrow H^1(\mathbb{RP}^m; \mathbb{F}_2)$$

is nontrivial as well.

Let $x \neq 0$ be the nonzero element in $H^1(\mathbb{RP}^n; \mathbb{F}_2)$. Then $f^*(x) \neq 0$ is nonzero in $H^1(\mathbb{RP}^m; \mathbb{F}_2)$. By the calculation of the $H^*(\mathbb{RP}^m; \mathbb{F}_2)$, we have

$$0 \neq (f^*(x))^m = f^*(x^m).$$

Thus, $x^m \neq 0$ in $H^m(\mathbb{RP}^n; \mathbb{F}_2)$ which implies $m \leq n$. □

Lemma: Paths between antipodal points

Let $p \in S^n$ and let $\sigma: \Delta^1 \rightarrow S^n$ be a 1-simplex on S^n which connects p and its antipodal point $-p$ in S^n , i.e., $\sigma(e_0) = p$ and $\sigma(e_1) = -p$. Let

$$\pi: S^n \rightarrow \mathbb{RP}^n$$

be the quotient map.

Then $\pi_*(\sigma) = \pi \circ \sigma$ is a **cycle** on \mathbb{RP}^n which represents a **nonzero element** in $H_1(\mathbb{RP}^n; \mathbb{F}_2)$.

Proof: First, that $\pi_*(\sigma)$ is a **cycle on \mathbb{RP}^n** just follows from the fact

$$[\pi(\sigma(e_0))] = [-\pi(\sigma(e_0))] = [\pi(\sigma(e_1))] \text{ in } \mathbb{RP}^n.$$

It remains to show that it is **not a boundary**.

Recall that there is a cell structure on S^n with skeleta

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n.$$

By symmetry, we can assume that p and $-p$ are the points of S^0 .

• For $n = 1$, we have a **homeomorphism** $\mathbb{RP}^1 \approx S^1$ (for example, one could use the stereographic projection). Since σ connects p and $-p$ on S^1 , there is an

integer k such that σ walks around S^1 $(k + 1/2)$ -many times. Thus $\pi_*(\sigma)$ walks around \mathbb{RP}^1 $(2k + 1)$ -many times, i.e., an **odd** number of times.

Now recall that we showed

$$\mathbb{Z} \xrightarrow{\cong} H_1(S^1; \mathbb{Z}), m \mapsto (z \mapsto z^m)$$

where we use the identification $\pi(S^1) = H_1(S^1; \mathbb{Z})$ that we showed in the exercises. This implies that with \mathbb{F}_2 -coefficients, even numbers correspond to 0 in $H_1(\mathbb{RP}^1; \mathbb{F}_2)$ and odd numbers correspond to the nonzero element in $H_1(\mathbb{RP}^1; \mathbb{F}_2)$.

Thus, the image of $\pi_*(\sigma)$ in $H_1(\mathbb{RP}^1; \mathbb{F}_2)$ is nonzero.

• For $n > 1$, we first choose a path τ on $S^1 \subset S^n$ which connects p and $-p$ on S^1 . By the previous case, we know $[\pi_*(\tau)] \neq 0$ in $H_1(\mathbb{RP}^1; \mathbb{F}_2)$. The **inclusion** map $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ induces an **isomorphism**

$$H_1(\mathbb{RP}^1; \mathbb{F}_2) \xrightarrow{\cong} H_1(\mathbb{RP}^n; \mathbb{F}_2).$$

Hence $[\pi_*(\tau)] \neq 0$ in $H_1(\mathbb{RP}^n; \mathbb{F}_2)$ as well.

But for $n > 1$, the difference $\sigma - \tau$ is a boundary, since it is homotopic to a constant map. This implies

$$[\pi_*(\sigma)] = [\pi_*(\tau)] \neq 0 \text{ in } H_1(\mathbb{RP}^n; \mathbb{F}_2).$$

Lemma: No antipodal maps

For any n , there is no continuous map $f: S^{n+1} \rightarrow S^n$ with

$$f(-p) = -f(p) \text{ for all } p \in S^{n+1}.$$

Proof: Assume there was such a map f . Since $f(-p) = -f(p)$ for all p , f induces a map

$$\bar{f}: \mathbb{RP}^{n+1} \rightarrow \mathbb{RP}^n$$

which fits into a commutative diagram

$$\begin{array}{ccc} S^{n+1} & \xrightarrow{f} & S^n \\ \pi^{n+1} \downarrow & & \downarrow \pi^n \\ \mathbb{RP}^{n+1} & \xrightarrow{\bar{f}} & \mathbb{RP}^n. \end{array}$$

Now we take a 1-simplex σ which connects two antipodal points on S^{n+1} . Its image $f_*(\sigma) = f \circ \sigma$ is then a 1-simplex which connects two antipodal points on S^n , since $f(-p) = f(-p)$.

By the previous lemma, $\pi_*(f_*(\sigma)) \neq 0$ in $H_1(\mathbb{RP}^n; \mathbb{F}_2)$. Thus

$$\bar{f}_*(\pi_*^{n+1}(\sigma)) = \pi_*(f_*(\sigma)) \neq 0.$$

In other words,

$$\bar{f}_* \neq 0: H_1(\mathbb{RP}^{n+1}; \mathbb{F}_2) \rightarrow H_1(\mathbb{RP}^n; \mathbb{F}_2)$$

is nontrivial. By the other lemma, this is not possible. Hence f cannot exist. \square

The Borsuk-Ulam Theorem

Let $g: S^n \rightarrow \mathbb{R}^n$ be a continuous map. Then there is a point $p \in S^n$ with

$$g(p) = g(-p).$$

Proof: If there is no such point, we can define a continuous map

$$f: S^n \rightarrow S^{n-1}, p \mapsto \frac{g(p) - g(-p)}{|g(p) - g(-p)|}.$$

But this map satisfies

$$f(-p) = -f(p).$$

This contradicts the previous lemma. \square

Orientation and fundamental classes

We now leave the world of \mathbb{F}_2 -coefficients and contemplate on what we need for a Poincaré duality theorem with other coefficients. Since we will only sketch the main ideas anyway, we will just look at \mathbb{Z} -coefficients.

We start with the following observation on the homology groups of a manifold at a point:

Lemma: Local homology on manifolds

Let M be an n -dimensional topological manifold. For any point $x \in M$, there is an isomorphism

$$H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}.$$

and $H_i(M, M - \{x\}; \mathbb{Z}) = 0$ for all $i \neq n$.

Proof: Since M is a manifold, there is an open neighborhood U around x in M such that $U \cong \mathbb{R}^n$. We set $Z = M - U$ and apply excision to get

$$\begin{aligned} H_i(M, M - \{x\}; \mathbb{Z}) &\cong H_i(M - Z, (M - \{x\}) - Z; \mathbb{Z}) \text{ (by excision)} \\ &= H_i(U, U - \{x\}; \mathbb{Z}) \\ &\cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \\ &\cong H_{i-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) \text{ (by homotopy invariance and long ex. seq.)} \\ &\cong H_{i-1}(S^{n-1}; \mathbb{Z}) \text{ (by homotopy invariance).} \end{aligned}$$

This implies

$$H_i(M, M - \{x\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

Local orientation

The group $H_n(M, M - \{x\}; \mathbb{Z})$ is often called the **local homology of M at x** . It is an infinite cyclic group and therefore has **two generators**. A **choice** of a generator $\mu_x \in H_n(M, M - \{x\}; \mathbb{Z})$ is a **local orientation** of M at x .

For every point $x \in M$, we can choose such a generator. Note that such a choice was not necessary in \mathbb{F}_2 , since there is only one generator. That makes \mathbb{F}_2 -coefficients quite special.

The **natural question** is how all these choices are related. In other words, is it possible to **choose** these generators **in a compatible way**?

