Math 231b
Lecture 03

G. Quick

3. Lecture 3: Families of sections

We have seen in the proof that the canonical line bundle over the projective space is nontrivial that it can be very helpful to study the sections of a bundle. Today we want to push this idea a little further.

Definition 3.1. Let $\{s_1, \ldots, s_n\}$ be a collection of sections of a vector bundle $\pi\colon E \to B$. The sections $s_1, \ldots, s_n$ are called *nowhere linearly dependent* if, for each $b \in B$ the vectors $s_1(b), \ldots, s_n(b)$ are linearly independent.

The existence of nowhere dependent sections is rather special.

Theorem 3.2. *An n-dimensional vector bundle ξ is trivial if and only if ξ admits n sections $s_1, \ldots, s_n$ which are nowhere linearly dependent.*

The proof will depend on the following basic result.

Lemma 3.3. *Let ξ and η be vector bundles over B and let $f : E(\xi) \to E(\eta)$ be a continuous function which maps each vector space $E_b(\xi)$ isomorphically onto the corresponding vector space $E_b(\eta)$. Then f is necessarily a homeomorphism and ξ is isomorphic to η.*

Proof. The hypothesis on what f does with the fibers implies that f is bijective. Hence it remains to show that f^{-1} is continuous. This is a local question so let $b_0 \in B$ be any point and choose local trivializations (U,g) for ξ and (V,h) for η with $b_0 \in U \cap V$. Then we want to show that the composition

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{h^{-1} \circ f \circ g} (U \cap V) \times \mathbb{R}^n$$

is a homeomorphism. Setting

$$h^{-1}(f(g(b,x))) = (b,y)$$

it is evident that $y = (y_1, \ldots, y_n)$ can be expressed in the form

$$y_i = \sum_j f_{ij}(b) x_j$$

where $(f_{ij}(b))$ denotes an invertible $n \times n$-matrix of real numbers. Furthermore, since h^{-1}, f and g are continuous maps, the entries $f_{ij}(b)$ depend continuously on b.

Let $(F_{ji}(b))$ denote the inverse matrix. Then we have

$$g^{-1} \circ f^{-1} \circ h(b,y) = (b,x)$$

where

$$x_j = \sum_i F_{ji}(b) y_i.$$

Since the inverse of a mtrix A is given by $1/\det(A)$ times the adjoint matrix, the numbers $F_{ji}(b)$ depend continuously on the entries $f_{ij}(b)$. Hence they depend continuously on b. Thus $g^{-1} \circ f^{-1} \circ h$ is continuous. This completes the proof of the lemma □

Proof of Theorem 3.2. Let $s_1, \ldots, s_n$ be sections of ξ which are nowhere linearly dependent. Define

$$f\colon B \times \mathbb{R}^n \to E$$

by

$$f(b,x) = x_1 s_1(b) + \ldots + x_n s_n(b).$$

Evidently, f is continuous and maps each fiber of the trivial bundle ϵ^n_B isomorphically onto the corresponding fiber of ξ. The previous lemma implies that f is an isomomorphism of bundles and ξ is trivial.

Conversely, suppose that ξ is trivial, with trivialization (B,h). Defining

$$s_i(b) = h(b, (0, \ldots, 0, 1, 0, \ldots, 0)) \in E_b(\xi)$$

(with the 1 in the i-th place), it is evident that $s_1, \ldots, s_n$ are nowhere linearly dependent sections. This completes the proof of Theorem 3.2. □

Example 3.4. The tangent bundle of the circle $S^1 \subset \mathbb{R}^2$ admits one nowhere zero section

$$s(x_1, x_2) = ((x_1, x_2), (-x_2, x_1)).$$

We can rewrite this in terms of complex numbers. If we set $z = x_1 + ix_2$ then the section s is given by

$$z \mapsto iz.$$

Example 3.5. The tangent bundle to the 3-sphere $S^3 \subset \mathbb{R}^4$ admits three nowhere linearly dependent sections $s_i(x) = (x,\bar{s}_i(x))$ where

$$\begin{aligned} \bar{s}_1(x) &= (-x_2, x_1, -x_4, x_3) \\ \bar{s}_2(x) &= (-x_3, x_4, x_1, -x_2) \\ \bar{s}_3(x) &= (-x_4, -x_3, x_2, x_1). \end{aligned}$$

It is easy to check that the three vectors $\bar{s}_1(x)$, $\bar{s}_2(x)$, and $\bar{s}_3(x)$ are orthogonal to each other and to $x = (x_1, x_2, x_3, x_4)$. Hence s_1, s_2, and s_3 are nowhere linearly dependent sections of the tangent bundle of S^3 in $\mathbb{R}^4$.

The above formulas come in fact from the quaternion multiplication in $\mathbb{R}^4$. For let $\mathbb{H}$ be the quaternions, i.e., the division algebra whose elements are expressions of the form $z = x_1 + ix_2 + jx_3 + kx_4$ with $x_1, \ldots, x_4 \in \mathbb{R}$ subject to the multiplication rules

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i, \text{ and } ik = -j.$$

If we identify $\mathbb{H}$ with $\mathbb{R}^4$ via the coordinates (x_1, x_2, x_3, x_4) then we can describe the three sections s_1, s_2, and s_3 of the tangent bundle of S^3 in $\mathbb{H}$ by the formulas

$$\begin{aligned} \bar{s}_1(z) &= iz \\ \bar{s}_2(z) &= jz \\ \bar{s}_3(z) &= kz. \end{aligned}$$

Remark 3.6. If the tangent bundle of a manifold is trivial then one says that the manifold is parallelizable. Hence the last two examples show that S^1 and S^3 are parallelizable.