Math 231b
Lecture 04

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4. Lecture 4: Constructing new bundles out of old

We already have a bunch of examples of bundles at hand. But we'd like to be able to construct new bundles out of known ones. We will see some basic constructions for new bundles today.

4.1. **Restricting a bundle to a subset of the base space.** Let ξ be a vector bundle with projection $\pi\colon E \to B$ and let U be a subset of B. Setting $E|U = \pi^{-1}(U)$, and letting

$$\pi|U\colon E|U = \pi^{-1}(U) \to U$$

be the restriction of π to $E|U$, one obtains a new vector bundle which will be denoted by $\xi|U$, and called the *restriction* of ξ to U.

Each fiber $E_b(\xi|U)$ is just equal to the corresponding fiber $E_b(\xi)$, and is given the same vector space structure.

4.2. **Induced or pullback bundles.** Let ξ be a vector bundle over B and let B_1 be an arbitrary topological space. Given a continuous map $f\colon B_1 \to B$ one can construct the *induced bundle* or *pullback bundle* $f^*\xi$ over B_1 as follows. The total space E_1 of $f^*\xi$ is the subset $E_1 \subset B_1 \times E$ consisting of all pairs (b,e) such that $f(b) = \pi(e)$, or in a formula

$$E_1 = \{(b,e) \in B_1 \times E |\ f(b) = \pi(e)\}.$$

The projection map $\pi_1\colon E_1 \to B_1$ is defined by $\pi_1(b,e) = b$. Thus one has a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\hat{f}"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ B_1 \arrow[r, "f"] & B \end{tikzcd}$$

where $\hat{f}(b,e) = e$. The vector space structure in $\pi^{-1}(b)$ is defined by

$$t_1(b,e_1) + t_2(b,e_2) = (b, t_1e_1 + t_2e_2).$$

Thus $\hat{f}$ carries the vector space $E_b(f^*(\xi)$ isomorphically onto the vector space $E_{f(b)}(\xi)$.

It remains to specify the local trivializations of $f^*\xi$. If (U, h) is a local trivialization for ξ, we set $U_1 = f^{-1}(U)$ and define

$$h_1 \colon U_1 \times \mathbb{R}^n \to \pi_1^{-1}(U_1) \text{ by } h_1(b,x) = (b, h(f(b), x)).$$

Then (U_1, h_1) is a local trivialization of $f^*\xi$.

Example 4.1. If ξ is trivial, then $f^*\xi$ is trivial. For if $E = B \times \mathbb{R}^n$ then the total space E_1 of $f^*(\xi)$ consists of the triples (b_1, b, x) in $B_1 \times B \times \mathbb{R}^n$ with $b = f(b_1)$. Hence b does not induce any restriction and E_1 is just the product $B_1 \times \mathbb{R}^n$.

Remark 4.2. If $f \colon B_1 \to B$ is an inclusion map, then there is an isomorphism

$$E|B_1 \cong f^*(E)$$

given by sending $e \in E$ to the point $(\pi(e),e)$.

We still have not yet said what a map between bundles over different base spaces should be. The above construction inspires the following definition.

Definition 4.3. Let ξ and η be two vector bundles. A *bundle map* from η to ξ is a continuous map

$$g \colon E(\eta) \to E(\xi)$$

which carries each vector space $E_b(\eta)$ isomorphically onto one of the vector spaces $E_{b'}(\xi)$ for some $b' \in B(\xi)$.

Remark 4.4. Setting $\bar{g}(b) = b'$, we obtain a map

$$\bar{g} \colon B(\eta) \to B(\xi).$$

This map is continuous. For $\bar{g}$ is completely determined by g, since the projection map π_η of η is surjective:

$$\begin{tikzcd} E(\eta) \arrow[r, "g"] \arrow[d, "\pi_\eta"'] & E(\xi) \arrow[d, "\pi_\xi"] \\ B(\eta) \arrow[r, "\bar{g}"] & B(\xi). \end{tikzcd}$$

Now since the question is local, we can choose a local trivialization (U,h) of ξ. Then it suffices to prove the assertion for a map of trivial bundles and a diagram

$$\begin{tikzcd} V \times \mathbb{R}^n \arrow[r, "g"] \arrow[d, "\pi_\eta"'] & U \times \mathbb{R}^n \arrow[d, "\pi_\xi"] \\ V \arrow[r, "\bar{g}"] & U. \end{tikzcd}$$

But now it is clear that $\bar{g}$ is continuous since g is continuous and $\bar{g}(b)$ is just the first coordinate of $g(b,x)$.

Lemma 4.5. *If $g\colon E(\eta) \to E(\xi)$ is a bundle map, and if $\bar{g}\colon B(\eta) \to B(\xi)$ is the corresponding map of base spaces, then η is isomorphic to the induced bundle $\bar{g}^*\xi$.*

Proof. Define

$$h\colon E(\eta) \to E(\bar{g}^*\xi) \text{ by } h(e) = (\pi(e), g(e))$$

where π denotes the projection map of η. Since h is continuous and maps each fiber $E_b(\eta)$ isomorphically onto the corresponding fiber $E_b(\bar{g}^*\xi)$, it follows from the lemma of the previous lecture that h is an isomorphism. □

The previous lemma shows the following uniqueness statement.