More precisely, let x and y be two points in M which both lie in some subset $U \subset M$. The inclusions $i_x: \{x\} \hookrightarrow M$ and $i_y: \{y\} \hookrightarrow M$ induce maps

$$H_n(M, M - \{x\}; \mathbb{Z}) \xleftarrow{i_{x*}} H_n(M, M - U; \mathbb{Z}) \xrightarrow{i_{y*}} H_n(M, M - \{y\}; \mathbb{Z}).$$

A class $\mu_U \in H_n(M, M - U; \mathbb{Z})$ which maps to generators in $H_n(M, M - \{x\}; \mathbb{Z})$ and $H_n(M, M - \{y\}; \mathbb{Z})$. Such an μ_U would define local orientations $\mu_x := i_{x*}(\mu_U)$ and $\mu_y := i_{y*}(\mu_U)$ at x and y , respectively. We call such an element μ_U a **fundamental class at U** .

Around every point in M there is a little neighborhood U with a fundamental class at U . The crucial question is: how large can we choose such a U ? Ideally, we would like to be able to choose $U = M$ such that $H_n(M, M - U) = H_n(M)$.

Unfortunately, this is not always possible. This leads to an important concept:

Orientations

Let M be a compact connected n -dimensional manifold.

- An **orientation** of M is a function $x \mapsto \mu_x$, where $\mu_x \in H_n(M, M - \{x\}; \mathbb{Z})$ is a generator, which satisfies the following condition:
At any point $x \in M$, there is a neighborhood U around x and an element $\mu_U \in H_n(M, M - U; \mathbb{Z})$ such that $i_{y*}(\mu_U) = \mu_y$ for all $y \in U$.
- If such an orientation exists, we say that M is **orientable**.

If M is orientable, then there are **exactly two orientations**. If M is orientable, and we have **chosen an orientation**, then we say that M is **oriented**.

We can reformulate this in terms of a particular class in homology, the fundamental class. The following statement is both a definition and proposition. We skip the proof, since we only have time for a rough sketch of the story.

Fundamental classes and orientability

Let M be a compact connected n -dimensional manifold.

- A fundamental class of M is an element $\mu \in H_n(M; \mathbb{Z})$ such that, for every point $x \in M$, the image of μ under the map

$$H_n(M; \mathbb{Z}) \rightarrow H_n(M, M - \{x\}; \mathbb{Z})$$

induced by the inclusion $(M, \emptyset) \hookrightarrow (M, M - \{x\})$ is a generator.

- M is orientable if and only if M has a fundamental class.
- M is orientable if and only if $H_n(M; \mathbb{Z}) = \mathbb{Z}$.

For **example**, \mathbb{RP}^{2n} is not orientable, since $H_{2n}(\mathbb{RP}^{2n}; \mathbb{Z}) = 0$. Whereas \mathbb{RP}^{2n+1} is orientable with $H_{2n+1}(\mathbb{RP}^{2n+1}; \mathbb{Z}) = \mathbb{Z}$.

Spheres and tori are orientable. The Klein bottle is not orientable.

The cap product

There is another type of product that has elements in both cohomology and homology and has a homology class as output. Actually, there are several other such products. But that is a story for another day.

Cap products are defined for arbitrary spaces. So we leave the world of manifolds for a moment and get back to it afterwards. Again we only discuss \mathbb{Z} -coefficients, but everything works for any ring R as coefficients.

Definition: Cap product

Let X be any space. The **cap product** is defined to be the \mathbb{Z} -bilinear map

$$S^q(X) \times S_p(X) \xrightarrow{\cap} S_{p-q}(X)$$

defined by sending a q -cochain $\varphi \in S^q(X)$ and a p -simplex $\sigma: \Delta^p \rightarrow X$ to the $p - q$ -chain

$$\varphi \cap \sigma := \varphi(\sigma|_{[e_0, \dots, e_q]})\sigma|_{[e_q, \dots, e_p]}.$$

If $p < q$, then the cap product is defined to be 0.

After checking the relation

$$\partial(\varphi \cap \sigma) = \varphi \cap (\partial\sigma)$$

we see that the cap product descends to a \mathbb{Z} -linear map on cohomology and homology

$$H^q(X) \otimes H_p(X) \xrightarrow{\cap} H_{p-q}(X).$$

Given a continuous map $f: X \rightarrow Y$, there is the following formula which expresses the naturality of the cap product:

$$\varphi \cap f_*(\sigma) = f_*(f^*\varphi \cap \sigma).$$

The cap product is important for us, since (one form of) Poincaré duality can be formulated by saying that the cap product with the fundamental class is an isomorphism:

Poincaré duality

Let M be a compact n -dimensional oriented manifold. Let $[M] \in H_n(M; \mathbb{Z})$ be its fundamental class. Then taking the cap product with $[M]$ yields an isomorphism

$$D: H^p(M; \mathbb{Z}) \xrightarrow{\cong} H_{n-p}(M; \mathbb{Z}), \varphi \mapsto \varphi \cap [M].$$

Note that there are many different ways to formulate Poincaré duality. In particular, there is also the stronger statement in terms of perfect pairings on cohomology groups that we have seen in the mod 2-case.

The **idea of the proof** of this theorem is to study the case of open subsets of \mathbb{R}^n first. Then we use that every point in M has an open neighborhood which is homeomorphic to an open subset in \mathbb{R}^n . Since M is compact, we only need to take finitely many such open neighborhoods to cover M . The Mayer-Vietoris sequence then allows to patch the overlapping open subsets together. Unfortunately, there are some technical difficulties to take care of along the way, e.g., that certain diagrams actually commute.

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APPENDIX A

Exercises

1. Exercises after Lecture 2

- [1] Let $f: X \rightarrow Y$ be a continuous map between topological spaces X and Y .
- a) Let $K \subseteq X$ be compact. Show that $f(K) \subseteq Y$ is compact.
 - b) Give an example of a map f and a compact subset $K \subseteq Y$ such that $f^{-1}(K) \subseteq X$ is not compact.
- [2] Draw a picture of S^2 as a cell complex with six 0-cells, twelve 1-cells and eight 2-cells.

- [3] Show that the stereographic projection

$$\phi: S^1 \rightarrow \mathbb{R} \cup \{\infty\}, (x, y) \mapsto \begin{cases} \frac{x}{1-y} & y \neq 1 \\ \infty & y = 1 \end{cases}$$

defines a homeomorphism from S^1 to the one-point compactification $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of \mathbb{R} .

- [4] a) Let X and Y be topological spaces. Show that homotopy defines an equivalence relation on the set $C(X, Y)$ of continuous maps $X \rightarrow Y$.
- b) Show that *being homotopy equivalent* defines an equivalence relation on topological spaces.
- [5] a) Show that S^1 is a strong deformation retract of $D^2 \setminus \{0\}$.
- b) Show that $D^2 \setminus \{0\}$ is not contractible.

