**Math 231b**
**Lecture 02**

G. Quick

## 2. Lecture 2: Vector bundles and sections

We have seen the definition and first examples of vector bundles. Today we will first continue our list of examples. Let us get started.

**Example 2.1.** Recall that the real projective $n$-space $\mathbb{R}\mathrm{P}^n$ is the space of lines in $\mathbb{R}^{n+1}$ through the origin. Since each such line intersects the unit sphere $S^n$ in a pair of antipodal points, we can also regard $\mathbb{R}\mathrm{P}^n$ as the quotient space of $S^n$ in which antipodal pairs of points are identified, i.e., $\mathbb{R}\mathrm{P}^n = S^n/x \sim (-x)$. The topology of $\mathbb{R}\mathrm{P}^n$ is then the topology as a quotient of $S^n$. Let $\{\pm x\}$ denote the equivalence class of $x$ in $S^n/\sim$

The *canonical line bundle* $\gamma_n^1$ *over* $\mathbb{R}P^n$ is the line bundle $\pi\colon E \to \mathbb{R}\mathrm{P}^n$ with total space

$$E(\gamma_n^1) = \{(\{\pm x\}, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} | v = tx \text{ for some } t \in \mathbb{R}\} \subset \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}.$$

In other words, $E$ is consisting of all pairs $(\ell, v)$ such that the vector $v$ lies on the line $\ell$.

The map $\pi\colon E \to \mathbb{R}\mathrm{P}^n$ is just the projection sending $(\{\pm x\}, v)$ to $\{\pm x\}$.

Now we need to find local trivializations for $\gamma_n^1$. Let $U \subset S^n$ be any open set which is small enough so as to contain no pair of antipodal points, and let $U_1$ denote the image of $U$ in $\mathbb{R}\mathrm{P}^n$. Then a homeomorphism

$$h\colon U_1 \times \mathbb{R} \to \pi^{-1}(U_1)$$

is defined by the requirement that

$$h(\{\pm x\}, t) = (\{\pm x\}, tx)$$

for each $(x,t) \in U \times \mathbb{R}$. The pair $(U_1,h)$ is a local trivialization of $\gamma_n^1$.

After seeing some examples of vector bundles we would like to be able to say when two bundles are isomorphic.

**Definition 2.2.** 1) Let $\xi$ and $\eta$ be two vector bundles over some base space $B$. Then we say that $\xi$ *is isomorphic to* $\eta$, written $\xi \cong \eta$, if there exists a homeomorphism

$$f : E(\xi) \to E(\eta)$$

between the total spaces which maps each vector space $E_b(\xi)$ isomorphically onto the corresponding vector space $E_b(\eta)$.

2) We say that a bundle is trivial if it is isomorphic to the product bundle $B \times \mathbb{R}^n$ for some $n \geq 0$.

**Example 2.3.** 1) The tangent bundle $\tau_1$ to $S^1$ is isomorphic to the trivial bundle $S^1 \times \mathbb{R}$. The isomorphism is given by the map

$$\tau_1 \to S^1 \times \mathbb{R}, (e^{i\theta}, ie^{i\theta}) \mapsto (e^{i\theta}, t) \text{ for } e^{i\theta} \in S^1 \text{ and } t \in \mathbb{R}.$$

Recall that the total space of $\tau^1$ is given by the space

$$E(\tau_1) = \{(x,v) \in S^1 \times \mathbb{R}^1 |\ x \perp v\} = \{(e^{i\theta}, ie^{i\theta}) |\ t \in \mathbb{R}, \theta \in [0,2\pi]\}.$$

Note: The triviality of $\tau_1$ is special to the case $n = 1$. Though the situation is simpler for the normal bundle.

2) The normal bundle $\nu$ of $S^n$ in $\mathbb{R}^{n+1}$ is isomorphic to the product line bundle $S^n \times \mathbb{R}$. The isomorphism is given by the map

$$(x,tx) \mapsto (x,t).$$

Hence $\nu$ is trivial.