Proposition 4.6. *Given a map $f\colon B_1 \to B$ and a vector bundle ξ over B, then $f^*\xi$ is up to isomorphism the unique vector bundle ξ' over B_1 which is equipped with a map to ξ which takes the fiber of ξ' over b isomorphically onto the fiber of ξ over $f(b)$ for each $b \in B_1$.*

Moreover, the pullback construction is natural in the following sense: If we have another continuous map $g\colon B_2 \to B_1$, then there is a natural isomorphism

$$g^* f^*(\xi) \cong (f \circ g)^*(\xi)$$

given by sending each point of the form

$$(b, e) \text{ to the point } (b, g(b), e), \text{ where } b \in B_2, e \in E.$$

Conclusion 4.7. For a space B let $\mathrm{Vect}^n(B)$ denote the set of isomorphism classes of n-dimensional vector bundles over B. Then a continuous map

$$f\colon B_1 \to B$$

induces a map

$$f^*\colon \mathrm{Vect}^n(B) \to \mathrm{Vect}^n(B_1) \text{ sending } \xi \text{ to } f^*\xi.$$

4.3. **Cartesian products.** Given two vector bundles ξ_1, ξ_2 with projection maps $\pi_i\colon E_i \to B_i$, $i = 1, 2$, the *Cartesian product* $\xi_1 \times \xi_2$ is defined to be the bundle with projection map

$$\pi_1 \times \pi_2\colon E_1 \times E_2 \to B_1 \times B_2$$

where each fiber

$$(\pi_1 \times \pi_2)^{-1}(b_1, b_2) = E_{b_1}(\xi_1) \times E_{b_2}(\xi_2)$$

is given the obvious vector space structure.

4.4. **Whitney sums.** Now let ξ_1, ξ_2 be two vector bundles over the same space B. Let

$$d\colon B \to B \times B$$

denote the diagonal embedding. The bundle $d^*(\xi_1 \times \xi_2)$ over B is called the *Whitney sum* of ξ_1 and ξ_2, and will be denoted $\xi_1 \oplus \xi_2$. Each fiber $E_b(\xi_1 \oplus \xi_2)$ is canonically isomorphic to the direct sum of the fibers $E_b(\xi_1) \oplus E_b(\xi_2)$.

Definition 4.8. Consider two vector bundles ξ and η over the same base space B with $E(\xi) \subset E(\eta)$. Then ξ is a *sub-bundle* of η, written $\xi \subset \eta$, if each fiber $E_b(\xi)$ is a sub-vector space of the corresponding fiber $E_b(\eta)$.

Lemma 4.9. *Let ξ_1 and ξ_2 be sub-bundles of η such that each vector space $E_b(\eta)$ is equal to the direct sum of the sub-spaces $E_b(\xi_1)$ and $E_b(\xi_2)$. Then η is isomorphic to the Whitney sum $\xi_1 \oplus \xi_2$.*

Proof. Define a map

$$f\colon E(\xi_1 \oplus \xi_2) \to E(\xi) \text{ by } f(b, e_1, e_2) = e_1 + e_2.$$

The lemma of the previous lecture shows that f is an isomorphism of bundles since it maps the fibers isomorphically onto each other. □

4.5. **Euclidian vector bundles.** Let V be a finite dimensional real vector space. Recall that a real valued function $q\colon V \to \mathbb{R}$ is called *quadratic* if q satisfies $q(av) = a^2 q(v)$ for every $v \in V$ and $a \in \mathbb{R}$ and the map $b\colon V \times V \to \mathbb{R}$ defined by

$$b(v,w) := \frac{1}{2}(q(v+w) - q(v) - q(w))$$

is a symmetric bilinear pairing. We also write $v \cdot w$ for $b(v,w)$. We have in particular: $v \cdot v = q(v)$. The quadratic function q is called *positive definite* if $q(v) > 0$ for every $v \neq 0$.

Definition 4.10. A *Euclidean vector space* is a real vector space V together with a positive definite quadratic function

$$q\colon V \to \mathbb{R}.$$

The real number $v \cdot w$ is called *inner product* of the vectors v and w. The number $q(v) = v \cdot v$ is also denoted by $|v|^2$.

Definition 4.11. A *Euclidean vector bundle* is a real vector bundle ξ together with a continuous map

$$q\colon E(\xi) \to \mathbb{R}$$

such that the restriction of q to each fiber of ξ is positive definite and quadratic. The map q is called a *Euclidian metric* on ξ.

In the case of the tangent bundle τ_M of a smooth manifold, a Euclidian metric $q\colon DM \to \mathbb{R}$ is called a *Riemannian metric*, and M together with q is called a *Riemannian manifold*.

Example 4.12. a) The trivial bundle ϵ^n_B on a space B can be given the Euclidean metric

$$q(b,x) = x_1^2 + \ldots + x_n^2.$$

b) Since the tangent bundle of $\mathbb{R}^n$ is trivial it follows that the smooth manifold $\mathbb{R}^n$ possesses a standard Riemannian metric. Moreover, any smooth manifold $M \subset \mathbb{R}^n$, the composition

$$DM \subset D\mathbb{R}^n \xrightarrow{q} \mathbb{R}$$

makes M into a Riemannian manifold.