2. Exercises after Lecture 6

- 1** Let $f \in C((X,A),(Y,B))$ be a map of pairs.
- Show that, for every $n \geq 0$, f induces a homomorphism $H_n(X,A) \rightarrow H_n(Y,B)$.
 - Show that the connecting homomorphisms fit into a commutative diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{H_n(f)} & H_n(Y,B) \\ \partial \downarrow & & \downarrow \partial \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f|_A)} & H_{n-1}(B). \end{array}$$

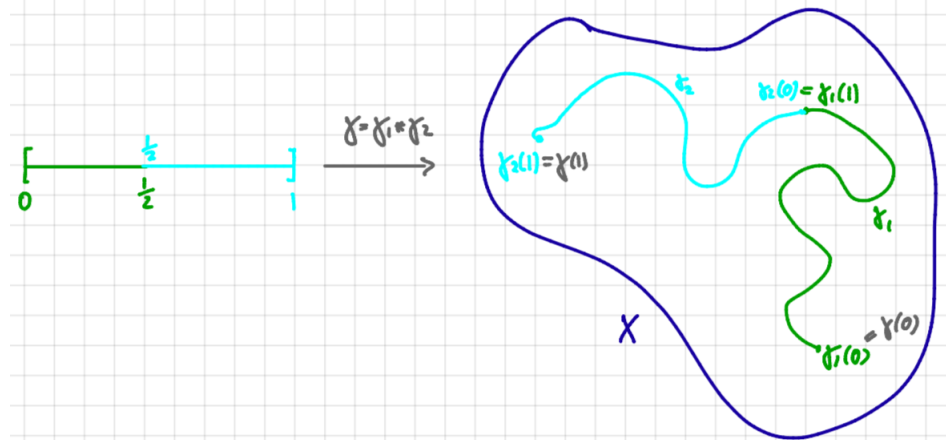
- 2** Let X be a nonempty topological space. Recall that if ω is a path on X , i.e., a continuous map $\omega: [0,1] \rightarrow X$, then we define an associated 1-simplex σ_ω by

$$\sigma_\omega(t_0, t_1) := \omega(1 - t_0) = \omega(t_1) \text{ for } t_0 + t_1 = 1, 0 \leq t_0, t_1 \leq 1.$$

- Show that if ω is a constant path, then σ_ω is a boundary.
- Let γ_1 and γ_2 be paths in X , and let $\gamma := \gamma_1 * \gamma_2$ be the path given by first walking along γ_1 and then walking along γ_2 , i.e., the map

$$\gamma = \gamma_1 * \gamma_2: [0,1] \rightarrow X, t \mapsto \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that the 1-chain $\sigma_\gamma - \sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary.



For the next two exercises, recall that, given a topological space X and a subspace $A \subset X$, A is called a *retract* of X if there is a retraction $\rho: X \rightarrow A$, i.e., a continuous map $\rho: X \rightarrow A$ with $\rho|_A = \text{id}_A$. Moreover, we can consider ρ also as a map $X \rightarrow X$ via the inclusion $X \xrightarrow{\rho} A \subset X$. If ρ is then in addition homotopic to the identity of X , then A is called a *deformation retract* of X .

[3] For every $n \geq 2$, show that S^{n-1} is not a deformation retract of the unit disk D^n .

[4] Show that if A is a retract of X then the map $H_n(i): H_n(A) \rightarrow H_n(X)$ induced by the inclusion $i: A \subset X$ is injective.

[5] In this bonus exercise we show that the additivity axiom is needed only for *infinite* disjoint unions:

For two topological spaces X and Y , let $i_X: X \hookrightarrow X \sqcup Y$ and $i_Y: Y \hookrightarrow X \sqcup Y$ be the inclusions into the disjoint union of X and Y . Without referring to the additivity axiom show that the remaining Eilenberg-Steenrod axioms imply that the induced map

$$H_n(i_X) \oplus H_n(i_Y): H_n(X) \oplus H_n(Y) \rightarrow H_n(X \sqcup Y)$$

is an isomorphism for every n . (Hint: You may want to apply the long exact sequence and excision with $U = X \subset X \sqcup Y$.)

3. Exercises after Lecture 7

Recall the definition from Lecture 7: For $n \geq 1$, let $f: S^n \rightarrow S^n$ be a continuous map. The *degree of f* , denoted by $\deg(f)$, is the integer determined by $H_n(f)([\sigma]) = \deg(f) \cdot [\sigma]$ for a generator $[\sigma] \in H_n(S^n) \cong \mathbb{Z}$. Since we know $H_{n+1}(D^{n+1}, S^n) \cong \mathbb{Z}$, we can apply the same definition also to selfmaps of the pair (D^{n+1}, S^n) : Let $f: (D^{n+1}, S^n) \rightarrow (D^{n+1}, S^n)$ be a continuous map of pairs. The *degree of f* , again denoted by $\deg(f)$, is the integer determined by $H_{n+1}(f)([\sigma]) = \deg(f) \cdot [\sigma]$ for a generator $[\sigma] \in H_{n+1}(D^{n+1}, S^n) \cong \mathbb{Z}$.

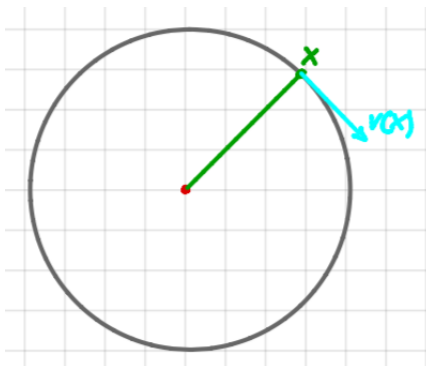
- 1** Show that the degree has the following properties:
- a) The identity has degree 1, i.e., $\deg(\text{id}) = 1$.
 - b) The degree of a constant map is 0.
 - c) If $f, g: S^n \rightarrow S^n$ are two continuous maps, then $\deg(f \circ g) = \deg(f) \deg(g)$.
 - d) If f_0 and f_1 are homotopic, then $\deg(f_0) = \deg(f_1)$.
 - e) If $f: S^n \rightarrow S^n$ is a homotopy equivalence, then $\deg(f) = \pm 1$.
 - f) For $f: (D^{n+1}, S^n) \rightarrow (D^{n+1}, S^n)$, let $f|_{S^n}$ denote the restriction of f to S^n . Then $\deg(f) = \deg(f|_{S^n})$.

- 2** Let $a: S^n \rightarrow S^n$ be the antipodal map, i.e.,

$$a: (x_0, x_1, \dots, x_n) \mapsto (-x_0, -x_1, \dots, -x_n).$$

- a) Show $\deg(a) = (-1)^{n+1}$.
(Hint: Use the result from Lecture 7 on the degree of a reflection.)
 - b) For n even, show that the antipodal map is not homotopic to the identity on S^n .
(Hint: Use what you have just learned in the previous exercises.)
- 3**
- a) If $f: S^n \rightarrow S^n$ is a continuous map *without fixed points*, i.e., $f(x) \neq x$ for all $x \in S^n$, then $\deg(f) = (-1)^{n+1}$.
(Hint: Show that f is homotopic to the antipodal map.)
 - b) If $f: S^n \rightarrow S^n$ is a continuous map *without an antipodal point*, i.e., $f(x) \neq -x$ for all $x \in S^n$, then $\deg(f) = 1$.
(Hint: Show that f is homotopic to the identity map.)
 - c) If n is even and $f: S^n \rightarrow S^n$ is any continuous map, show that there is a point $x \in S^n$ with $f(x) = \pm x$.
(Hint: Apply the previous observations.)

A *vector field* on S^n is a continuous map $v: S^n \rightarrow \mathbb{R}^{n+1}$ with $x \perp v(x)$ for all $x \in S^n$ (x and $v(x)$ are orthogonal to each other).



- 4 Prove the following theorem: The n -dimensional sphere S^n admits a vector field v *without zeros*, i.e., $v(x) \neq 0$ for all $x \in S^n$, if and only if n is odd.

In particular, every vector field on S^2 must have a zero. This is often rephrased as: you cannot comb a hairy ball without leaving a bald spot.

(Hint: If n even, show that the assumption $v(x) \neq 0$ would allow to define a homotopy between the identity and the antipodal map. When you write down the homotopy make sure the image lies on S^n .)

