Math 231b
Lecture 01

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1. Lecture 1: Vector bundles

We start with the basic theory of vector bundles. For the moment there is nothing special about the complex case, we could also consider real vector bundles. But later when we define K-theory it will matter if we work with complex or real bundles. Our references for the next lectures are the book of Milnor and Stasheff and Hatcher's online notes.

We introduce our first main character of the story.

Definition 1.1. Let B be a topological space.

1) A *family of real vector spaces* ξ over B consists of the following data:

- a topological space $E = E(\xi)$ called the *total space*
- a continuous $\pi\colon E \to B$ called the *projection map*, and
- for each $b \in B$ the structure of a vector space over the real numbers $\mathbb{R}$ in the set $E_b := \pi^{-1}(b)$.

2) The family ξ is called a *real vector bundle* over B if these data are subject to the following condition:

- *Local triviality*: For each point $b \in B$ there should exist a neighborhood $U \subset B$, an integer $n \geq 0$, and a homeomorphism

$$h\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$$

  such that, for each $b \in U$, the correspondence $x \mapsto h(b,x)$ defines an isomorphism between the vector space $\mathbb{R}^n$ and the vector space $\pi^{-1}(b)$.

3) A *family of complex vector spaces* ζ over B consists of the data:

- a topological space $E = E(\zeta)$ called the *total space*
- a continuous $\pi\colon E \to B$ called the *projection map*, and

- for each $b \in B$ the structure of a vector space over the complex numbers $\mathbb{C}$ in the set $\pi^{-1}(b)$.

4) The family ζ is called a *complex vector bundle* over B if these data are subject to the following condition:

- *Local triviality*: For each point $b \in B$ there should exist a neighborhood $U \subset B$, an integer $n \geq 0$, and a homeomorphism

$$h\colon U \times \mathbb{C}^n \to \pi^{-1}(U)$$

such that, for each $b \in U$, the correspondence $z \mapsto h(b,z)$ defines an isomorphism between the vector space $\mathbb{C}^n$ and the vector space $\pi^{-1}(b)$.

For vector bundles, we will use some further terminology:

- A pair (U,h) as in Definition 1.1 will be called a *local trivialization* about b.
- If it is possible to choose U equal to the entire space B of a vector bundle, then the vector bundle will be called a *trivial bundle*.
- We often refer to a vector bundle $\pi : E \to B$ by just mentioning the total space E.
- The vector space $\pi^{-1}(b)$ is called the *fiber* over b. It will also be denoted by E_b.
- The fiber $E_b = \pi^{-1}(b)$ is never vacuous, but it may consist of a single point. The dimension n of E_b is allowed to vary, but it is always a *locally constant* function. Though in most cases of interest the dimension is constant. In this case one speaks of an *n-dimensional bundle* and call n the *rank* of the bundle.
- A 1-dimensional bundle is also called a *line bundle*.

Now that we have the basic notions at hand, we will focus for a while on real vector bundles and we will often refer to a real vector bundle just as a vector bundle. Later, when we introduce K-theory we will look at complex bundles again.

So let us have a look at some examples of (real) vector bundles.

Example 1.2. There is an obvious example of a vector bundle over any topological space B: The *product* or *trivial* bundle $E = B \times \mathbb{R}^n$ with π the projection onto the first factor.

Example 1.3. Let $I = [0,1]$ be the unit interval, and let E be the quotient space of $I \times \mathbb{R}$ under the identification $(0,t) \sim (1,-t)$. Then the projection $I \times \mathbb{R} \to I$ induces a map

$$\pi \colon E \to S^1$$

which is a line bundle. Since E is homeomorphic to a Möbius band, i.e., a cylinder cut open, twisted once and glued back together, with its boundary circle deleted, we call this bundle the *Möbius bundle.*

Example 1.4. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. The *tangent bundle* τ *to* S^n is the vector bundle $\pi : E \to S^n$ where

$$E = \{(x,v) \in S^n \times \mathbb{R}^n |\ x \perp v\}.$$

We think of v as a tangent vector to S^n by translating it so that its tail is at the head of x on S^n. The map $\pi \colon E \to S^n$ sends (x,v) to x.

The vector space structure on $\pi^{-1}(x)$ is defined by

$$t_1(x,v_1) + t_2(x,v_2) = (x, t_1v_1 + t_2v_2).$$

In order to show that this is a vector bundle we have to construct local trivializations. So let $x \in S^n$ be any point and let $U_x \subset S^n$ be the open hemisphere which contains x and is bounded by the hyperplane through the origin orthogonal to x.

DRAW A PICTURE FOR S^2.

Define

$$h_x \colon \pi^{-1}(U_x) \to U_x \times \pi^{-1}(x) \cong U_x \times \mathbb{R}^n$$

by

$$h_x(y,v) = (y, p_x(v))$$

where p_x is the orthogonal projection onto the hyperplane $\pi^{-1}(x)$. PICTURE!

Then h_x is a local trivialization, since p_x restricts to an isomorphism of $\pi^{-1}(y)$ onto $p^{-1}(x)$ for each $y \in U_x$.

Example 1.5. The *normal bundle* ν *to* S^n *in* $\mathbb{R}^{n+1}$ is the line bundle $\pi \colon E \to S^n$ with E consisting of pairs

$(x,v) \in S^n \times \mathbb{R}^{n+1}$ such that v is perpendicular to the tangent plane to S^n at x,

or in other words,

$$v = tx \text{ for some } t \in \mathbb{R}.$$

DRAW A PICTURE FOR S^2!

The map $\pi\colon E \to S^n$ is just given by $\pi(x,v) = x$ and the vector space structure on $\pi^{-1}(x)$ is again defined by

$$t_1(x,v_1) + t_2(x,v_2) = (x, t_1v_1 + t_2v_2).$$

As in the previous example, local trivializations $h_x\colon \pi^{-1}(U_x) \to U_x \times \mathbb{R}$ can be obtained by orthogonal projection of the fibers $\pi^{-1}(y)$ onto $\pi^{-1}(x)$ for $y \in U_x$ and U_x as in the previous example.