Lemma 4.13. *Let ξ be a trivial bundle of dimension n over a space B and let q be any Euclidean metric on ξ. Then there exist n sections $s_1, \ldots, s_n$ of ξ which are normal and orthogonal in the sense that*

$$s_i(b) \cdot s_j(b) = \delta_{ij}$$

for each $b \in B$ where δ_{ij} is the Kronecker symbol.

Proof. The lemma of the previous lecture shows that ξ admits n nowhere dependent sections. Pointwise application of the Gram-Schmidt orthonormalization process yields orthonormal sections. $\square$

4.6. **Orthogonal complements.** Given a sub-bundle $\xi \subset \eta$, is there a complementary sub-bundle so that η splits as a Whitney sum? If η is a Euclidean bundle, we can always find such a complement. We can construct it as follows.

Let $E_b(\xi^\perp)$ denote the subspace of $E_b(\eta)$ consisting of all vectors v such that $v \cdot w = 0$ for all $E_b(\xi)$. Let $E(\xi^\perp)$ denote the union of all $E_b(\xi^\perp)$.

Theorem 4.14. *The space $E(\xi^\perp)$ is the total space of a sub-bundle $\xi^\perp \subset \eta$, and η is isomorphic to the Whitney sum $\xi \oplus \xi^\perp$. The bundle $\xi^\perp$ is called the* orthogonal complement *of ξ in η.*

Proof. It is clear that each fiber $E_b(\eta)$ is the direct sum of the subspaces $E_b(\xi)$ and $E_b(\xi^\perp)$. Thus it remains to show the local triviality of $\xi^\perp$. The lemma of the previous lecture then implies that the map $(v,w) \mapsto v + w$ is an isomorphism of vector bundles.

Given any point $b_0 \in B$, let U be a neighborhood of b_0 which is sufficiently small that both $\xi|U$ and $\eta|U$ are trivial. Since $\xi|U$ is trivial, we can choose orthonormal sections $s_1, \ldots, s_m$ of $\xi|U$. We may enlarge this set of sections to a set of n

independent local sections of $\eta|U$ by first choosing $s'_{m+1}, \ldots, s'_n$ first in the fiber $E_{b_0}(\eta)$. By the continuity of the determinant function, there is a neighborhood $V \subset U$ of b_0 such that $s_1(b), \ldots, s_m(b), s'_{m+1}(b), \ldots, s'_n(b)$ are linearly independent for all $b \in V$ and such that the $s_i(b)$ vary continuously with b in V. Applying the Gram-Schmidt orthonormalization process to $s_1, \ldots, s_m, s'_{m+1}, \ldots, s'_n$ in each fiber to obtain new sections $s_1, \ldots, s_n$. The formulae for this process show that the s_i vary continuously with $b \in V$. We can now define a trivialization

$$h\colon V \times \mathbb{R}^{n-m} \to E(\xi^{\perp})$$

by the formula

$$h(b,x) = x_1 s_{m+1}(b) + \ldots + x_{n-m} s_n(b).$$

$\square$

4.7. **Stably trivial bundles.** The direct sum of two trivial bundles is of course again trivial. But the direct sum of two nontrivial bundles can also be trivial. If one bundle is trivial, this phenomenon has been given a name.

Definition 4.15. A vector bundle ξ over B is called *stably trivial* if the direct sum $\xi \oplus \epsilon^n$ is a trivial bundle for some n.

Example 4.16. The direct sum of the tangent bundle τ and the normal bundle ν to S^{n-1} in $\mathbb{R}^n$ is the trivial bundle $S^{n-1} \times \mathbb{R}^n$. For the elements of the direct sum $\tau \oplus \nu$ are triples $(x,v,tx) \in S^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n$ with $x \perp v$, and the map

$$(x, v, tx) \mapsto (x, v + tx)$$

gives an isomorphism of $\tau \oplus \nu$ with $S^{n-1} \times \mathbb{R}^n$. Since the normal bundle ν is trivial, this shows that τ is stably trivial.

But there are also examples where both bundle are nontrivial whereas their Whitney sum is trivial.

Example 4.17. Let γ_n^1 be the canonical line bundle on $\mathbb{R}\mathrm{P}^n$. Then the map $(\ell, v, w) \mapsto (\ell, v + w)$ for $v \in \ell$ and $w \perp \ell$ defines an isomorphism $\gamma_n^1 \oplus (\gamma_n^1)^{\perp} \cong \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$.

Example 4.18. Specializing the previous example to the case $n = 1$, we see that

$$\gamma_1^1 \oplus (\gamma_1^1)^{\perp} \cong \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^2 \cong S^1 \times \mathbb{R}^2.$$

The map that rotates a vector by 90 degrees defines an isomorphism between $(\gamma_1^1)^{\perp}$ and γ_1^1. Since γ_1^1 is isomorphic to the Möbius bundle over S^1, this shows that the direct sum of the Möbius bundle with itself is the trivial bundle.