4. Exercises after Lecture 9

- [1] Let X be a space, $A \subset X$ be a subspace and $j: (X, \emptyset) \hookrightarrow (X, A)$ be the inclusion map. Suppose A is contractible.
- a) Show that the natural homomorphism $H_n(j): H_n(X) \rightarrow H_n(X, A)$ is an isomorphism for all $n \geq 2$.
 - b) Show that $H_n(j)$ is an isomorphism for all $n \geq 1$ if A and X are path-connected.
 - c) For $n \geq 1$, let $p \in S^n$ be a point. Show that $S^n \setminus \{p\}$ is contractible.
 - d) For two distinct points $p_1, p_2 \in S^n$, is $S^n \setminus \{p_1, p_2\}$ contractible?

- [2] Let $f: S^n \rightarrow S^n$ be a continuous map. If f is not surjective, then $\deg(f) = 0$.

- [3] Our goal in this exercise is to construct a surjective map $f: S^1 \rightarrow S^1$ with $\deg(f) = 0$.

- a) Start with a map

$$g: S^1 \rightarrow S^1, e^{is} \mapsto \begin{cases} e^{-is} & \text{if } s \in [0, \pi) \\ e^{is} & \text{if } s \in [\pi, 2\pi). \end{cases}$$

Show that g has degree 0.

- b) Compose g with a another map such that the composition becomes a surjective map $f: S^1 \rightarrow S^1$ of degree 0.

- [4] Let $f: S^n \rightarrow S^n$ be a continuous map with $\deg(f) = 0$. Show that there must exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

- [5] With this exercise we would like to refresh our memory on real projective spaces and connect it to questions on the existence of fixed points.

Recall from Lecture 2 that the real projective space \mathbb{RP}^k is defined to be the quotient of $\mathbb{R}^{k+1} \setminus \{0\}$ under the equivalence relation $x \sim \lambda x$ for $\lambda \in \mathbb{R} \setminus \{0\}$. The topology on \mathbb{RP}^k is the quotient topology.

- a) Show that any invertible \mathbb{R} -linear map $F: \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$ induces a continuous map $f: \mathbb{RP}^k \rightarrow \mathbb{RP}^k$.
- b) Show that for any invertible \mathbb{R} -linear map $F: \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$ with an eigenvector, the induced map $f: \mathbb{RP}^k \rightarrow \mathbb{RP}^k$ has a fixed point.
- c) Show that any continuous map $f: \mathbb{RP}^{2n} \rightarrow \mathbb{RP}^{2n}$ that is induced by an invertible \mathbb{R} -linear map $F: \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ has a fixed point.

- d) Show that there are continuous maps $f: \mathbb{RP}^{2n-1} \rightarrow \mathbb{RP}^{2n-1}$ without fixed points.

- 6 Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients in \mathbb{C} . The goal of this exercise is to prove the Fundamental Theorem Algebra, i.e., we would like to show that there is a $z \in \mathbb{C}$ with $p(z) = 0$.

We are going to show this as follows:

Consider p as a map $\mathbb{C} \rightarrow \mathbb{C}$. **Assume that p had no root.** Then we can define a new map

$$\hat{p}: S^1 \rightarrow S^1, z \mapsto \frac{p(z)}{|p(z)|}.$$

We are going to show that this assumption leads to a **contradiction**.

- a) Show that \hat{p} is homotopic to a constant map. What is the degree of \hat{p} ?
b) Show that the map

$$H: S^1 \times (0,1] \rightarrow S^1, (z,t) \mapsto \frac{t^n p(\frac{z}{t})}{|t^n p(\frac{z}{t})|}$$

can be continuously extended to a map $S^1 \times [0,1]$, i.e., analyze $H(z,t)$ for $t \rightarrow 0$. What is the degree of \hat{p} ?

- c) Deduce that p must have a root, i.e., there must be a $z \in \mathbb{C}$ with $p(z) = 0$.

- 7 In this exercise we continue our study of the Fundamental Theorem Algebra. Our goal is to connect the degree and the multiplicity of a root of a polynomial.

- a) Let $f: S^1 \rightarrow S^1$ be a continuous map. Show that if f can be extended to a map on D^2 , i.e., if there is a continuous map $F: D^2 \rightarrow S^1$ such that $F|_{S^1} = f$, then $\deg(f) = 0$.

Now let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients in \mathbb{C} .

- b) Assume that p has no root z with $|z| \leq 1$. Then we can define the map

$$\hat{p}: S^1 \rightarrow S^1, z \mapsto \frac{p(z)}{|p(z)|}.$$

Show that the degree of \hat{p} is 0.

- c) Assume that p has exactly one root z_0 with $|z_0| < 1$ and no root z with $|z| = 1$. Then we can define the map

$$\hat{p}: S^1 \rightarrow S^1, z \mapsto \frac{p(z)}{|p(z)|}.$$

Show that the degree of \hat{p} equals the multiplicity of the root z_0 , i.e., $\deg(\hat{p}) = m$ where $m \geq 0$ is the unique number such that $p(z) = (z - z_0)^m q(z)$ with $q(z_0) \neq 0$.

Finally, we switch perspectives a bit. We know that the polynomial p satisfies $\lim_{|z| \rightarrow \infty} |p(z)| = \infty$. Hence we can extend the map $p: \mathbb{C} \rightarrow \mathbb{C}$ to a map $p: S^2 \rightarrow S^2$ where we think of S^2 as the one-point-compactification of $\mathbb{C} \cong \mathbb{R}^2$. We are going to use the following fact: Let $f_m: S^2 \rightarrow S^2, z \mapsto z^m$. The effect of $H_2(f_m)$ as a selfmap of $H_2(S^2)$ and as a selfmap of $H_2(S^2, S^2 \setminus \{0\})$ is given by multiplication by m .

- d) Let z_i be a root of p . Show that the local degree $\deg(p|_{z_i})$ of p at z_i is equal to the multiplicity of z_i as a root of p .

5. Exercises on the Hurewicz map

This will be a guided tour to the fundamental group and the Hurewicz homomorphism. You can read about this topic in almost every textbook. But you could also take the time to solve the following exercises and enjoy the fun of developing the maths on your own.

Note that we already seen some of the following problems in previous exercises. But feel free to do them again. :)

We fix the following notation:

Let X be a nonempty topological space and let x_0 be a point in X . We write $I = [0,1]$ for the unit interval and $\partial I = \{0,1\}$.

We denote by

$$\Omega(X, x_0) := \{\gamma \in C([0,1], X) : \gamma(0) = x_0 = \gamma(1)\}$$

the set of continuous loops based at x_0 .

We are going to call two loops γ_1 and γ_2 based at x_0 are *homotopic relative to ∂I* if there is a continuous map

$$h: [0,1] \times [0,1] \rightarrow X \text{ such satisfies } \begin{cases} h(s,0) = \gamma_1(s) \text{ for all } s \\ h(s,1) = \gamma_2(s) \text{ for all } s \\ h(0,t) = x_0 = h(1,t) \text{ for all } t. \end{cases}$$

On this exercise set, we will always use the word *loop* to denote a loop based at x_0 and say that two loops are *homotopic* when they are homotopic relative to ∂I .

Let γ_1 and γ_2 be two loops based at x_0 . We define the loop $\gamma_1 * \gamma_2$ to be the loop given by first walking along γ_1 and then walking along γ_2 with doubled speed, i.e., the map

$$\gamma_1 * \gamma_2: [0,1] \rightarrow X, s \mapsto \begin{cases} \gamma_1(2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ \gamma_2(2s - 1) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Our first goal is to prove the following theorem:

The fundamental group

The set of equivalence classes of loops modulo homotopy

$$\pi_1(X, x_0) := \Omega(X, x_0) / \simeq$$

becomes a group with group operation

$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 * \gamma_2].$$

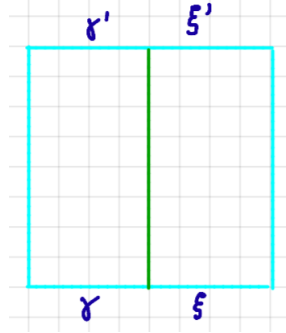
The class of the constant loop ϵ_{x_0} is the neutral element. The inverse of $[\gamma]$ is $[\bar{\gamma}]$, where $\bar{\gamma}$ denotes the loop in reverse direction $s \mapsto \gamma(1-s)$.