But, of course, not all bundles are trivial.

**Proposition 2.4.** *The canonical line bundle $\gamma_n^1$ over $\mathbb{R}\mathrm{P}^n$ is not trivial for $n \geq 1$.*

We prove this claim by studying the sections of $\gamma_n^1$.

**Definition 2.5.** A *section* of a vector bundle $\pi \colon E \to B$ is a continuous map

$$s \colon B \to E$$

which takes each $b \in B$ into the corresponding fiber $\pi^{-1}(b)$. In other words, $s$ is a continuous map such that $\pi \circ s = \mathrm{id}_B$.

A section is called *nowhere zero* if $s(b)$ is a non-zero vector of $\pi^{-1}(b)$ for each $b$.

**Example 2.6.**
- Every vector bundle has a *zero section* whose value is the zero vector in each fiber.
- A trivial bundle possesses a nowhere zero section.

From the last point we see that in order to proof Proposition 2.4 it suffices to show that $\gamma_n^1$ does not have nowhere zero section:

Let

$$s\colon \mathbb{R}\mathrm{P}^n \to E(\gamma_n^1)$$

be any section, and consider the composition

$$S^n \to \mathbb{R}\mathrm{P}^n \xrightarrow{s} E(\gamma_n^1)$$

which carries each $x \in S^n$ to some pair

$$(\{\pm x\}, t(x)x) \in E(\gamma_n^1).$$

Since this map is the composite of continuous maps it is itself continuous and hence the map $x \mapsto t(x)$ is a continuous map $S^n \to \mathbb{R}$, i.e. it is a continuous real valued function. Moreover, it satisfies

$$t(-x) = -t(x).$$

Since $S^n$ is connected it follows from the intermediate value theorem that $t(x_0) = 0$ for some $x_0$. Hence

$$s(\{\pm x_0\}) = (\{\pm x_0\}, 0)$$

and $s$ cannot be nowhere zero. Thus $\gamma_n^1$ is not trivial. □

**Example 2.7.** Let us have a closer look at the space $E(\gamma_n^1)$ for the special case $n = 1$. In this case, each point $e = (\{\pm x\}, v)$ of $E(\gamma_n^1)$ can be written as

$$e = (\{\pm(\cos\theta, \sin\theta)\}, t(\cos\theta, \sin\theta)) \text{ with } 0 \le \theta \le \pi, t \in \mathbb{R}.$$

This representation is unique except that for the point

$$(\{\pm(\cos 0, \sin 0)\}, t(\cos 0, \sin 0)) = (\{\pm(\cos\pi, \sin\pi)\}, -t(\cos\pi, \sin\pi)) \text{ for each } t \in \mathbb{R}.$$

In other words, $E(\gamma_n^1)$ can be obtained from the strip $[0, \pi] \times \mathbb{R}$ in the $(\theta, t)$-plane by identifying the left hand boundary $\{0\} \times \mathbb{R}$ with the right hand boundary $\{\pi\} \times \mathbb{R}$ under the correspondence

$$(0,t) \mapsto (\pi, -t).$$

Thus $E(\gamma_n^1)$ is an open Möbius band over $\mathbb{R}\mathrm{P}^1$. Since $\mathbb{R}\mathrm{P}^1$ is just $S^1$ we see that in this case $\gamma_1^1$ is just the Möbius bundle over $S^1$ we defined in the previous lecture. And we see once again that $\gamma_1^1$ is non-trivial.

Another strategy to distinguish non isomorphic bundles is to look at the complement of the zero section. For any vector bundle isomorphism must the zero section to the zero section. Hence it induces a homeomorphism on the complements of the zero sections.

**Example 2.8.** This gives us another way to see that the Möbius bundle is nontrvival. The complement of the zero section of the Möbius bundle is connected but the complement of the zero section of the product bundle $S^1 \times \mathbb{R}$ is not connected.