The group $\pi_1(X, x_0)$ is called the fundamental group of X at x_0 .

- 1** In this exercise we are going to show that $\pi_1(X, x_0)$ together with the above described operation is a group.

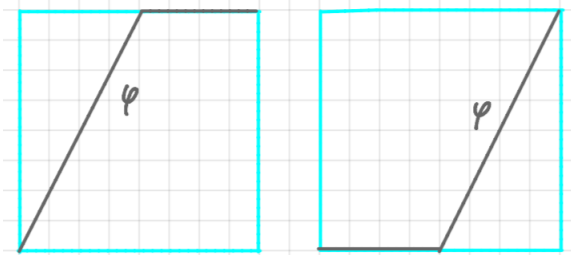
Let $\gamma, \gamma', \xi, \xi', \zeta$ denote loops based at x_0 . Let ϵ_{x_0} with $\epsilon_{x_0}(t) = x_0$ for all t be the constant loop at x_0 . Recall that we use the notation $\gamma \simeq \gamma'$ to say that the two loops γ and γ' are homotopic relative to ∂I .

- a)** Show that if $\gamma \simeq \gamma'$ and $\xi \simeq \xi'$, then $\gamma * \xi \simeq \gamma' * \xi'$.

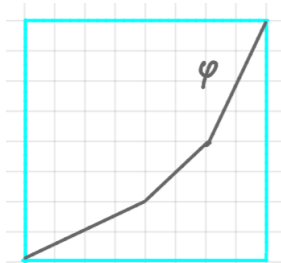


Let $\varphi: [0,1] \rightarrow [0,1]$ be a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$. The composition $\gamma \circ \varphi$ is called a *reparametrization* of γ .

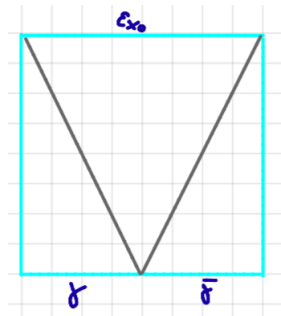
- b)** Show that φ is homotopic to the identity map of $[0,1]$. Deduce that $\gamma \circ \varphi$ is homotopic to γ .
- c)** Choose appropriate reparametrizations of the paths involved to show $\epsilon_{x_0} * \gamma \simeq \gamma \simeq \gamma * \epsilon_{x_0}$.



- d) Choose a φ corresponding to the following picture to show that $\gamma * (\xi * \zeta)$ is a reparametrization of $(\gamma * \xi) * \zeta$ by φ . Conclude that $(\gamma * \xi) * \zeta \simeq \gamma * (\xi * \zeta)$.



- e) Show that $\gamma * \bar{\gamma} \simeq \epsilon_{x_0} \simeq \bar{\gamma} * \gamma$ by writing down precise formulae for the following picture.



Our next goal is to construct a homomorphism $\pi_1(X, x_0) \rightarrow H_1(X)$.

- 2 Recall that if γ is a loop on X we define an associated 1-simplex σ_γ by

$$\sigma_\gamma(1-t, t) := \gamma(t) \text{ for } 0 \leq t \leq 1.$$

Note that if γ is a loop, then σ_γ is a 1-cycle.

A brief **reminder** before we start. For solving the following problems remember that if you want to construct a 2-simplex with a certain boundary, you need to define a map on all of Δ^2 and not just its boundary. Omitting to describe the map on all of Δ^2 would make the tasks trivial. Now let us get to work:

- Show that if $\gamma = \epsilon_{x_0}$ is the constant loop at x_0 , then σ_γ is a boundary.
- Show that the 1-chain $\sigma_{\gamma_1} + \sigma_{\gamma_2} - \sigma_{\gamma_1 * \gamma_2}$ is a boundary.
- Show that if γ_1 and γ_2 are homotopic loops, then $\sigma_{\gamma_1} - \sigma_{\gamma_2}$ is a boundary.

(Hint: For a homotopy h between γ_1 and γ_2 , think of $I \times I$ as a square. Then you can either collapse it to a triangle or divide it

along the diagonal to get two triangles. This will give you a way to construct 2-simplices out of h .)

- d) Show that the 1-chain $\sigma_\gamma + \sigma_{\bar{\gamma}}$ is a boundary.
- e) Conclude that the map

$$\phi: \pi_1(X, x_0) \rightarrow H_1(X), [\gamma] \mapsto [\sigma_\gamma]$$

is a homomorphism of groups. It is called the *Hurewicz homomorphism*.

The fundamental group $\pi_1(X, x_0)$ is in general not abelian. Since $H_1(X)$ is by definition an abelian group, the homomorphism φ factors through the maximal abelian quotient of $\pi_1(X, x_0)$. Recall that this quotient is defined as follows:

For a group G , the commutator subgroup $[G, G]$ of G is the smallest subgroup of G containing all commutators $[g, h] = [ghg^{-1}h^{-1}]$ for all $g, h \in G$. Note that $[G, G]$ is a normal subgroup. The quotient $G_{\text{ab}} := G/[G, G]$ is the maximal abelian quotient of G and is called the *abelianization* of G .

The abelianization has the following universal property: Let $q: G \rightarrow G/[G, G]$ be the quotient map. If H is an abelian and $\eta: G \rightarrow H$ a homomorphism of groups, then there is a unique homomorphism of abelian groups $\eta_{\text{ab}}: G_{\text{ab}} \rightarrow H$ such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\eta} & H \\ q \downarrow & \nearrow \eta_{\text{ab}} & \\ G_{\text{ab}} & & \end{array}$$

- 3] Assume that X is path-connected. We are going to show that the induced homomorphism

$$\phi_{\text{ab}}: \pi_1(X, x_0)_{\text{ab}} \rightarrow H_1(X),$$

which is also called the *Hurewicz homomorphism*, is an isomorphism.

We are going to construct an inverse ψ of ϕ_{ab} as follows:

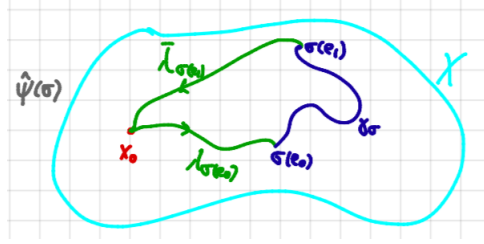
For any $x \in X$, we choose a continuous path λ_x from x_0 to x . If $x = x_0$, then we choose λ_{x_0} to be the constant path at x_0 .

Let $\sigma: \Delta^1 \rightarrow X$ be a 1-chain in X . Denote the associated path in X by

$$\gamma_\sigma: [0, 1] \rightarrow X, t \mapsto \sigma(1-t, t).$$

Then we define a loop

$$\hat{\psi}(\sigma): [0,1] \rightarrow X, t \mapsto \lambda_{\sigma(e_0)} * \gamma_\sigma * \bar{\lambda}_{\sigma(e_1)}.$$



We extend this definition \mathbb{Z} -linearly to obtain homomorphism of abelian groups

$$\hat{\psi}: S_1(X) \rightarrow \pi_1(X, x_0)_{\text{ab}}.$$

- a) As a preparation show the following: Let $\beta: \Delta^2 \rightarrow X$ be a 2-simplex. Let α_i be the path corresponding to the i th face $\beta \circ \phi_i$ of β . Show that the loop $\alpha_2 * \alpha_0 * \bar{\alpha}_1$ based at $y_0 := \beta(e_0)$ is homotopic to the constant loop ϵ_{y_0} at y_0 .

Note: If you do not want to derive formulae, draw a picture to convince yourself that the statement makes sense and describe in words why it is true.

- b) Show that $\hat{\psi}$ sends the group $B_1(X)$ of 1-boundaries to the neutral element $1 \in \pi_1(X, x_0)_{\text{ab}}$.

(Hint: Use that $\pi_1(X, x_0)_{\text{ab}}$ is abelian and that the loop given by walking along the boundary of a 2-simplex is homotopic to the constant loop.)

- c) Conclude that $\hat{\psi}$ induces a homomorphism of abelian groups

$$\psi: H_1(X) \rightarrow \pi_1(X, x_0)_{\text{ab}}.$$

- d) Show that if γ is a loop, then $\psi(\phi_{\text{ab}}([\gamma])) = [\gamma]$.
e) Let σ be a 1-simplex. Show that $\phi_{\text{ab}}(\psi([\sigma])) = [\sigma + \kappa_{\sigma(e_0)} - \kappa_{\sigma(e_1)}]$, where κ_y denotes the constant 1-simplex with value y .
f) Show that, if c is a 1-cycle, then $\phi_{\text{ab}}(\psi([c])) = [c]$.
g) Conclude that ψ is an inverse of ϕ_{ab} and hence that ϕ_{ab} is an isomorphism.

6. Exercises after Lecture 12

- 1** In this exercise we give another proof of the exactness of the Mayer-Vietoris sequence. We start with an algebraic lemma which provides good practice in diagram chasing and then we use this result to deduce the MVS from the Excision Axiom.

a) Assume we have a map of long exact sequences

$$(32) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & K'_n & \xrightarrow{i'_n} & L'_n & \xrightarrow{a'_n} & M'_n & \xrightarrow{b'_n} & K'_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_n & & \downarrow g_n & & \cong \downarrow k_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & K_n & \xrightarrow{i_n} & L_n & \xrightarrow{a_n} & M_n & \xrightarrow{b_n} & K_{n-1} & \longrightarrow & \cdots \end{array}$$

such that $k_n: M'_n \xrightarrow{\cong} M_n$ is an isomorphism for every n . For each n , we define the homomorphism ∂_n to be

$$\partial_n: L_n \xrightarrow{a_n} M_n \xrightarrow{k_n^{-1}} M'_n \xrightarrow{b'_n} K'_{n-1}.$$

Show that the sequence

$$\cdots \rightarrow K'_n \xrightarrow{\begin{bmatrix} f_n \\ -i'_n \end{bmatrix}} K_n \oplus L'_n \xrightarrow{\begin{bmatrix} i_n & g_n \end{bmatrix}} L_n \xrightarrow{\partial_n} K'_{n-1} \rightarrow \cdots$$

is exact.

- b) Let $\{A, B\}$ be a cover of X . Apply the previous algebraic observation to the long exact sequences of the pairs (X, A) and $(B, A \cap B)$ and use the excision isomorphism to deduce the Mayer-Vietoris sequence.

- 2** Let A and B be two disjoint closed subsets of \mathbb{R}^2 .

a) Show that there is an isomorphism

$$H_1(\mathbb{R}^2 - (A \cup B)) \cong H_1(\mathbb{R}^2 - A) \oplus H_1(\mathbb{R}^2 - B).$$

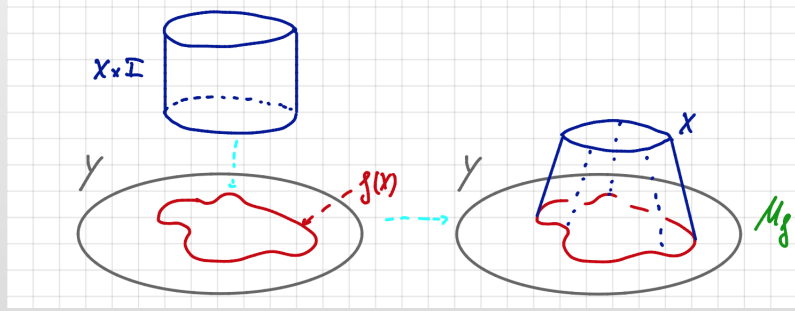
Recall that a path-component of a space X is a maximal path-connected subspace (where the ordering is given by inclusion). For example, if X is path-connected itself, then it has one path-component. If X is the disjoint union of two path-connected spaces U and V , then U and V are the path-components of X .

- b) Show that the number of path-components of $\mathbb{R}^2 - (A \cup B)$ is one less than the sum of the numbers of path-components of $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - B$.

Definition: Mapping cylinder

Let $f: X \rightarrow Y$ be a continuous map. The **mapping cylinder of f** is defined to be the quotient space

$$M_f := (X \times [0,1] \sqcup Y) / ((x,0) \sim f(x)).$$



The mapping cylinder fits into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow f_1 & \nearrow g \\ & M_f & \end{array}$$

where f_1 maps x to $(x,1)$ and g maps (x,t) to $f(x)$ for all $x \in X$ and $t \in [0,1]$ and $y \in Y$ to y .

- 3 a) Show that the inclusion $i: Y \hookrightarrow M_f$ is a deformation retract and g is a deformation retraction.
 b) We can construct the **Möbius band** $M := M_f$ as the mapping cylinder of the map

$$f: S^1 \rightarrow S^1, z \mapsto z^2 \quad (S^1 \subset \mathbb{C}).$$

Determine the homology of the Möbius band.

- c) For $n \geq 1$ and $m \in \mathbb{Z}$, let M_f be the mapping cylinder of a map

$$f: S^n \rightarrow S^n \text{ with } \deg(f) = m.$$

Show that $H_n(f_1)$ is given by multiplication with m .

- d) For $n \geq 1$ and $m \geq 2$, let M_f be the mapping cylinder of a map

$$f: S^n \rightarrow S^n \text{ with } \deg(f) = m.$$

Show that $X = S^n$ is not a weak retract of M_f .

- 4 We can consider the **real projective plane** \mathbb{RP}^2 as a two dimensional disk D^2 with a Möbius band M attached at its boundary. Writing $A = D^2$ and $B = M$, we have $A \cap B \simeq S^1$. Calculate the homology groups of \mathbb{RP}^2 .

7. Exercises after Lecture 13

On this exercise set we are going to explore some additional, important topics. We will use them later in the lectures.

We start with an application of the excision property:

- 1** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty open subsets. Show that if there is a homeomorphism $\varphi: U \xrightarrow{\cong} V$, then we must have $n = m$.
(Hint: Take a point $x \in U$ and compare $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ and $H_n(U, U - \{x\})$.)

Now we are going to introduce a slight modification of homology:

Definition: Reduced homology

Let X be a nonempty topological space. We define the **reduced homology of X** to be the homology of the **augmented** complex of singular chains

$$\cdots S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\tilde{\epsilon}} \mathbb{Z} \rightarrow 0$$

where $\tilde{\epsilon}(\sum_i n_i \sigma_i) := \sum_i n_i$ (and \mathbb{Z} is placed in degree -1). Recall that we checked before (in Lecture 4) that $\tilde{\epsilon} \circ \partial_1 = 0$. Hence the above sequence is a chain complex. Moreover, the construction of the augmented chain complex is **functorial**.

The n th reduced homology group of X is denoted by $\tilde{H}_n(X)$.

Reduced homology does not convey any new information, but is convenient for stating things. It also helps focusing on the important information, since it disregards the contribution in $H_0(X)$ which comes from a single point.

Here are some basic properties of reduced homology:

- 2** Let X be a nonempty topological space. Show that reduced homology satisfies the following properties:
- a) $\tilde{H}_0(X) = \text{Ker}(\epsilon: H_0(X) \rightarrow H_0(\text{pt}))$.
 - b) $\tilde{H}_n(X) = H_n(X)$ for all $n \geq 1$.
 - c) If X is path-connected, then $\tilde{H}_0(X) = 0$.
 - d) For any point $x \in X$, $H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\{x\})$ and $\tilde{H}_n(X) \cong H_n(X, \{x\})$ for all $n \geq 0$.

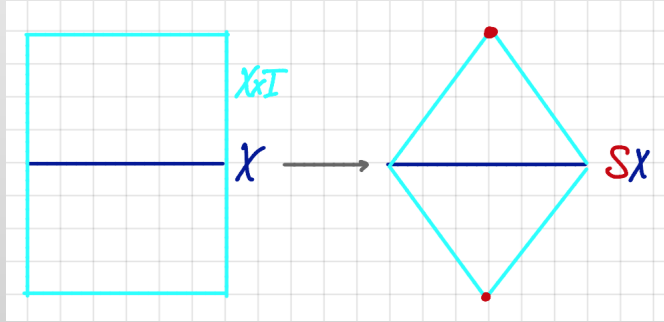
Now we move on towards an important construction on spaces, the suspension. It does not look spectacular, but will prove extremely useful and important later on in our studies of Algebraic Topology:

Definition: Suspension of a space

Let X be a topological space. The **suspension of X** is defined to be the quotient space

$$(X \times [0,1]) / ((x_1,0) \sim (x_2,0) \text{ and } (x_1,1) \sim (x_2,1) \text{ for all } x_1, x_2 \in X).$$

In other words, SX is constructed by taking a cylinder over X and then collapsing all points $X \times \{0\}$ to a point p_0 and all points $X \times \{1\}$ to a point p_1 . The topology on SX is the quotient topology.



For any continuous map $f: X \rightarrow Y$, there is an induced continuous map

$$S(f): SX \rightarrow SY, [x,s] \mapsto [f(x),s].$$

- 3 Our goal in this exercise is to understand SX a bit better and to show that there are isomorphisms

$$\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X) \text{ for all } n \geq 0$$

(Note that reduced homology makes it much easier to state this result. For, without reduced homology we would have to write $H_1(SX) \cong \text{Ker}(H_0(X) \rightarrow H_0(\text{pt}))$ for $n = 0$.)

- Show that SX is path-connected and hence $H_0(SX) \cong \mathbb{Z}$.
- Show that $SX - \{p_1\}$ is contractible.
- Show that $SX - \{p_0, p_1\}$ is homotopy equivalent to X .
- Use the Mayer-Vietoris sequence to determine $\tilde{H}_{n+1}(SX)$ for all $n \geq 0$.

A crucial example is the suspension of the sphere.

- 4**
- a) Show that the suspension SS^{n-1} of the $(n-1)$ -sphere is homeomorphic to the n -sphere.
 - b) For $n \geq 1$, let $f: S^n \rightarrow S^n$ be a continuous map and let $S(f): SS^n \rightarrow SS^n$ be the induced map on suspensions. By using either $SS^n \approx S^{n+1}$ or $H_{n+1}(SS^n) \cong H_n(S^n)$ show that $H_{n+1}(S(f))$ is given by multiplication by an integer which we denote by $\deg(S(f))$. Show that $\deg(S(f)) = \deg(f)$.
 - c) For $n \geq 1$, show that, for any given $k \in \mathbb{Z}$, there is a map $f: S^n \rightarrow S^n$ with $\deg(f) = k$.

8. Exercises after Lecture 15

- 1 Show that the two different cell structures on S^n we discussed in the lecture lead to cellular chain complexes which have the same homology groups.
- 2 Show the statement of the lecture that the isomorphism between the homology of the cellular chain complex is functorial in the following sense: Let $f: X \rightarrow Y$ be a **cellular** (or filtration-preserving) map between cell complexes, i.e., $f(X_n) \subseteq Y_n$ for all n . Show that f induces a homomorphism of cellular chain complexes $C_*(f): C_*(X) \rightarrow C_*(Y)$ which fits into a commutative diagram

$$\begin{array}{ccc}
 H_*(C_*(X)) & \xrightarrow{H_*(C_*(f))} & H_*(C_*(Y)) \\
 \cong \downarrow & & \downarrow \cong \\
 H_*(X) & \xrightarrow{H_*(f)} & H_*(Y).
 \end{array}$$

- 3 Let X be a cell complex and A a subcomplex. Show that the quotient X/A inherits a cell structure such that the quotient map $q: X \rightarrow X/A$ is cellular.
- 4 Consider S^1 with its standard cell structure, i.e. one 0-cell e^0 and one 1-cell e^1 . Let X be a cell complex obtained from S^1 by attaching two 2-cells e_1^2 and e_2^2 to S^1 by maps f_1 and f_2 of degree 2 and 3, respectively. We may express this construction as

$$X = S^1 \cup_{f_1} e_1^2 \cup_{f_2} e_2^2.$$

- Determine all the subcomplexes of X .
- Determine the cellular chain complex of X and compute the homology of X .
- For each subcomplex Y of X , compute the homology of Y and of the quotient space X/Y .
- As a more challenging task show that the only subcomplex Y of X for which $X \xrightarrow{q} X/Y$ is a homotopy equivalence is the trivial subcomplex consisting only of the 0-cell.
(Hint: Study the effect of $H_2(q)$.)

Note that one can nevertheless show that X is homotopy equivalent to S^2 . But we are lacking some results in homotopy theory to prove this.

For the next exercise, note that if X and Y are cell complexes, then $X \times Y$ is a cell complex with cells the products $e_{\alpha,X}^n \times e_{\beta,Y}^m$ where $e_{\alpha,X}^n$ ranges over the cells of X and $e_{\beta,Y}^m$ ranges over the cells of Y .

5 Show that the Euler characteristic has the following properties:

a) If X and Y are finite cell complexes, then

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

b) Assume the finite cell complex X is the union of the two union of two subcomplexes A and B . Then

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

9. Exercises after Lecture 17

We start with proving some properties about the Tor-functor that we mentioned in the lecture.

1 Let M be an abelian group.

a) Let A be an abelian group and $0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ be a free resolution of M . Consider the chain complex K_* given by

$$(33) \quad 0 \rightarrow A \otimes F_1 \rightarrow A \otimes F_0 \rightarrow 0$$

with $A \otimes F_1$ in dimension one and $A \otimes F_0$ in dimension zero.

Show that $H_1(K_*) = \text{Tor}(A, M)$ and $H_0(K_*) = A \otimes M$.

b) Show that for any short exact sequence of abelian groups

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

there is an associated long exact sequence

$$0 \rightarrow \text{Tor}(A, M) \rightarrow \text{Tor}(B, M) \rightarrow \text{Tor}(C, M) \rightarrow A \otimes M \rightarrow B \otimes M \rightarrow C \otimes M \rightarrow 0.$$

c) For any abelian group A , show that $\text{Tor}(A, M) = 0$ if A or M is a free abelian group.

d) Show that, for any abelian group A , Tor is symmetric:

$$\text{Tor}(A, M) \cong \text{Tor}(M, A).$$

(Note: It takes a while to show this without any additional tools from homological algebra. It is important though that you think about what you have to do to prove the statement and that you try. The arguments used in the lecture are useful here, too.)

e) For any abelian group A , show that $\text{Tor}(A, M) = 0$ if A or M is torsion-free, i.e., the subgroup of torsion elements vanishes.

(Hint: You may want to use the fact that a finitely generated torsion-free abelian group is free.)

f) Let A be an abelian group and let $T(A)$ denote the subgroup of torsion elements in A . Show $\text{Tor}(A, M) = \text{Tor}(T(A), M)$.

(Hint: $A/T(A)$ is torsion-free.)

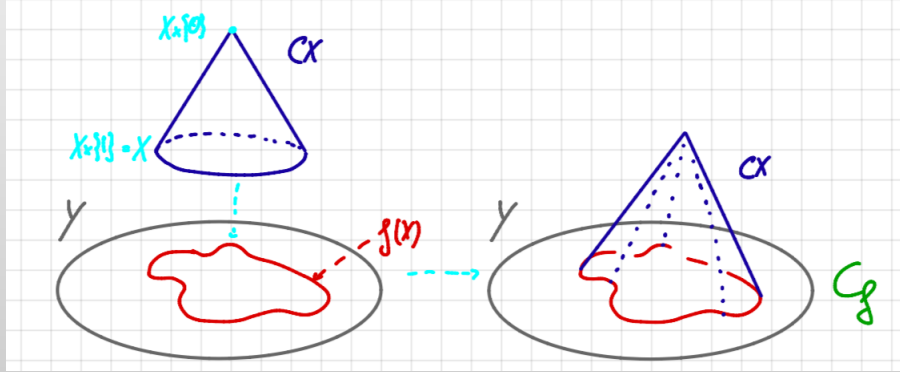
Definition: Mapping cone

For a space X , the **cone** over X is defined as the quotient space

$$CX := (X \times [0, 1]) / (X \times \{0\}).$$

Let $f: X \rightarrow Y$ be a continuous map. The **mapping cone of f** is defined to be the quotient space

$$C_f := (CX \sqcup Y) / ((x, 1) \sim f(x)) = Y \cup_f CX.$$



- 2 Let $f: X \rightarrow Y$ be a continuous map, and let M be an abelian group.
- a) Show that the homology of the mapping cone of f fits into a long exact sequence

$$\cdots \rightarrow \tilde{H}_{n+1}(C_f; M) \rightarrow \tilde{H}_n(X; M) \xrightarrow{f_*} \tilde{H}_n(Y; M) \xrightarrow{i_*} \tilde{H}_n(C_f; M) \rightarrow \tilde{H}_{n-1}(X; M) \rightarrow \cdots$$

(Hint: Relate C_f to the mapping cylinder M_f from a previous exercise set.)

- b) Show that f induces an isomorphism in homology with coefficients in M if and only if $\tilde{H}_*(C_f; M) = 0$.

The first part of the next exercise requires some familiarity with Tor beyond the discussion of the lecture. But you should think about it anyway and definitely note the statement.

- 3 Let X and Y be topological spaces.
- a) Show that $\tilde{H}_*(X; \mathbb{Z}) = 0$ if and only if $H_*(X; \mathbb{Q}) = 0$ and $H_*(X; \mathbb{F}_p) = 0$ for all primes p .
(Hint: Use the UCT with \mathbb{F}_p -coefficients to control the torsion part and with \mathbb{Q} -coefficients to control the torsion-free part. Then apply suitable points of the first exercise.)
- b) Show that a map $f: X \rightarrow Y$ induces an isomorphism in integral homology if and only if it induces an isomorphism in homology

with rational coefficients and in homology with \mathbb{F}_p -coefficients for all primes p .

- 4** Let X be a finite cell complex, and let \mathbb{F}_p be a field with p elements. Show that the Euler characteristic $\chi(X)$ can be computed by the formula

$$\chi(X) = \sum_i \dim_{\mathbb{F}_p} (-1)^i H_i(X; \mathbb{F}_p).$$

In other words, $\chi(X)$ is the alternating sum of the dimensions of the \mathbb{F}_p -vector spaces $H_i(X; \mathbb{F}_p)$.

(Hint: Use the UCT.)

- 5** Use the Künneth Theorem of the lecture to show:

a) The homology of the product $X \times S^k$ satisfies

$$H_n(X \times S^k) \cong H_n(X) \oplus H_{n-k}(X).$$

b) The homology of the n -torus T^n defined as the n -fold product $T^n = S^1 \times \dots \times S^1$ is given by

$$H_i(T^n) \cong \mathbb{Z}^{\binom{n}{i}}.$$

10. Exercises after Lecture 21

- 1 Let M be an abelian group. Let X be a cell complex and let X_n denote the n -skeleton of X . We set

$$C^*(X; R) := H^n(X_n, X_{n-1}; M).$$

We would like to turn this into a cochain complex. We define the differential

$$d^n: C^n(X; M) \rightarrow C^{n+1}(X; M)$$

as the composite

$$\begin{array}{ccccc} C^n(X; M) = H^n(X_n, X_{n-1}; M) & \xrightarrow{d^n} & H^{n+1}(X_{n+1}, X_n; M) = C^{n+1}(X; M) \\ & \searrow j^n & \nearrow \partial^n \\ & H^n(X_n; M) & \end{array}$$

where ∂^n is the connecting homomorphism in the long exact sequence of cohomology groups of pairs and j^n is the homomorphism induced by the inclusion $(X_n, \emptyset) \hookrightarrow (X_n, X_{n-1})$. Define the **cellular cochain complex** of X with coefficients with M to be the cochain complex $(C^*(X; M), d^*)$.

Note that the cup product defines a product on the cellular cochain complex.

- a) Show that $C^*(X; M)$ is in fact a complex, i.e., $d^n \circ d^{n-1} = 0$.
- b) Show that $C^*(X; M)$ is isomorphic to the cochain complex $\text{Hom}(C_*(X), M)$ where $C_*(X)$ is the cellular chain complex of X .
(Hint: Remember the Kronecker map κ .)
- c) Use the UCT for cohomology and the isomorphism between $H_n(X)$ and $H_n(C_*(X))$ to show

$$H^n(X; M) \cong H^n(C^*(X; M)).$$

Note that the isomorphism we produce this way is not functorial.

- 2 Let $X = M(\mathbb{Z}/m, n)$ be a Moore space constructed by starting with an n -sphere S^n and then forming X by attaching an $n+1$ -dimensional cell to it via a map $f: S^n \rightarrow S^n$ of degree m

$$X = S^n \cup_f D^{n+1}.$$

Let

$$q: X \rightarrow X/S^n \approx S^{n+1}$$

be the quotient map.

Recall that we showed that q induces a trivial map on $\tilde{H}_i(-; \mathbb{Z})$ for all i .

- a) Show $H^{n+1}(X; \mathbb{Z}/m) \cong \mathbb{Z}/m$ and that $H^{n+1}(q; \mathbb{Z}/m)$ is nontrivial. (Hint: Use the UCT for cohomology.)
- b) Use the previous example to show that the splitting in the UCT for cohomology cannot be functorial. (Hint: You need to show that a certain square induced by the UCT does not commute.)

- 3 Show that if a map $g: \mathbb{RP}^n \rightarrow \mathbb{RP}^m$ induces a nontrivial homomorphism
- $$g^*: H^1(\mathbb{RP}^m; \mathbb{Z}/2) \rightarrow H^1(\mathbb{RP}^n; \mathbb{Z}/2),$$
- then $n \geq m$.

- 4 Show that there does not exist a homotopy equivalence between \mathbb{RP}^3 and $\mathbb{RP}^2 \vee S^3$.

The next exercise is a bit more challenging.

- 5 Let X be the cell complex obtained by attaching a 3-cell to \mathbb{CP}^2 via a map

$$S^2 \rightarrow S^2 = \mathbb{CP}^1 \subset \mathbb{CP}^2$$

of degree p . Let $Y = M(\mathbb{Z}/p, 2) \vee S^4$ where $M(\mathbb{Z}/p, 2)$ is a Moore space. We observe that the cell complexes X and Y have the same 2-skeleton, but the 4-cell is attached via different maps.

- a) Show that X and Y have isomorphic cohomology rings with \mathbb{Z} -coefficients.
- b) Show that the cohomology rings of X and Y with \mathbb{Z}/p -coefficients are not isomorphic.